Lecture 15 & 16 : Examples of Hilbert Spaces. Projection Theorem. Riesz Representation Theorem. Adjoint Operators.

Example 1. 1. *The space $\mathbb{R}^n$ is a Hilbert space over $\mathbb{R}$ with the standard inner product defined by*

$$\langle x, y\rangle := \sum_{k=1}^{n} x_k y_k \quad \text{for } x, y \in \mathbb{R}^n.$$

2. *The space $\mathbb{C}^n$ is a Hilbert space over $\mathbb{C}$ with inner product defined by*

$$\langle x, y\rangle := \sum_{k=1}^{n} x_k \overline{y_k} \quad \text{for } x, y \in \mathbb{C}^n.$$

3. *The space $L_2[a,b] = \left\{ f : [a,b] \to \mathbb{K} \quad s.t \quad \int_a^b |f(t)|^2 dt < \infty \right\}$ is a Hilbert space over $\mathbb{K}$ with inner product defined by $\langle f, g\rangle := \int_a^b f(t)\overline{g(t)}\, dt$.*

4. *The space $\ell_2 = \left\{ x = (x_1, x_2, \dots) \quad s.t \quad \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$ is a Hilbert space with inner product defined by*

$$\langle x, y\rangle := \sum_{k=1}^{\infty} x_k \overline{y_k} \quad \text{for } x = (x_k) \in \ell_2, y = (y_k) \in \ell_2.$$

5. *The space $(\ell^p, \|\cdot\|_p)$ with $p \neq 2$ is not an inner product space.*

Proof. We will show that the norm does not satisfy the parallelogram equality.
Take $x = (1, 1, 0, 0, \cdots) \in \ell_p$ and $y = (1, -1, 0, 0, \cdots) \in \ell_p$. Then

$$\|x\| = \|y\| = 2^{1/p}, \|x + y\| = \|x - y\| = 2$$

So the parallelogram equality is not satisfied. □

6. *The space $(C[a,b], \|\cdot\|_\infty)$ is not an inner product space, hence not a Hilbert space.*

Proof. Take $f(t) = 1$ and $g(t) = \dfrac{t-a}{b-a}$. We have $\|f\| = \|g\| = 1$ and $\|f + g\| = 2, \|f - g\| = 1$. So the parallelogram equality is not satisfied. □

Proposition 22. *If in an inner product space, $x_n \to x$ and $y_n \to y$ then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Proof. We have

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \leq$$
$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \leq \|x_n\|\, \|y_n - y\| + \|x_n - x\|\, \|y\|$$

Let $n \to \infty$, we get

$$0 \leq \lim_{n\to\infty} |\langle x_n, y_n\rangle - \langle x, y\rangle| \leq \|x\| 0 + 0 \|y\| = 0.$$

Hence, $\lim_{n\to\infty} |\langle x_n, y_n\rangle - \langle x, y\rangle| = 0$, $\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle$, which completes the proof. □

3.2 Orthogonal Projection

Definition 1. *Let A be a subset of an inner product space X. The orthogonal complement of A is defined as*

$$A^{\perp} := \{x \in X : \langle x, a\rangle = 0 \quad \text{for all} \;\; a \in A\}.$$

Proposition 23. *Let A be a subset of an inner product space X. Then $A^{\perp}$ is a closed linear subspace of X and $A \cap A^{\perp} \subset \{0\}$.*

Proof. Exercise. □

Theorem 1. *Let Y be a closed linear subspace of the real or complex Hilbert space X and $x \in X$ be given. Then the following holds*

(i) There exists a unique $y \in Y$ such that

$$\|x - y\| = \min_{z \in Y} \|x - z\|$$

(ii) The point y in part (i) is the unique vector in Y such that $x - y \in Y^{\perp}$.

The point y is called the orthogonal projection of x onto the subspace Y.

Proof. (i) *Existence.* Denote $d = \inf_{z\in Y} \|x - z\|$. By the definition of the infimum, there exists a sequence $\{y_n\} \subset Y$ such that $\|x - y_n\| \to d$ as $n \to \infty$. We will prove that $\{y_n\}$ is a Cauchy sequence.
Using parallelogram law, we have

$$\|y_n - y_m\|^2 + 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 = 2(\|x - y_n\|^2 + \|x - y_m\|^2) \quad \text{for} \;\; n, m \geq 1.$$

Since Y is a linear subspace of X and $y_n, y_m \in Y$, we have $\frac{1}{2}(y_n + y_m) \in Y$. Therefore, $\left\|x - \frac{1}{2}(y_n + y_m)\right\| \geq d$. Hence

$$\|y_n - y_m\|^2 = 2(\|x - y_n\|^2 + \|x - y_m\|^2) - 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 \leq 2(\|x - y_n\|^2 + \|x - y_m\|^2) - 4d^2.$$

Let $n, m \to \infty$, we have $\|x - y_n\| \to d$, $\|x - y_m\| \to d$ and

$$0 \leq \lim_{n,m\to\infty} \|y_n - y_m\|^2 \leq 4d^2 - 4d^2 = 0.$$

Therefore, $\lim_{n,m\to\infty} \|y_n - y_m\| = 0$ and $\{y_n\}$ is a Cauchy sequence. Since X is a Hilbert space, there exists $y \in X$ such that $\lim_{n\to\infty} y_n = y$. Since $y_n \in Y$ and Y is closed, $y \in Y$. In conclusion, we have

$$\|x - y\| = \min_{z\in Y} \|x - z\|.$$

Uniqueness. Suppose there is also $\hat{y} \in Y$ such that

$$\|x - \hat{y}\| = \min_{z\in Y} \|x - z\|.$$

Applying the parallelogram law and using $\left\|x - \frac{1}{2}(y + \hat{y})\right\| \geq d$ (since $\frac{1}{2}(y + \hat{y}) \in Y$), we have

$$\|y - \hat{y}\|^2 + 4\left\|x - \frac{1}{2}(y + \hat{y})\right\|^2 = 2(\|x - y\|^2 + \|x - \hat{y}\|^2) = 4d^2,$$

$$0 \leq \|y - \hat{y}\|^2 = 4d^2 - 4\left\|x - \frac{1}{2}(y + \hat{y})\right\|^2 \leq 0.$$

So $\|y - \hat{y}\|^2 = 0,\ y = \hat{y}$.
(ii). *Orthogonality.* Clearly, $\langle x - y, 0\rangle = 0$. Take $z \in Y, z \neq 0$. We will prove that $\langle x - y, z\rangle = 0$. By the construction of y in part (i), we have

$$\|x - y\|^2 \leq \|x - (y + \lambda z)\|^2 = \|x - y\|^2 + |\lambda|^2\|z\|^2 - \lambda\langle z, x - y\rangle - \overline{\lambda}\langle x - y, z\rangle,$$

$$0 \leq |\lambda|^2\|z\|^2 - \lambda\overline{\langle x - y, z\rangle} - \overline{\lambda}\langle x - y, z\rangle.$$

Plugging $\lambda = \frac{\langle x - y, z\rangle}{\|z\|^2}$ into the above inequality, we conclude

$$\frac{|\langle x - y, z\rangle|^2}{\|z\|^2} \leq 0,$$

which only happens when $\langle x - y, z\rangle = 0$.
Uniqueness. Assume there is also $y^* \in Y$ such that $x - y^* \in Y^\perp$. Then $y - y^* = (x - y^*) - (x - y) \in Y^\perp$. On the other hand, $y - y^* \in Y$ since $y, y^* \in Y$. So $y - y^* \in Y \cap Y^\perp \subset \{0\}$. Therefore, $y - y^* = 0$ and $y = y^*$. □

Note in part (i), we only need the condition that if $y_n, y_m \in Y$ then its average is also in Y. Therefore, we have a more general result for part (i).

Theorem 2 (Hilbert's Projection Theorem)**.** *Given a closed convex set Y in a Hilbert space X and $x \in X$. There exists a unique $y \in Y$ such that*

$$\|x - y\| = \min_{z \in Y} \|x - z\|.$$

Corollary 5 (Orthogonal Decomposition)**.** *Let Y be a closed linear subspace of the real or complex Hilbert space X. Then every vector $x \in X$ can be uniquely represented as*

$$x = y + w, \quad y \in Y,\ w \in Y^\perp.$$

The orthogonal decomposition is usually written as $X = Y \oplus Y^\perp$.

Note: We can prove that $X/Y \cong Y^\perp$ (linear isomorphism).

Definition 2 (Orthogonal Projection)**.** *Let Y be a closed linear subspace of the real or complex Hilbert space X. The map $P_Y : X \to X, P_Y(x) = y$, where $x = y + w$ and $(y, w) \in Y \times Y^\perp$, is called the orthogonal projection in X onto Y.*

3.3 Riesz Representation Theorem

Lemma 7. *Let $(X, \langle,\rangle)$ be an inner product space. Then*

1. $\langle x, 0\rangle = \langle 0, x\rangle = 0, \quad \forall x \in X$
2. *If there are $y_1, y_2 \in X$ such that $\langle x, y_1\rangle = \langle x, y_2\rangle$ for all $x \in X$, then $y_1 = y_2$.*

Proof. Exercise. □

Theorem 1 (Riesz Representation Theorem)**.** *Let X be a Hilbert space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

1. *For every $y \in X$, the functional $f : X \to \mathbb{K}$, $f(x) = \langle x, y\rangle$ is an element in X^* and $\|f\| = \|y\|$.*
2. *Conversely, for every $f \in X^*$, there exists a unique $y \in X$ such that $f(x) = \langle x, y\rangle$ for every $x \in X$. Moreover, $\|f\| = \|y\|$.*

Proof. (1). If $y = 0$, then the function $f(x) = \langle x, 0\rangle = 0$, for every $x \in X$, is an element in X^* and $\|f\| = 0 = \|y\|$.
If $y \neq 0$, we first verify that $f(x) = \langle x, y\rangle$ is linear. (prove this).
Using Cauchy-Schwarz inequality, we have

$$|f(x)| = |\langle x, y\rangle| \leq \|x\|\|y\|.$$

So f is bounded and $\|f\| \leq \|y\|$.
On the other hand, $\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(y)|}{\|y\|} = \|y\|$. Therefore, $\|f\| = \|y\|$.
(2). *Existence.* Consider $f \in X^*$. If $f = 0$, we can take $y = 0$ and $f(x) = \langle x, 0\rangle$ for every $x \in X$.
Consider $f \in X^*, f \neq 0$.

- Step 1: Prove that $\dim(X/\ker f) = 1$.
  Since $f \neq 0$, $\mathrm{Im} f \neq \{0\}$. Moreover, since $\mathrm{Im} f$ is a subspace of $\mathbb{K}$ and $\dim_{\mathbb{K}} \mathbb{K} = 1$, we have $\dim \mathrm{Im} f \leq 1$. So $\dim \mathrm{Im} f = 1$. From the linear isomorphism $X/\ker f \cong \mathrm{Im}\, f$, we have $\dim(X/\ker f) = 1$.

- Step 2: Verify $\ker f$ is a closed subspace of X. Indeed, we know that $\ker f$ is a linear subspace of X. Also, since f is continuous and $\{0\} \subset \mathbb{K}$ is a closed set of $\mathbb{K}$, $\ker f = f^{-1}\{0\}$ is a closed set of X.

- Step 3: By the orthogonal decomposition theorem, we have $X = \ker f \oplus (\ker f)^{\perp}$ and $\dim(\ker f)^{\perp} = \dim(X/\ker f) = 1$. Without loss of generality (by scaling), we can assume that $(\ker f)^{=}Span(y_0)$ for some $y_0 \in X$ and $f(y_0) = 1$ (note that $y_0 \notin \ker f$ so $f(y_0) \neq 0$, therefore we do the scaling).

- Step 4: Find y.
  Take $x \in X$. Then

$$f(x - f(x)y_0) = f(x) - f(x)f(y_0) = f(x) - f(x) = 0.$$

Therefore, $w = x - f(x)y_0 \in \ker f$. So $\langle w, y_0\rangle = 0$. Now compute

$$\langle x, y_0\rangle = \langle w + f(x)y_0, y_0\rangle = \langle w, y_0\rangle + f(x)\langle y_0, y_0\rangle = f(x)\langle y_0, y_0\rangle,$$

$$f(x) = \langle x, \frac{y_0}{\|y_0\|^2}\rangle.$$

Set $y = \dfrac{y_0}{\|y_0\|^2}$.

Uniqueness If there is alo $\hat{y}$ such that $f(x) = \langle x, \hat{y}\rangle$ for all $x \in X$, then

$$\langle x, y\rangle = \langle x, \hat{y}\rangle \quad \text{for every} \quad x \in X.$$

By above lemma, $y = \hat{y}$.

□

3.4 Hilbert Adjoint Operator

Definition 1. *Let $T : H \to H$ be a bounded linear operator, where H is a Hilbert space. Then the Hilbert adjoint operator T^* of T is an operator $T^* : H \to H$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \textit{for all } x, y \in H.$$

Theorem 1. *Let $T : H \to H$ be a bounded linear operator, where H is a Hilbert space. Then the Hilbert adjoint operator T^* of T exists and moreover, T^* is also a bounded linear operator and $\|T^*\| = \|T\|$.*

Proof.

- Step 1: Construct T^*. Let $y \in H$. Define $l : H \to \mathbb{K}$, $l(x) := \langle Tx, y\rangle$. We can verify that $l \in H^*$ (check this). Moreover, using Cauchy-Schwarz inequality and the boundedness of T, we have

$$|l(x)| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\| = |T\|\|y\|\|x\|.$$

Therefore l is bounded. By the Riesz representation theorem, there exists a unique $y^* \in H$ such that

$$l(x) = \langle x, y^*\rangle \quad \text{for every} \quad x \in X.$$

and $\|l\| = \|y^*\|$. We define $T^* : H \to H, T^*(y) = y^*$. Then clearly,

$$\langle Tx, y\rangle = l(x) = \langle x, y^*\rangle = \langle x, T^*y\rangle.$$

- Step 2: Verify T^* is linear. Indeed, let $\alpha, \beta \in \mathbb{K}$ and $y, z \in H$. We have

$$\begin{aligned}\langle x, T^*(\alpha y + \beta z)\rangle &= \langle Tx, \alpha y + \beta z\rangle = \langle Tx, \alpha y\rangle + \langle Tx, \beta z\rangle = \overline{\alpha}\langle Tx, y\rangle + \overline{\beta}\langle Tx, z\rangle \\ &= \overline{\alpha}\langle x, T^*y\rangle + \overline{\beta}\langle x, T^*z\rangle = \langle x, \alpha T^*y + \beta T^*z\rangle.\end{aligned}$$

Since this equality holds for every $x \in H$, from Lemma 7, we have

$$T^*(\alpha y + \beta z) = \alpha T^*y + \beta T^*z \quad \text{for all } y, z \in H, \alpha, \beta \in \mathbb{K}.$$

- Verify T^* is bounded. Indeed, we have

$$\|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \leq \|TT^*y\|\|y\| \leq \|T\|\|T^*y\|\|y\|.$$

  So $\|T^*y\| \leq \|T\|\|y\|$ for every $y \in Y$. Therefore T^* is bounded and $\|T^*\| \leq \|T\|$.

- Show that $\|T^*\| = \|T\|$.
  Since $T^* \in B(H)$, we can apply steps 1,2,3 and have $T^{**} \in B(H)$ and $\|T^{**}\| \leq \|T^*\|$.
  On the other hand, for every $x, y \in H$, we get

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, T^{**}x\rangle} = \langle T^{**}x, y\rangle,$$

  where the first and the third equalities come from the definition of T^* and T^{**}. Since $\langle Tx, y\rangle = \langle T^{**}x, y\rangle$ for all $y \in H$, we have $Tx = T^{**}x$ for all $x \in H$. Therefore, $T = T^{**}$. In conclusion, we have

$$\|T\| \geq \|T^*\| \geq \|T^{**}\| = \|T\|,$$

  which implies $\|T\| = \|T^*\|$.

□

Lemma 8. *Let X and Y be inner product spaces and $T \in B(X, Y)$. Then*

1. *$T = 0$ if and only if $\langle Tx, y\rangle = 0$ for all $x \in X$, $y \in Y$.*

2. *If $T : X \to X$ and X is a complex inner product space and $\langle Tx, x\rangle = 0$ for all $x \in X$, then $T = 0$.*

Proof. Exercise. □

Definition 2. *A bounded linear operator $T : H \to H$ on a Hilbert space H is said to be*

- *self-adjoint (Hermitian) if $T^* = T$, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$.*

- *unitary if T is bijective and $T^* = T^{-1}$.*

- *normal if $TT^* = T^*T$.*

Clearly, if T is self-adjoint or unitary, T is normal.

Proposition 24. *Let $T : H \to H$ be a bounded linear operator on a Hilbert space H and T be onto. Then T is unitary if and only if $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$.*

Proof. In class. □