Properties of the adjoint operator

Let H_1, H_2 Hilbert spaces, $S: H_1 \to H_2$, $T: H_1 \to H_2$ bounded linear operators, $\alpha \in \mathbb{K}$. Then

$$\langle T^* y, x\rangle = \langle y, Tx\rangle.$$

$$(S+T)^* = S^* + T^*$$

$$(\alpha T)^* = \bar{\alpha} T^*$$

$$(T^*)^* = T$$

$$\|T^*T\| = \|TT^*\| = \|T\|^2$$

$$T^*T = 0 \Leftrightarrow T = 0.$$

$$(ST)^* = T^* S^* \quad (H_1 = H_2)$$

Definition A bounded linear operator $T: H \to H$ on a Hilbert space H. is said to be
1. self-adjoint or Hermitian if $T^* = T$, $\langle Tx, y\rangle = \langle x, Ty\rangle$
2. unitary if T is bijective and $T^* = T^{-1}$.
3. normal if $TT^* = T^*T$.

Clearly, If T is self-adjoint or unitary, T is normal.

Prop $T: H \to H$, H Hilbert space, T bounded linear operator and T is onto. Then T is unitary iff $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$.

Proof ($\Rightarrow$) $\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, y\rangle$.

($\Leftarrow$) $\langle x, y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle$ so $T^*T = Id$ ~~fors~~

T is onto
T is one-to-one. $Tx = 0$ then $x = 0$ so T is bijective, T^{-1} exists

$$TT^* = TT^*TT^{-1} = TT^{-1} = Id \quad \text{so } T^* = T^{-1}.$$

- Recall Riesz Representation Theorem — Theorem 2.E

Let H be a Hilbert space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) (note $\langle \alpha u+\beta v, w\rangle = \alpha\langle u,w\rangle+\beta\langle v,w\rangle$ $\forall \alpha,\beta\in\mathbb{K}$, $u,v,w\in H$) (Eq1)

Then for every $f\in H^*$, $\exists!\, y\in H$ s.t

$$f(x)=\langle x,y\rangle, \quad \forall x\in H.$$

Note some books use the following definition of inner product (

$$\langle u, \alpha v+\beta w\rangle = \alpha\langle u,v\rangle+\beta\langle u,w\rangle$$

instead of Eq1. Then the Riesz-Representation Theorem will be

$$\forall f\in H^*, \exists!\, y\in H \text{ s.t } f(x)=\langle y,x\rangle, \quad \forall x\in H.$$

Recall orthogonal projection theorem — Theorem 2.D

Let $\underline{Y}$ be a nonempty closed linear subspace of a Hilbert space H and $x\in H$. Then ① $\exists!\, y\in \underline{Y}$ s.t $\|x-y\| = \min_{z\in Y}\|x-z\|$

Denote $y = P_{\underline{Y}}\, x$

② $\langle x-P_{\underline{Y}}\,x, z\rangle = 0$ for every $z\in Y$.

Today, we talk about another application of Riesz-Representation Theorem, which is a simple and efficient tool for solving linear partial differential equations.

Lax-Milgram Theorem Let H be a real Hilbert space. Consider a function $a: H \times H \to \mathbb{R}$ that has the following properties

1. bilinearity : $a(\alpha u + \beta v, w) = \alpha a(u,w) + \beta a(v,w)$

$$a(w, \alpha u + \beta v) = \alpha a(w,u) + \beta a(w,v)$$

for all $\alpha, \beta \in \mathbb{R}$, $u, v, w \in H$

2. continuity : There is a constant $C > 0$ s.t

$$|a(u,v)| \leq C \|u\| \, \|v\| \quad , \text{for all } u, v \in H$$

3. coercivity : There is a constant $\alpha > 0$ s.t

$$a(v,v) \geq \alpha \|v\|^2 \quad \text{for all } v \in H.$$

Then given any $\varphi \in H^*$, there is a unique element $u \in H$ s.t

$$\boxed{a(v,u) = \varphi(v), \quad \forall v \in H.} \qquad \text{Eq 2}$$

~~Moreover, if a is symmetric (that is $a(u,v) = a(v,u)$, $\forall u, v \in H$) then u is characterised by the property~~

$$\text{Eq3} \qquad \boxed{u \in H, \text{ and } \quad \frac{1}{2} a(u,u) - \varphi(u) = \min_{v \in H} \left\{ \frac{1}{2} a(v,v) - \varphi(v) \right\}}$$

~~In the language of calculus of variations, Eq2 is the Euler equation associated with the minimization problem Eq3.~~

Note: If H is a Hilbert space over $\mathbb{C}$, the function $a: H \times H \to \mathbb{C}$ is continuous, coercive, and sesiquilinear, we have the conclusion in Eq2

<u>Proof</u> $\boxed{\text{Step 1:}}$ apply the Riesz Representation Theorem for certain bounded linear functionals on H.

Since $\varphi \in H^*$, by the Riesz Representation theorem, there exists a unique $y \in H$ s.t. $\boxed{\varphi(v) = \langle v, y\rangle}$, $\forall v \in H$.

For each fixed $w \in H$, the map $v \overset{\psi_w}{\mapsto} a(v,w)$ is linear since a is bilinear. Moreover, ψ_w is bounded with

$$|\psi_w(v)| = |a(v,w)| \leq C\|v\|\|w\|,$$

hence $|\psi_w| \leq C\|w\|$.

By the Riesz Representation theorem for $\psi_w \in H^*$, $\exists!\ Aw \in H$

s.t. $\psi_w(v) = \langle v, Aw\rangle$, $\forall v \in H$

$$\boxed{a(v,w) = \langle v, Aw\rangle}, \quad \forall v \in H.$$

$\boxed{\text{Step 2}}$ Claim: A is linear, bounded, $\|Aw\| \leq C\|w\|$, and $\langle v, Av\rangle \geq \alpha\|v\|^2$.

$\|Av\| \geq \alpha\|v\|$

So $\|Av\| \geq \alpha\|v\|$

We have $a(v,w) = \langle v, Aw\rangle$. For $\alpha, \beta \in \mathbb{R}$, $w_1, w_2 \in H$, we have

$$a(v, \alpha w_1 + \beta w_2) = \langle v, A(\alpha w_1 + \beta w_2)\rangle$$

$\|$ Since a is bilinear

$$\alpha a(v,w_1) + \beta a(v,w_2) = \alpha\langle v, Aw_1\rangle + \beta\langle v, Aw_2\rangle$$

(note $\alpha, \beta \in \mathbb{R}$)

$$= \langle v, \alpha Aw_1 + \beta Aw_2\rangle$$

So $\langle v, A(\alpha w_1 + \beta w_2)\rangle = \langle v, \alpha Aw_1 + \beta Aw_2\rangle$, $\forall v, w_1, w_2 \in H$, $\alpha, \beta \in \mathbb{R}$

Hence $A(\alpha w_1 + \beta w_2) = \alpha Aw_1 + \beta Aw_2$.

Therefore A is linear.

From $|a(v,w)| \leq c\|v\|\|w\|$, we get

$$|\langle v, Aw\rangle| \leq c\|v\|\|w\| \quad , \forall v, w \in H$$

Choose $v = Aw$, we have

$$\|Aw\|^2 \leq c\|Aw\|\|w\|$$

Therefore, $\|Aw\| \leq c\|w\|$, $\forall w \in H$ (including the case $\|Aw\| = 0$)

So A is bounded.

Finally, from $a(v,v) \geqslant \alpha\|v\|^2$, we get

$$\langle v, Av\rangle \geqslant \alpha\|v\|^2.$$

<u>Step3</u> The problem turns into finding $u \in H$ s.t.

$$\langle v, Aw\rangle = \langle v, y\rangle \quad \forall v \in H$$

$$\langle v, Au-y\rangle = 0 \quad \forall v \in H.$$

Choose $v = Au-y$, So $Au-y = 0$

$$Au = y.$$

Since A is bounded below, ~~A^{-1} exists~~ A is one-to-one

A is onto. i.e. Range $A = H$

Suppose Range $A \subsetneq H$. Range A is a closed subspace of H (Prove this)

$\exists z \in (\text{Range } A)^{\perp}$. $z \neq 0$

$$\langle Av, z\rangle = 0 \quad \forall v \in H.$$

$$\cancel{a(z,v) = \langle z, Av\rangle = 0}$$

$$\cancel{a(z+v, z+v) \geqslant \alpha\|z+v\|^2}$$

$$\cancel{a(z)^2 + a(v,v) \geqslant}$$

$\langle Az, z\rangle = 0 \geqslant \alpha\|z\|^2 > 0$

a contradiction.

So A is bijective, A^{-1} exist

$u = A^{-1}y$ is the solution

Main Theorem on Quadratic Variational Problem (Theorem 2. A)

Let H be a real Hilbert space

Suppose that $a: H \times H \to \mathbb{R}$ is symmetric, bilinear, continuous, coercive and $b \in H^*$

① Then the variational problem $\boxed{\min_{v \in H} \left\{ \frac{1}{2} a(v,v) - b(v) \right\}}$ (Eq3)

has a unique solution $u \in H$

② The minimization problem (Eq3) is equivalent to the following variational equation: (Eq4)

~~Find $u \in H$ s.t~~ $\boxed{a(v,u) = b(v), \quad \forall v \in H}$, for a fixed $u \in H$.

Proof Prove the equivalence of Eq3 and Eq4.

$Eq3 \Rightarrow Eq4$

Suppose u is the solution of Eq3. We will prove $a(v,u) = \varphi(v)$ $\forall v \in H$.

Set $F(v) = \frac{1}{2} a(v,v) - b(v)$, $\forall v \in H$

For fixed v, consider $\phi : \mathbb{R} \to \mathbb{R}$

$$\phi(t) = F(u + tv), \quad \phi(t) \geq \phi(0).$$

Using the symmetry of a, we have

$$\phi(t) = \frac{1}{2} a(u + tv, u + tv) - b(u + tv)$$

$$= \frac{1}{2} a(u,u) - b(u) + t a(u,v) + \frac{t^2}{2} a(v,v) - t b(v).$$

$a(v,v) \geq \alpha \|v\|^2$ So $\phi(t)$ attains infimum at $t = 0$ iff

$$\phi'(0) = 0$$

$$0 \cdot a(v,v) + a(u,v) - b(v) = 0$$

$$\text{So } a(u,v) - b(v) = 0$$

Eq4 ⇒ Eq3 If $u \in H$ satisfies $a(v,u) = b(v)$ $\forall v \in H$

we will prove that $F(v) \geq F(u)$ $\forall v \in H$.

$v = u + w$

$$F(v) = \frac{1}{2} a(u+w, u+w) - b(u+w)$$

$$= \frac{1}{2} a(u,u) - b(u) + \underbrace{a(u,w) + \frac{1}{2} a(w,w) - b(w)}$$

$$= F(u) + \frac{1}{2} a(w,w) \geq F(u) + \frac{\alpha}{2} \|w\|^2 \geq F(u).$$

Step 2 Existence & uniqueness
From the lax Milgram Theorem.

Note The following three existence principles are mutually equivalent:

1. The existence principle for quadratic minimum problems
2. The orthogonal projection theorem
3. The Riesz Representation Theorem.

variants of the linear othogonality principle in Hilbert spaces.

Another variant of Lax-Milgram theorem for variational inequality

Theorem (Stampachia) Let H be a real Hilbert space. Assume $a: H \times H \to \mathbb{R}$ is a continuous, coercive, bilinear form on H. Let $K \subset H$ be a nonempty closed, and convex subset. Then given any $\varphi \in H^*$, $\exists!\, u \in K$ s.t

$$\boxed{a(v-u, u) \geq \varphi(v-u), \quad \forall v \in K.}$$

Moreover if a is symmetric, then u is characterized by the property

$$\boxed{u \in K, \quad \frac{1}{2} a(u,u) - \varphi(u) = \min_{v \in K} \left\{ \frac{1}{2} a(v,v) - \varphi(v) \right\}}$$

Nonlinear Monotone Operators (Banach Fixed Point Theorem + Riesz Representation)

Problem Solve the nonlinear operator equation

$$Au = z \ , \ u \in H$$

Theorem 1 Let H be a real Hilbert space. Assume $A: H \to H$ is strongly monotone, that is, there is a constant $c > 0$ such that

$$\langle u - v, Au - Av \rangle \geq c \|u - v\|^2 \ , \quad \forall u, v \in H$$

Also assume A is Lipschitz continuous, i.e., $\exists$ a constant $L > 0$ s.t

$$\|Au - Av\| \leq L \|u - v\|, \quad \forall u, v \in H.$$

Then for each given $z \in H$, the equation $Au = z$ has a unique solution.

This theorem "marks the beginning of the modern theory of monotone operators." (from Zeidler)

Proof

Fixed $t \in \mathbb{R}$, $t > 0$ (t will be specified later)

Consider $B: H \to H$, $B(u) = u - t(Au - z)$.

Case 1 $H = \{0\}$, the statement is trivial

Case 2 $H \neq \{0\}$. For all $u, v \in H$

$$\|Bu - Bv\|^2 = \|u - v - t(Au - Av)\|^2 = \|u - v\|^2 - 2t \langle u - v, Au - Av \rangle + t^2 \|Au - Av\|^2$$

$$\leq (1 - 2tc + t^2 L^2) \|u - v\|^2.$$

Choose $t > 0$ s.t $1 - 2tc + t^2 L^2 < 1$ $\left(t^2 L^2 < 2tc, \ 0 < t < \frac{2c}{L^2} \right)$

B is k-contractive, $k = \sqrt{1 - 2tc + t^2 L^2}$.

By the Banach fixed point theorem, $\exists !\, u \in H$ s.t $u = Bu$

$$t(Au - z) = 0$$

$$Au - z = 0.$$

Uniqueness $Au_1 = z$, $Au_2 = z$

$$\langle u_1 - u_2, Au_1 - Au_2 \rangle \geq c\|u_1 - u_2\|^2$$ So $u_1 - u_2 = 0$

$$0 = \langle u_1 - u_2, 0 \rangle \geq c\|u_1 - u_2\|^2.$$

Application

Nonlinear Lax-Milgram Theorem and the nonlinear Orthogonality Principle

Theorem 2 Let H be a real Hilbert space. Consider a function $b \in H^*$ and $a: H \times H \to \mathbb{R}$ that has the following properties.

① For each $w \in H$, the map $v \mapsto a(v, w)$ is an element in H^*

② There is a constant $L > 0$ s.t

$$|a(w,u) - a(w,v)| \leq L\|u-v\|\|w\|$$

③ $\exists\ \alpha > 0$ s.t. $\quad a(u-v, u) - a(u-v, v) \geq \alpha\|u-v\|^2$

Then the equation $\quad a(v, u) = b(v) \quad \forall v \in H$ has a unique solution

Proof Sketch

$b(v) = \langle v, y \rangle$, $\forall v \in H$

$\exists\, y, Aw \in H$ s.t $\quad \Psi_w(v) = a(v, w) = \langle v, Aw \rangle$, $\forall v \in H$

By Riesz Theorem

Step 2 $\quad A: H \to H$ is strongly monotone since

From ③ $\quad \langle u-v, Au \rangle - \langle u-v, Av \rangle = \langle u-v, Au - Av \rangle \geq \alpha\|u-v\|^2$

A is Lipschitz continuous

From ② $\quad |\langle w, Au \rangle - \langle w, Av \rangle| \leq L\|u-v\|\|w\|$

$$|\langle w, Au - Av \rangle| \leq L\|u-v\|\|w\|$$

Choose $w = Au - Av$ $\quad \|Au - Av\|^2 \leq L\|u-v\|\,\|Au - Av\|$

So $\|Au - Av\| \leq L\|u-v\|$