Generalized Fourier Series

<u>Definition</u> ① A sequence $\{x_k\}$ in a Hilbert space H is called an <u>orthogonal system</u> if $\langle x_k, x_\ell\rangle = 0$ for all $k \neq \ell$.

② A sequence $\{x_k\}$ in a Hilbert space H is called an <u>orthonormal system</u> if $\langle x_k, x_\ell\rangle = \delta_{k\ell}$

<u>Example</u> ① In $(\ell_2, \|.\|_2)$, $x_k = (0,..,0,\overset{k^{th}}{1},0,...)$

orthonormal system ② In $(L_2^{\mathbb{C}}[-\pi,\pi], \|.\|_2)$ $x_k(t) = \frac{1}{\sqrt{2\pi}}e^{ikt}$, $t \in [-\pi,\pi]$

③ $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos t, \frac{1}{\sqrt{\pi}}\sin t, \frac{1}{\sqrt{\pi}}\cos 2t, \frac{1}{\sqrt{\pi}}\sin 2t, \ldots\right\}$

<u>Lemma 1</u> An orthonormal set is linearly independent.

<u>Proof</u>: Exercise.

~~Theorem Bessel's inequality~~

<u>Definition</u> ③ Consider an orthonormal system $\{x_k\}_{k=1}^{\infty}$ in a Hilbert space H and $x \in H$. The <u>Fourier series</u> of x w.r.t $\{x_k\}$ is the formal series

$$\sum_{k=1}^{\infty} \langle x, x_k\rangle x_k .$$

The coefficients $\langle x, x_k\rangle$ are called the <u>Fourier coefficients</u> of x

<u>Lemma 2</u> $\sum_{k=1}^{n} \langle x, x_k\rangle x_k$ is the orthogonal projection of x onto $\mathrm{Span}(x_1,..,x_n)$ where $\{x_k\}_{k=1}^{\infty}$ is an orthonormal system

<u>Sketch of the proof</u> $\mathrm{Span}(x_1,\ldots,x_n)$ is a linear subspace of H.

① Verify $\mathrm{Span}(x_1,\ldots,x_n)$ is a closed set in H.

② Verify $\left(x - \sum_{k=1}^{n}\langle x, x_k\rangle x_k\right) \in \mathrm{Span}(x_1,\ldots,x_n)^{\perp}$

Bessel's Inequality Let $\{x_k\}$ be an orthonormal system in a Hilbert space H. Then for every $x \in H$, we have

$$\sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2 \le \|x\|^2 .$$

Proof Denote $S_n = \sum_{k=1}^{n} \langle x, x_k\rangle x_k$

From lemma 2, $x - S_n \in \mathrm{Span}(x_1, \dots, x_n)^{\perp}$

In particular, $x - S_n \perp S_n$.

$$\|x\|^2 = \|x - S_n\|^2 + \|S_n\|^2 \geq \|S_n\|^2 = \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 , \quad \forall n$$

let $n \to \infty$, $\|x\|^2 \geq \sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2$.

Theorem (convergence) Let $\{x_k\}$ be an orthonormal sequence in a Hilbert space H. Then

ⓐ The series $\sum_{k=1}^{\infty} a_k x_k$ converges (in the norm on H) iff $\sum_{k=1}^{\infty} |a_k|^2$ converges. iff $\sum a_k x_k$ converges unconditionally in H, i.e., for every reordering of terms

ⓑ If $\sum_{k=1}^{\infty} a_k x_k$ converges then $a_k = \langle x, x_k\rangle$ where $x = \sum_{k=1}^{\infty} a_k x_k$

ⓒ Conversely, for every $x \in H$, the Fourier series $\sum_k \langle x, x_k\rangle x_k$ converges in H

Before proving the convergence theorem, we need the following lemma:

Lemma 3 If $\sum_{n=1}^{\infty} a_n w_n$ converges in H then for each $y \in H$

$$\left\langle \sum_{n=1}^{\infty} a_n w_n , y \right\rangle = \sum_{n=1}^{\infty} a_n \langle w_n, y\rangle$$

Sketch of the Proof. Given $y \in H$, $L_y(x) = \langle x, y \rangle$ is in H^*.

$$|L_y(x)| \le \|y\| \|x\|, \text{ so } \|L_y\| \le \|y\|.$$

$$\left| L_y(x) - \sum_{k=1}^{m} L_y(a_k w_k) \right| = \left| L_y\left(x - \sum_{k=1}^{m} a_k w_k\right) \right| \le \|L_y\| \underbrace{\left\| x - \sum_{k=1}^{m} a_k w_k \right\|}_{\xrightarrow{m \to \infty} 0}$$

So $L_y(x) = \sum_{k=1}^{\infty} L_y(a_k w_k)$

Proof of the Convergence theorem

(a) $s_n = \sum_{k=1}^{n} a_k x_k$, $\sigma_n = \sum_{k=1}^{n} |a_k|^2$

For $n > m$, $\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^{n} a_k x_k \right\|^2 = \sum_{k=m+1}^{n} |a_k|^2 = \sigma_n - \sigma_m$.

So $\{s_n\}$ is Cauchy in H iff σ_n is Cauchy in $\mathbb{R}$

(b) $x = \sum_{k=1}^{\infty} a_k x_k$. From lemma 3

$$\langle x, x_\ell \rangle = \left\langle \sum_{k=1}^{\infty} a_k x_k, x_\ell \right\rangle = \sum_{k=1}^{\infty} a_k \langle x_k, x_\ell \rangle = a_\ell$$

(c) Since $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \le \|x\|^2$ (Bessel Inequality)

from part a), we conclude that the Fourier series $\sum \langle x, x_k \rangle x_k$ converges.

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(4) From Bessel Inequality, $\sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \le \|x\|^2 \; \forall n$, we have the following.

<u>Lemma 4</u> <u>Fourier coefficients</u> Let X be an inner product space and $\{x_k\}_{k \in I}$ (I could be uncountable) is an orthonormal family in X. Then any $x \in X$ can have at most countably many nonzero Fourier coefficients $\langle x, x_k \rangle$

Proof Since $\sum_{k=1}^{n} |\langle x, x_k\rangle|^2 \le \|x\|^2$, $\forall n$

For each m, $\#\{k\in I : |\langle x, x_k\rangle| > \frac{1}{m}\}$ is finite.

$\{k\in I : |\langle x, x_k\rangle| \neq 0\} = \bigcup_{m\geq 1} \{k\in I : |\langle x,x_k\rangle| > \frac{1}{m}\}$ has at most countable cardinal.

From lemma 4, with fixed $x\in H$, we can associate a series $\sum_{k\in I} \langle x, x_k\rangle x_k$ and we can arrange the x_k with $\langle x, x_k\rangle \neq 0$ in a sequence so that $\sum_{k\in I} \langle x,x_k\rangle x_k$ takes the form $\sum_{k=1}^{\infty} \langle x, x_k\rangle x_k$

Note that from the convergence theorem part ⓐ, the sum $\sum_{k=1}^{\infty}\langle x,x_k\rangle x_k$ does not depend on the order of $\{x_k\}$

Next question Let H be a Hilbert space.

$\{x_k\}_{k=1}^{\infty}$ is an orthonormal set.

$$\text{Is } \quad x = \sum_{k=1}^{\infty} \langle x, x_k\rangle x_k \quad ?$$

Lemma 5

Answer In general, it is not. Indeed, $\sum_{k=1}^{\infty}\langle x, x_k\rangle x_k$ is the orthogonal projection of x onto $\overline{\mathrm{Span}(x_1, x_2, \dots)}$.

Proof Verify $\left(x - \sum_{k=1}^{\infty}\langle x, x_k\rangle x_k\right) \perp \overline{\mathrm{Span}(x_1, x_2, \dots)}$

Note $\mathrm{Span}(x_1,x_2,\dots) := \left\{\sum_{k=1}^{m} a_k x_{i_k},\ a_k\in\mathbb{C},\ x_{i_k}\in\{x_1,x_2,\dots\}\right\}$

Definition 5 Let H be a Hilbert space. An orthonormal sequence $\{x_k\}_{k=1}^{\infty}$ in H is maximal if $\langle x, x_k\rangle = 0 \ \forall k$ implies $x = 0$.

Lemma 6 If $\{x_k\}_{k=1}^{\infty}$ is a maximal orthonormal sequence in H ~~then~~

iff $\overline{\text{Span}\{x_k\}} = H$.

Proof ($\Leftarrow$) Assume $H = \overline{\text{Span}\{x_k\}}$ and $\langle x, x_k\rangle = 0 \ \forall k$.

$x \in H = \overline{\text{Span}\{x_k\}}$ So there exists a sequence $\{y_n\} \subset \text{Span}\{x_k\}$

s.t $y_n \to x$.

Since $y_n \in \text{Span}\{x_k\}$ and $\langle x, x_k\rangle = 0 \ \forall k$, we have

$$\langle x, y_n\rangle = 0 \longrightarrow \langle x, x\rangle = \|x\|^2$$

So $x = 0$

($\Rightarrow$) Assume If $\exists x \in H$: $\langle x, x_k\rangle = 0 \ \forall k$ then $x = 0$.

We need to prove $H = \overline{\text{Span}\{x_k\}}$

$\{x \in H: \langle x, x_k\rangle = 0 \ \forall k\} = \{0\}$. So $(\text{Span}\{x_k\})^{\perp} = \{0\}$

$$\Downarrow$$

$$\left(\overline{\text{Span}\{x_k\}}\right)^{\perp} = \{0\}$$

$\overline{\text{Span}\{x_k\}}$ is a closed subspace of H.

$$H = \overline{\text{Span}\{x_k\}} \oplus \left(\overline{\text{Span}\{x_k\}}\right)^{\perp} = \overline{\text{Span}\{x_k\}}$$

Combining lemma 5 and Lemma 6, we have the following theorem

Theorem (Fourier expansion) ~~Let $\{x_k\}$ be an orthonormal sequence in~~

Let $\{x_k\}$ be a maximal orthonormal sequence in a Hilbert space H. Then every vector $x \in X$ can be expanded in its Fourier series

$$x = \sum_k \langle x, x_k\rangle x_k, \quad \|x\|^2 = \sum_k |\langle x, x_k\rangle|^2 \ \text{(Parseval Identity)}$$