19/NOV/19

Note:

Hilbert space $H$, $T: H \to H$ operator, $\langle Tu, u\rangle = 0$ $\forall u \in H$, Is $T=0$? No.

( $T$ has to have a condition, e.g., bnded & linear)

Continuation of the proof last time:

($\Leftarrow$) If $H = \overline{\text{span}\{x_k\}}$, we need to show $\{x_k\}_{k=1}^{\infty}$ is maximal. Take $x \in H$ s.t. $\langle x, x_k\rangle = 0$ $\forall k$. Since $H = \overline{\text{span}\{x_k\}_{k=1}^{\infty}}$, there exists a sequence $\{y_n\} \subset \text{span}\{x_k\}$ s.t. $\lim_{n\to\infty} y_n = x$.

For each $n$, $y_n \in \text{span}\{x_k\}$, $\langle x, x_k\rangle = 0$, $\forall k$. So, $\langle x, y_n\rangle = 0$, $\forall n$. On the other hand,

↳ $y_n = \sum \alpha_k x_k$

$y_n \to x$, $\overset{0}{\overset{\shortparallel}{\langle x, y_n\rangle}} \to \langle x, x\rangle$. So, $\langle x, x\rangle = 0$. It means $x = 0$. □

Thm (Fourier Expansion)

Let $H$ be a Hilbert, $\{x_k\}_{k=1}^{\infty}$ be a maximal orthonormal seq. in $H$.

Then, $\forall x \in H$ can be written as $x = \sum_{k=1}^{\infty} \langle x, x_k\rangle x_k$.

Note: ① The Thm relies on the existence of a maximal orthonormal seq.

② Combining w/ the previous lemmas, with the same assumptions as the Fourier expansions Thm, $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, x_k\rangle|^2$ (Parseval's Identity)

↳ use the fact that $\|x\|^2 = \langle x, x\rangle$ & $x_k x_m = 0$, $\forall k \neq m$.

Thm Let $H$ be a Hilbert space. Then $H$ has a maximal orthonormal seq. iff $H$ is seperable.

↳ (It means $H$ has a dense countable subset)

$$H = \overline{\bigcup_{k=1}^{\infty} z_k}, \quad z_k \in H$$

FS $H = \overline{\text{span}\{x_k\}}$

$\{x_k\}$ orthonorm.

$$\|x\|^2 = \langle x, x\rangle$$
$$= \Big\langle \sum_k \langle x, x_k\rangle x_k, \cdots \Big\rangle$$
$$= \sum_k \big\langle \langle x, x_k\rangle x_k, \cdots \big\rangle$$
$$= \sum_k x_k \big\langle \langle x, x_k\rangle, \cdots \big\rangle$$
$$= \sum_k \big\langle \langle x, x_k\rangle, x_k \sum_k \cdots$$
$$= \sum_k |\langle x, x_k\rangle|^2$$

All 0's except for one $k$

Pf. $H = \overline{\bigcup_{k=1}^{\infty} z_k}$.

$\{z_k\}$ vectors in $H$ $\xrightarrow{\text{construct}}$ a maximal linearly independent set $\{z_{k_i}\}_{i=1}^{\infty}$

↓ Gram-Schmit procedure

at most countable orthonorm. set $\{u_k\}_{k=1}^{\infty}$

So, $H = \overline{\bigcup_{k=1}^{\infty} z_k} = \overline{\text{span}\{\bigcup_{k=1}^{\infty} u_k\}}$. □

Example Seperable Hilbert space

$L_2[a,b] = \overline{C[a,b]} = \overline{P_{\mathbb{Q}}(x)}$

where $P_{\mathbb{Q}}(x) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_k \in \mathbb{Q}\}$

a maximal orthonorm. basis.

**Thm** $\{\Phi_n\} = \{1, \sqrt{2}\cos 2\pi t, \sqrt{2}\sin 2\pi t, \dots, \sqrt{2}\cos 2\pi k t, \sqrt{2}\sin 2\pi k t, \dots\}$

is a maximal orthonorm. seq. for $L_2[0,1]$.

$\downarrow$ orthonormal basis

$H = \overline{\text{span}\{\Phi_k\}_{k=1}^{\infty}}$

**Pf.** See Siegel.

**Spectral theory** (for compact self-adjoint operators)

$\langle Lx, y\rangle = \langle x, Ly\rangle$ implies $L$ bdded & linear, $L^* = L$.

For the remainder of today, we only work w/ $L: H \to H$ compact, self-adj. and $H$ Hilbert.

$\hookrightarrow$ If $\{z_n\} \subset H$ bdded, then $\exists$ a subseq $\{Lz_{n_k}\}_k$ converges.

**Definitions** $\lambda$ is called an eigenvalue of $L$, if $\exists v \in H$, $v \neq 0$ s.t. $Lv = \lambda v$.

**Example** $\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}$

After integration, bla bla, $y(x) = \lambda \int_0^1 g(x, s) y(s)\, ds$ where $g(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \le 1 \\ x(1-s) & 0 \le x \le s \le 1 \end{cases}$.

Define $L: L_2[0,1] \to L_2[0,1]$, $L$ compact, $L^* u(x) = \int_0^1 \overline{g(s,x)}\, u(s)\, ds$, $\overline{g(s,x)} = g(s,x) = g(x,s)$.

$Lu(x) := \int_0^1 g(x,s) u(s)\, ds$ $\quad$ $L^* = L$ self-adjoint

If $y$ is a solution of the ODE, $y = \lambda L y$ ($y$ is an eigenvector of $L$ w/ eigvalue $1/\lambda$).

**Prop 1:** Eigenvectors belonging to distinct eigvalues are orthogonal. All eigenvalues of $L$ are real.

**Pf.** $Lv_1 = \lambda_1 v_1$, $Lv_2 = \lambda_2 v_2$, $v_1, v_2 \neq 0$; $\lambda_1 \neq \lambda_2$.

$\langle Lv_1, v_1\rangle = \langle \lambda_1 v_1, v_1\rangle = \lambda_1 \|v_1\|^2$

self-adj. $\|$ $\quad \Rightarrow \lambda_1 = \overline{\lambda_1} \Rightarrow \lambda_1 \in \mathbb{R}$

$\langle v_1, Lv_1\rangle = \langle v_1, \lambda_1 v_1\rangle = \overline{\lambda_1} \|v_1\|^2$

$x$ $\quad \int_0^s ds$ $\quad s\Big|_0^{\min(x,s)} = \min(x,s) - 0$ $\quad s \min(x,s) - 0$

Next, $\langle L v_1, v_2\rangle = \lambda_1 \langle v_1, v_2\rangle$

self-adj. $\|$ $\quad \Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle v_1, v_2\rangle = 0 \Rightarrow \langle v_1, v_2\rangle = 0 \Rightarrow$ orth. $\square$

$\langle v_1, Lv_2\rangle = \underbrace{\lambda_2}_{\in \mathbb{R}} \langle v_1, v_2\rangle$

**Prop 2** $\|L\| = \sup_{\substack{v\in H\\ \|v\|\le 1}} |\langle Lv,v\rangle| = \sup_{\substack{\|v\|=1\\ v\in H}} |\langle Lv,v\rangle|$ (Prove this)

self-adj, compact

**Prop 3** $\exists\, \phi_1 \in H$, $\|\phi_1\|=1$, $L\phi_1 = \mu_1\phi_1$ for some $\mu_1\in\mathbb{R}$. Indeed, $|\mu_1| = \|L\|$, and $\phi_1$ is a solution of $\max_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle|$.

$\|L\| =$

Here **Pf**. Construct $\phi_1$. Idea: Since $\|L\| = \sup_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle|$, $\exists\, \{v_n\}\subset H$ s.t. $|\langle Lv_n,v_n\rangle| \to \underbrace{\|L\|}_{\in\mathbb{R}\text{ or }\mathbb{C}}$.

So, $\{\langle Lv_n,v_n\rangle\}_n$ also converges to $\underbrace{\mu_1}_{\in\mathbb{R}}$ (some real number), $|\mu_1| = \|L\|$. From the uniqueness of limits.

**Step 1** Show that $Lv_n - \mu_1 v_n \to 0$. **Recall Lemma** If $L$ is self-adj, $\langle Lx,x\rangle\in\mathbb{R}$, $\forall x\in H$.

$$\|\underbrace{Lv_n}_{\in\mathbb{R}} - \underbrace{\mu_1}_{\in\mathbb{R}} v_n\| = \|Lv_n\|^2 - 2\mu_1\langle Lv_n,v_n\rangle + \mu_1^2\|v_n\|^2$$

$$\le \|L\|^2 \cancel{\|v_n\|^2}^{\,1} - 2\mu_1\langle Lv_n,v_n\rangle + \mu_1^2 \qquad |\mu_1| = \|L\|$$

$$\le 2\mu_1^2 - 2\mu_1\langle Lv_n,v_n\rangle$$

$$n\to\infty \quad 2\mu_1^2 - 2\mu_1\langle Lv_n,v_n\rangle \to 0$$

Therefore, $Lv_n - \mu_1 v_n \to 0$ as $n\to\infty$.

$\|v_n\|=1$. $\{v_n\}$ bdded. Since $L$ is compact, $\exists\,\{v_{n_k}\}$ s.t. $\{Lv_{n_k}\}_k$ converges.

$\{Lv_{n_k} - \mu_1 v_{n_k}\}_k$ also converges to 0. So, $\{\mu_1 v_{n_k}\}_k$ converges.

$|\mu_1| = \|L\| \ne 0$, $\mu_1\ne 0$. therefore, $\{v_{n_k}\}_k$ converges.

Denote $\phi_1 = \lim_{k\to\infty} v_{n_k}$. Since $L$ is continuous, $L\phi_1 = \lim_{k\to\infty} Lv_{n_k} = \lim_{k\to\infty}\mu_1 v_{n_k} = \mu_1\phi_1$.

Since $\|v_{n_k}\|=1$, $\|\phi_1\| = \lim_{k\to\infty}\|v_{n_k}\| = 1$. $\phi_1$ is an eigenvector of $L$.

$\langle L\phi_1,\phi_1\rangle = \mu_1\langle\phi_1,\phi_1\rangle = \mu_1$ ($\|L\|$). So, $\phi_1$ is a solution of $\max_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle| = \|L\|$

Summary, $\phi_1$: $\|\phi_1\|=1$, $\phi_1\in H$, $|\langle L\phi_1,\phi_1\rangle| = \max_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle|$

$L\phi_1 = \mu_1\phi_1$

$|\mu_1| = \|L\|$.

**Next step**

$\phi_2$: $H_1 = \{x\in H : \langle x,\phi_1\rangle = 0\} = \{\phi_1\}^\perp = \operatorname{span}\{\phi_1\}^\perp$ subspace closed. $L|_{H_1} : H_1 \to H$.

Operator $L$ restricted to $H_1$ in domain

Claim $\text{Im}(L|_{H_1}) \subseteq H_1$.

Take $x \in H_1$. Show that $Lx \in H_1$.

$$\langle Lx, \Phi_1 \rangle \overset{\text{selfadj.}}{=} \langle x, L\Phi_1 \rangle = \lambda_1 \langle x, \Phi_1 \rangle = \lambda_1 0 = 0$$

Therefore, $Lx \in H_1$.

So, $L_1 = L|_{H_1} : H_1 \to H_1$. $H_1$ closed subspace.

$L_1$ is also compact, self-adj. $\|L_1\| \leq \|L\|$

$$\exists \Phi_2 \in H_1,\ \|\Phi_2\| = 1,\ \underbrace{L_1\Phi_2}_{= L\Phi_2} = \mu_2 \Phi_2,\ |\mu_2| = \|L_2\| \leq |\mu_1|.$$

$$|\langle L\Phi_2, \Phi_2 \rangle| = \max_{\substack{v \in H_1 \\ \|v\|=1}} |\langle Lv, v \rangle|$$

$\langle \Phi_2, \Phi_1 \rangle = 0$

$\vdots$

$H_n = \{x \in H : \langle x, \Phi_k \rangle = 0 \text{ for all } k = 1, \dots, n\}$

$\Phi_{n+1} \in H_n$. eigenvector, $\langle \Phi_{n+1}, \Phi_k \rangle = 0,\ \forall k = 1, \dots, n$

$$|\langle L\Phi_{n_{k+1}}, \Phi_{n_{k+1}} \rangle| = \max_{\substack{v \in H_n \\ \|v\|=1}} |\langle Lv, v \rangle|$$

Observation $L\Phi_k = \mu_k \Phi_k$ ; $|\mu_k| = \|L_k\|$ where $L_k = L|_{H_k}$

$|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n| \geq \dots$ $\mu_k \in \mathbb{R}$. → Can an operator have inf. many eigvalues?

Lemma $\lim_{n \to \infty} \mu_n = 0$.

Pf. Assume $\lim_{n \to \infty} \mu_n \neq 0$. $\exists \varepsilon, \{n_k\} : |\mu_{n_k}| > \varepsilon,\ \forall k = 1, 2, \dots$

$$\left\| \frac{\Phi_{n_k}}{\mu_{n_k}} \right\| < \frac{1}{\varepsilon}. \qquad \underbrace{\|\Phi_{n_k}\|}_{=1} < \underbrace{\frac{|\mu_{n_k}|}{\varepsilon}}_{1<}$$

$\{\Phi_{n_k}/\mu_{n_k}\}_k$ is bnded in $H$.

L compact $\underbrace{L\left(\frac{\Phi_{m_k}}{\mu_{m_k}}\right)}_{\Phi_{m_k}} \to$ converges where $\left\{\frac{\Phi_{m_k}}{\mu_{m_k}}\right\} \subset \left\{\frac{\Phi_{n_k}}{\mu_{n_k}}\right\}$ why?

On the other hand, $\underbrace{\|\Phi_{m_k} - \Phi_{m_\ell}\|^2 = 2}_{\text{why?}}$. So, $\{\Phi_{m_k}\}$ could not converge. Contradiction. $\square$