Spectral Theorem (cont'd) Lecture 20

Recall Let H be a Hilbert space, $L: H \to H$ a compact, self-adjoint operator

1. all eigenvalues of L are real (using the self-adjoint property)
2. Eigenvectors belonging to different eigenvalues are orthogonal.
3. Eigenvalues and Eigenvectors Construction.

$\exists \phi_1 \in H$ s.t $\|\phi_1\| = 1$, $L\phi_1 = \mu_1 \phi_1$ where $|\mu_1| = \|L\|$

$$\text{Moreover } \underbrace{|\langle L\phi_1, \phi_1\rangle|}_{|\mu_1|} = \max_{\substack{v \in H \\ \|v\|=1}} |\langle Lv, v\rangle| -$$

$H_1 = \{x \in H : \langle x, \phi_1\rangle = 0\}$ closed subspace of H

$\exists \phi_2 \in H_1$ s.t $\|\phi_2\| = 1$, $L\phi_2 = \mu_2 \phi_2$ where

$$|\mu_2| = |\langle L\phi_2, \phi_2\rangle| = \max_{\substack{v \in H_1 \\ \|v\|=1}} |\langle Lv, v\rangle| = \|L|_{H_1}\| = \sup_{\substack{v \in H_1 \\ v \neq 0}} \frac{\|Lv\|}{\|v\|}$$

$\vdots$

$\exists \phi_{n+1} \in H_n = \{x \in H : \langle x, \phi_1\rangle = \dots = \langle x, \phi_n\rangle = 0\}$

s.t $\|\phi_{n+1}\| = 1$, $L\phi_{n+1} = \mu_{n+1}\phi_{n+1}$ where

$$|\mu_{n+1}| = |\langle L\phi_{n+1}, \phi_{n+1}\rangle| = \max_{\substack{v \in H_n \\ \|v\|=1}} |\langle Lv, v\rangle| = \|L|_{H_n}\| = \sup_{\substack{v \in H_n \\ v \neq 0}} \frac{\|Lv\|}{\|v\|}$$

Lemma ① $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n| \geq \dots$

② $\lim_{n \to \infty} \mu_n = 0$

3 The $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal set of eigenvectors.

4 For any $v \in H$, $\quad Lv = \sum_{k=1}^{\infty} \mu_k \langle v, \phi_k \rangle \phi_k = \sum_{k=1}^{\infty} \langle Lv, \phi_k \rangle \phi_k$.

5 The $\{\mu_n\}_{n=1}^{\infty}$ are all nonzero eigenvalues of L

7 The $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for H iff 0 is not an eigenvalue of $L \iff \operatorname{Ker} L = \{0\}$

6 $\{$ ~~eigenvectors associated with eigenvalue μ_n~~ $\} = \{$ $\}$ $\quad$ If $L\phi = \mu_n \phi$, $\phi \neq 0$ Then $\phi \in \operatorname{Span}\{\phi_J\}$ $\quad \mu_J = \mu_n$

Proof 3 From the construction of ϕ_n

4 Take $v \in H$. Let $\boxed{g_n = v - \sum_{k=1}^{n} \langle v, \phi_k \rangle \phi_k}$.

For $1 \le J \le n$

$$\langle g_n, \phi_J \rangle = \langle v, \phi_J \rangle - \sum_{k=1}^{n} \langle v, \phi_k \rangle \langle \phi_k, \phi_J \rangle = \langle v, \phi_J \rangle - \langle v, \phi_J \rangle = 0$$

So $\boxed{g_n \in H_n}$.

Therefore, $\boxed{\|Lg_n\| \le |\mu_{n+1}| \, \|g_n\| \le |\mu_{n+1}| \, \|v\|}$ since

$$\|g_n\|^2 = \langle v - \sum_{k=1}^{n} \langle v, \phi_k \rangle \phi_k \, , \, v - \sum_{J=1}^{n} \langle v, \phi_J \rangle \phi_J \rangle$$

$$= \|v\|^2 - \sum_{k=1}^{n} |\langle v, \phi_k \rangle|^2 \le \|v\|^2$$ ~~by the~~

So $\quad 0 \le \|Lg_n\| \le |\mu_{n+1}| \, \|v\|$

$$\xrightarrow{n\to\infty} 0$$

Therefore $\boxed{0 = \lim_{n\to\infty} Lg_n} = \lim_{n\to\infty} \left(Lv - \sum_{k=1}^{n} \langle v, \phi_k \rangle L\phi_k \right)$

$$Lv = \sum_{k=1}^{\infty} \langle v, \phi_k \rangle L\phi_k = \sum_{k=1}^{\infty} \langle v, \phi_k \rangle \mu_k \phi_k = \sum_{k=1}^{\infty} \langle v, \mu_k \phi_k \rangle \phi_k$$

$$= \sum_{k=1}^{\infty} \langle v, L\phi_k \rangle \phi_k \overset{\text{Since } L = L^*}{=} \sum_{k=1}^{\infty} \langle Lv, \phi_k \rangle \phi_k$$

5 $\{\mu_n\} = \{$ nonzero eigenvalues of $L\}$

"$\supset$" If $L\phi = \mu\phi$ for some $\mu \neq 0$, $\mu \notin \{\mu_n\}_{n=1}^{\infty}$. Then $\langle \phi, \phi_k \rangle = 0$
eigenvector $\phi \neq 0$ $\forall k = 1, 2, \dots$

Applying 4, $L\phi = \sum_{k=1}^{\infty} \mu_k \langle \phi, \phi_k \rangle \phi_k = 0$

$\mu\phi = 0$ ~~but ϕ~~

$\phi = 0$, a contradiction

"$\subset$" Obviously since $|\mu_n| = \| L|_{H_n} \|$ and $L \neq 0$.

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6 Suppose $L\phi = \mu_n \phi$, $\phi \neq 0$. Then

$$\mu_n \phi = L\phi = \sum_{k=1}^{\infty} \mu_k \langle \phi, \phi_k \rangle \phi_k \overset{\text{Since } \langle \phi, \phi_k \rangle = 0 \text{ if } \mu_k \neq \mu_n}{=\!=\!=}$$

$$= \sum_{\substack{k \\ \text{s.t. } \mu_k = \mu_n}} \mu_k \langle \phi, \phi_k \rangle \phi_k$$

So $\phi = \sum_{\substack{k \\ \text{s.t. } \mu_k = \mu_n}} \langle \phi, \phi_k \rangle \phi_k$

6a There exists a finite number of linearly independent eigenvectors corresponding to eigenvalue $\mu = \mu_n$ for any $n = 1, 2, \ldots$

Proof Suppose there is an infinite sequence of linearly independent vectors $\{x_k\}_{k=1}^{\infty}$ s.t $Lx_k = \mu x_k$, $k = 1, 2, \ldots$

Consider subspaces $E_n = \text{span}\{x_k\}_{k=1}^{n}$

Then $E_1 \subsetneq E_2 \subsetneq \ldots \subsetneq E_n \subsetneq \ldots$

Therefore, $\exists\, y_n \in E_n$, $\|y_n\| = 1$ and $\text{dist}(y_n, E_{n-1}) \geq \frac{1}{2}$

Recall a lemma Let Y be a finite dimensional subspace of X., $Y \subsetneq X$

Then $\exists\, x_0 \in X$ s.t $\|x_0\| = 1$ and $\text{dist}(x_0, Y) \geq \frac{1}{2}$. (See Assignment 2 Solution Question 5)

We will show that $\{Ty_n\}$ contains no Cauchy sequences.

$y_n = a_n x_n + u_{n-1}$, where $u_{n-1} \in E_{n-1}$

$Ty_n = a_n \mu x_n + Tu_{n-1}$, where $Tu_{n-1} \in E_{n-1}$

$n > m$ $\quad Ty_m \in E_m \subseteq E_{n-1}$

$\|Ty_n - Ty_m\| = \|a_n \mu x_n + w_{n-1}\|$ where $w_{n-1} \in E_{n-1}$

$= \mu \|y_n + \frac{w_{n-1}}{\mu}\|$ $\quad y_n \in E_n \setminus E_{n-1}$, $\frac{w_{n-1}}{\mu} \in E_{n-1}$

$\geq \frac{\mu}{2} > 0$

So $\{Ty_n\}$ contains no Cauchy sequence, contradiction

6b If $L\Phi = \mu_n \Phi$, $\Phi \neq 0$ for some $n = 1, 2, \ldots$

Then $\Phi \in \operatorname{Span}\{\Phi_j\}_{\mu_j = \mu_n}$ (See page 3)

6c Each eigenvalue μ_n of T has finite multiplicity

That is $\dim \ker(T - \mu_n I) < \infty$

7 $\{\Phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for H iff $\mu = 0$ is not an eigenvalue of L.

Proof Since $\sum_{k=1}^{\infty} |\langle v, \Phi_k \rangle|^2 \leq \|v\|^2$

the series $\sum_{k=1}^{\infty} \langle v, \Phi_k \rangle \Phi_k$ converges to $W \overset{\in H}{=} \sum_{k=1}^{\infty} \langle v, \Phi_k \rangle \Phi_k$

~~From 4, $LW = \sum_{k=1}^{\infty} \mu_k \langle w, \Phi_k \rangle \Phi_k$~~

Claim $LW = \sum_{k=1}^{\infty} \langle v, \Phi_k \rangle L\Phi_k$

Proof $\left\| LW - \sum_{k=1}^{n} \langle v, \Phi_k \rangle L\Phi_k \right\| = \left\| L\left(W - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \Phi_k\right) \right\|$

$$\leq \|L\| \underbrace{\left\| W - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \Phi_k \right\|}_{\xrightarrow{n \to \infty} 0}$$

Also, $Lv = \sum_{k=1}^{\infty} \langle v, \Phi_k \rangle L\Phi_k$

Let $h = v - W$. Then $Lh = 0$.

⑥

($\Leftarrow$) If $\mu = 0$ is not an eigenvalue, $Lh = 0$ implies $h = 0$

So $v = w$. Done

($\Rightarrow$) If $\{\phi_k\}$ is a maximal orthonormal set ~~Then $Lv=0$~~

Consider $v \in H$ s.t $Lv = 0$

$$\begin{gathered} Lv = 0 \\ \| \\ \sum_{k=1}^{\infty} \mu_k \langle v, \phi_k \rangle \phi_k = 0 \end{gathered}$$

So $\langle v, \phi_k \rangle = 0 \quad \forall k$

So $v = 0$ since $\{\phi_k\}$ is maximal.

Spectral Theorem Let $L: H \to H$ compact, self-adjoint., $\dim H = +\infty$

There exists orthonormal eigenvectors $\{\phi_k\}$ and nonzero eigenvectors $\{\mu_k\}$

$L\phi_k = \mu_k \phi_k$, $k = 1, 2, \ldots$, $|\mu_1| \geq |\mu_2| \geq \ldots$

$\lim \mu_n = 0$

and $Lv = \sum_{k=1}^{\infty} \mu_k \langle v, \phi_k \rangle \phi_k \quad \forall v \in H$

$$\begin{tikzcd} H \arrow[r, "L"] \arrow[d, "U"'] & H \\ \ell_2 \arrow[r, "D"] & \ell_2 \arrow[u, "U^{-1}"'] \end{tikzcd}$$

In addition, $\{\phi_k\}$ is an orthonormal basis iff 0 is not an eigenvalue of H.

Spectral Decomposition, Let $\{e_n\}$ be an orthonormal basis of all eigenvectors with eigenvalues $\{\mu_n\}_{n=1}^{\infty} \cup \{0\}$.

Define $U: H \to \ell_2$ $\qquad U^{-1}: \ell_2 \to H$

$Ux = (\langle x, e_1 \rangle, \langle x, e_2 \rangle, \ldots)$ $\qquad U^{-1}(y_1, y_2, \ldots, y_n \ldots) = \sum_{n=1}^{\infty} y_n e_n$

$D: \ell_2 \to \ell_2$

$D(y_1, y_2, \ldots) = (\lambda_1 y_1, \lambda_2 y_2, \ldots)$ Then $L = U^{-1} D U$