3.8 Sobolev Spaces

Definition 1. *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Then*

1. $C^k(G) = \{u : G \to \mathbb{R} \quad$ *s.t. u has continuous partial derivatives of orders $m = 0, 1, \ldots, k$*$\}$.
2. $C^\infty(G) = \{u : G \to \mathbb{R} \quad$ *s.t. u has continuous partial derivatives of orders $m = 0, 1, \ldots$*$\}$.
3. $C_0^\infty(G) = \{u \in C^\infty(G)$ *s.t. u vanishes outside a compact subset C of G that depends on u, i.e., $u(x) = 0$ for all $x \in G - C$*$\}$.

Proposition 25. *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Then $L_2(G) = \overline{C^\infty(G)} = \overline{C_0^\infty(G)}$. That is, for every $u \in L_2(G)$, there exists $\{u_n\} \subset C_0^\infty(G)$ such that $u_n \to u$ in $L_2(G)$.*

Sketch of the proof. Main idea: using mollifier, an important smoothing technique. The details can be found in Zeidler's book, pages 186-189.

- Consider
$$\Phi(x) = \begin{cases} c\, e^{\frac{1}{|x|^2 - 1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$
The constant c is chosen so that $\int_{\mathbb{R}^n} \Phi(x)dx = 1$. Verify that $\Phi \in C_0^\infty(\mathbb{R}^n)$.

- For each $\varepsilon > 0$, define
$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\Phi\left(\frac{x}{\varepsilon}\right), \quad G_\varepsilon = \{x \in G : \ \mathrm{dist}(x, \partial G) > \varepsilon\}.$$
Verify that $\Phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ and $\Phi_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$ for all $\varepsilon > 0$.

- For each $u \in L_2(G)$, set $u = 0$ outside G. Define
$$u_\varepsilon(x) := \int_{\mathbb{R}^n} \Phi_\varepsilon(x - y)u(y)dy.$$
Verify that $u_\varepsilon \in C^\infty(G_\varepsilon),\ u_\varepsilon \in L_2(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $L_2(G)$ as $\varepsilon \to 0$.

□

Lemma 9 (Variational Lemma)**.** *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$ and $u \in L_2(G)$ such that*
$$\int_G uv\,dx = 0 \quad \forall v \in C_0^\infty(G).$$
Then $u(x) = 0$ for almost all $x \in G$. In addition, if $u \in C(G)$ then $u(x) = 0$ for all $x \in G$.

Proof. Since $L_2(G) = \overline{C_0^\infty(G)}$, there exists $\{u_n\} \subset C_0^\infty(G)$ such that $u_n \to u$. Then

$$\langle u, u \rangle = \langle u, \lim_{n\to\infty} u_n \rangle = \lim_{n\to\infty} \langle u, u_n \rangle = 0.$$

So $u(x) = 0$ for almost all $x \in G$. $\square$

Recall Integration by Parts

1. In 1D, $u, v \in C^1[a, b]$, then $\int_a^b u'v dx = uv \mid_a^b - \int_a^b uv' dx$.
   In addition, if $v(a) = v(b) = 0$, then $\int_a^b u'v dx = -\int_a^b uv' dx$.

2. In $\mathbb{R}^n$, let G be an open set in $\mathbb{R}^n$. Then

$$\int_G u D^\alpha \Phi \, dx = (-1)^{|\alpha|} \int_G D^\alpha u \Phi \, dx \quad \text{for } u \in C^k(G),\ \Phi \in C_0^\infty(G),$$

   where $\quad \alpha = (\alpha_1, \ldots, \alpha_n)$ and $D^\alpha \Phi = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \Phi.$

Below is the definition of weak derivatives from Zeidler's book.

Definition 2 (Weak Derivatives)**.** *Let G be a nonempty open set in $R^n, n \geq 1$. Let $u, w \in L_2(G)$ and suppose*

$$\int_G u \partial_j \Phi \, dx = -\int_G w \Phi \, dx, \quad \textit{for all } \ \Phi \in C_0^\infty(G).$$

Then w is called an $\alpha^{th}-$weak partial derivative of u, where $\alpha = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $1's$ is at the j^{th}-position.

Here is the general definition of weak derivatives.

Definition 3. *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Let $u, w \in L^1_{loc}(G)$ where*

$$L^1_{loc}(G) = \{v : G \to \mathbb{R} \quad s.t. \quad v \in L_1(V) \textit{ for each } V \subset \overline{V}_{compact} \subset U\}.$$

Suppose

$$\int_G u \, D^\alpha \Phi \, dx = (-1)^{|\alpha|} \int_G w \, \Phi \, dx \quad \textit{for all } \ \Phi \in C_0^\infty(G).$$

Then w is called an α^{th}-weak partial derivative of u.

Lemma 10. *A weak α^{th}-partial derivative of u if exists, is uniquely defined up to a set of measure zero.*

Proof. Assume $w, \widetilde{w} \in L^1_{loc}(G)$ satisfying the formula. Then

$$\int_G (w - \widetilde{w}) \, \Phi \, dx = 0.$$

By the variational lemma, $w - \widetilde{w} = 0$ a.e. $\square$

Example 1. *Consider* $u : (-1,1) \to \mathbb{R},\ u(x) := |x|$ *for all* $x \in (-1,1)$. *Then the following function is the weak derivative of* u *in the weak sense.*

$$w(x) = \begin{cases} -1 & \text{if } \quad -1 < x < 0 \\ c & \text{if } \quad x = 0 \\ 1 & \text{if } \quad 0 < x < 1 \end{cases}$$

where c *is fixed, but otherwise arbitrary real number.*

Proof. Let $\Phi \in C_0^\infty(-1,1)$. Then

$$\int_{-1}^{1} u\Phi'\,dx = \int_{-1}^{0} u\Phi'\,dx + \int_{0}^{1} u\Phi'\,dx = -\int_{-1}^{0} x\Phi'\,dx + \int_{0}^{1} x\Phi'\,dx.$$

Using integration by parts, we have

$$-\int_{-1}^{0} x\Phi'\,dx + \int_{0}^{1} x\Phi'\,dx = \int_{-1}^{0} \Phi\,dx - \int_{0}^{1} \Phi\,dx = -\int_{-1}^{1} w\Phi\,dx,$$

which implies w is the derivative of u in the weak sense. □

Example 2. *Consider*

$$u(x) = \begin{cases} x & \text{if } \quad 0 < x \le 1 \\ 2 & \text{if } \quad 1 < x < 2 \end{cases}$$

The function u *does not have a weak derivative.*

Proof. Let $\Phi \in C_0^\infty(0,2)$. Suppose there exists $w \in L_2(0,2)$ such that

$$\begin{aligned} -\int_{0}^{2} w\Phi\,dx &= \int_{0}^{2} u\Phi' dx \\ &= \int_{0}^{1} u\Phi' dx + \int_{1}^{2} u\Phi' dx \\ &= -\int_{0}^{1} \Phi dx - \Phi(1), \end{aligned}$$

where the third line is obtained by integration by parts on the right hand side. Therefore,

$$\Phi(1) = \int_{0}^{2} w\Phi\,dx - \int_{0}^{1} \Phi dx \quad \forall \Phi \in C_0^\infty(0,2).$$

Consider $\{\Phi_n\} \subset C_0^\infty(0,2)$ such that $\Phi_n(1) = 1$, $0 \le \Phi_n(x) \le 1$ for all $x \in (0,2)$ and $\Phi_n(x) \to 0$ as $n \to \infty$ for all $x \in (0,2) \setminus \{1\}$. Then

$$\lim_{n\to\infty} \left(\int_{0}^{2} w\Phi_n\,dx - \int_{0}^{1} \Phi_n dx \right) = 0,$$

but

$$\lim_{n\to\infty} \Phi_n(1) = 1,$$

a contradiction. This completes the proof. □

Definition 4 (Sobolev space $W_p^k(G)$)**.** *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Denote*

$$W_2^1(G) = \{u \in L_2(G) \quad s.t. \quad \partial_j u \text{ exists in the weak sense and } \partial_j u \in L_2(G) \quad \forall j = 1, \ldots, n\}.$$

On $W_2^1(G)$, define

$$\langle u, v \rangle_{1,2} := \int_G \left(uv + \sum_{j=1}^n \partial_j u \, \partial_j v \right) dx,$$

$$\|u\|_{1,2} := \left(\int_G u^2 \, dx + \sum_{j=1}^n \int_G (\partial_j u)^2 \, dx \right)^{1/2}.$$

In general, fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. Define

$$W_p^k(G) = \{u \in L^1_{loc}(G) \ \ s.t. \ \ D^\alpha u \text{ exists in the weak sense for all } |\alpha| \leq k \text{ and } D^\alpha u \in L_p(G) \quad \forall j = 1, \ldots, n\}.$$

On $W_p^k(G)$, define

$$\|u\|_{k,p} := \left(\sum_{|\alpha| \leq k} \int_G |D^\alpha u|^p \, dx \right)^{1/p}.$$

Denote $H^k(G) = W_2^k(G)$.

Theorem 1. *For each $k = 1, 2 \ldots$ and $1 \leq p < \infty$, the Sobolev space $W_p^k(G)$ is a Banach space and $W_2^k(G)$ is a Hilbert space, provided we identify two functions whose values differ only on a set of measure zero.*

Sketch of the proof. We will sketch the proof for $W_2^1(G)$.

- Verify that $W_2^1(G)$ is an inner product space.
- Verify that $W_2^1(G)$ is a Banach space.
  - Let $\{u_n\} \subset W_2^1(G)$ be a Cauchy sequence. For every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

    $$\|u_n - u_m\|_{1,2} \leq \varepsilon \quad \forall n, m \geq N_\varepsilon.$$

    Since $\|v\|_{1,2} \geq \|v\|_2$ and $\|v\|_{1,2} \geq \|\partial_j v\|_2$ for all $v \in W_2^1(G)$, the sequences $\{\partial_j u\}_n$ for every $j = 1, \ldots, n$, and $\{u_n\}_n$ are Cauchy sequences in $L_2(G)$. Since $L_2(G)$ is a Banach space, there exists $w_j, u \in L_2(G)$ such that

    $$\lim_{n\to\infty} \|\partial_j u_n - w_j\|_2 = 0, \quad \forall j = 1, \ldots, n \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u\|_2 = 0.$$

- Show that $w_j = \partial_j u$ in the weak sense. Indeed, from

$$\int_G u_n \partial_j \Phi dx = -\int_G \partial_j u_n \, \Phi dx,$$

letting $n \to \infty$, we have

$$\int_G u \partial_j \Phi dx = -\int_G w_j \Phi \, dx,$$

which implies $w_j = \partial_j u$ in the weak sense.

- Finally, show that $\|u_n - u\|_{1,2} \to 0$ as $n \to \infty$.

□

Definition 5. *Let G be a nonempty open subset in $\mathbb{R}^n, n \geq 1$. Let $\overset{\circ}{W}{}_2^{1}(G)$ be the closure of $C_0^\infty(G)$ in the Hilbert space $W_2^1(G)$. That is, $u \in \overset{\circ}{W}{}_2^{1}(G)$ if and only if there exists $\{u_m\} \subset C_0^\infty(G)$ such that $\|u_m - u\|_{1,2} \to 0$ as $m \to \infty$.*

Proposition 26. *The space $\overset{\circ}{W}{}_2^{1}(G)$ is a real Hilbert space.*

Proof. Hint: $C_0^\infty(G)$ is a linear subspace of the Hilbert space $W_2^1(G)$ and the closure of a linear subspace of a Hilbert space is also a Hilbert space. □

Proposition 27. *Let $G = (a, b) \subset \mathbb{R}$, where $-\infty < a < b < \infty$. If $u \in \overset{\circ}{W}{}_2^{1}(G)$, there exists a unique continuous function $v : [a, b] \to \mathbb{R}$ such that $u(x) = v(x)$ for almost all $x \in (a, b)$ and $v(a) = v(b) = 0$. In addition*

$$\|v\|_\infty \leq (b-a)^{1/2} \left(\int_a^b (u')^2 \, dx \right)^{1/2} \leq (b-a)^{1/2} \|u\|_{1,2}.$$