Recall Some notations.

Let $G_{\text{open}} \subset \mathbb{R}^n$, $n \geq 1$.

$C^{\infty}(G) = \{ u : G \to \mathbb{R} : u \text{ is infinitely differentiable} \}$

$C_0^{\infty}(G) = \{ u \in C^{\infty}(G) \text{ s.t } u(x) = 0 \text{ for all } x \in G - C$, C : a compact subset of $G \}$

$L_p(G) = \{ u : G \to \mathbb{R} \text{ s.t } \int_G u^p \, dx < \infty \}$ for $1 \leq p < \infty$

$L^1_{loc}(G) = \{ u : G \to \mathbb{R} \text{ s.t } u \in L_1(V) \text{ for all } V \subset \overline{V}_{\text{compact}} \subset G \}$
$= \{ \text{locally integrable functions} \}$

$W_p^k(G) = \{ u \in L^1_{loc}(G) \text{ s.t } D^{\alpha}u \text{ exists in the weak sense and } D^{\alpha}u \in L_p(G) \text{ for all } |\alpha| \leq k \}$

Banach spaces

$H^k = W_2^k(G) = \{ u \in L^1_{loc}(G) \text{ s.t. } D^{\alpha}u \text{ exists in the weak sense and } D^{\alpha}u \in L_2(G) \text{ for all } |\alpha| \leq k \}$

Hilbert spaces

$H^0 = L_2(G)$

Norm on $W_p^k(G)$: $\|u\|_{k,p} = \left[\sum_{|\alpha| \leq k} \int |D^{\alpha}u|^p \, dx \right]^{1/p}$, $1 \leq p \leq \infty$, $k = 0, 1, 2, \dots$

Weak-derivatives : Let $u \in L^1_{loc}(G)$ Then $w \in L^1_{loc}(G)$ is the α^{th}-weak partial derivative of u if

$$\int_G u D^{\alpha}\phi \, dx = (-1)^{|\alpha|} \int_G w \phi \, dx \quad \text{for all } \phi \in C_0^{\infty}(G).$$

$$\overset{\circ}{W}{}_p^k(G) = \overline{C_0^{\infty}(G)} \text{ in } (W_p^k(G), \|.\|_{k,p})$$

$$L_2(G) = \overline{C_0^{\infty}(G)} \text{ in } (L_2(G), \|.\|_2)$$

Note: $\overset{\circ}{W}{}_p^k(G) = \{ u \in W_p^k(G) \text{ s.t } D^{\alpha}u = 0 \text{ on } \partial G \text{ for all } |\alpha| \leq k-1 \}$

Some Useful Inequalities

① <u>Hölder's Inequality</u>: Assume $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Then if $u \in L_p(G)$ and $v \in L_q(G)$, we have

$$\int_G |uv|\,dx \le \|u\|_{L_p(G)} \|v\|_{L_q(G)}$$

② <u>Minkowski's Inequality</u> Assume $1 \le p \le \infty$ and $u, v \in L_p(G)$. Then

$$\|u+v\|_{L_p(G)} \le \|u\|_{L_p(G)} + \|v\|_{L_p(G)}$$

③ <u>Cauchy-Schwarz Inequality</u> Let X be an inner product space and $x, y \in X$

Then $|\langle x, y\rangle| \le |\langle x, x\rangle|^{1/2} \langle y, y\rangle^{1/2}$

For example, $X = L_2(G)$, $f, g \in L_2(G)$ (real spaces)

$$\left|\int_G f g\,dx\right| \le \left(\int_G f^2\,dx\right)^{1/2} \left(\int_G g^2\,dx\right)^{1/2}.$$

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④ Generalized Hölder's Inequality

$$\int_G |u_1 \cdots u_m|\,dx \le \|u_1\|_{L_{p_1}(G)} \cdots \|u_m\|_{L_{p_m}(G)}$$

where $u_k \in L_{p_k}(G)$, $\infty > p_1, \ldots, p_m \ge 1$

$$\sum_{k=1}^{m} \frac{1}{p_k} = 1.$$

Sobolev's Inequalities - to ~~discover~~ discover embeddings of various Sobolev spaces, If $u \in W_p^1(G)$, does u belong to certain other space

1. Gagliardo - Nirenberg - Sobolev Inequality for $W_p^1(G)$, $G_{open} \subset \mathbb{R}^n$, $n \geq 1$.

Theorem 1 Assume $1 \leq p < n$. There exists a constant C, depending only on p and n such that

$$\boxed{\|u\|_{L_{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L_p(\mathbb{R}^n)} \quad \text{for all } u \in C_0^1(\mathbb{R}^n)}$$

where $p^* = \dfrac{np}{n-p}$.

Note: The constant C does not depend on the size of the support of u.

2. Poincaré - Friedrichs Inequality for $\overset{\circ}{W}{}_p^1(G)$, $G_{open, bounded} \subset \mathbb{R}^n$, $n \geq 1$.

Theorem 2 Assume $1 \leq p < n$. There exists a constant C, depending only on p, q, n, and G such that

$$\boxed{\|u\|_{L_q(G)} \leq C \|Du\|_{L_p(G)} \quad \begin{array}{l} \text{for all } u \in \overset{\circ}{W}{}_p^1(G) \\ \text{for all } q \in [1, p^*] \end{array}}$$

Theorem 3 $\overset{\circ}{W}{}_2^1(G)$, $G_{open, bounded} \subset \mathbb{R}^n$, $n \geq 1$, $p = 2$

Then there exists a constant C such that

$$\boxed{\|u\|_{L_2(G)} \leq C \|Du\|_{L_2(G)} \quad \text{for all } u \in \overset{\circ}{W}{}_2^1(G)}$$

References: L.C. Evans, Partial Differential Equations, page 277
Zeidler, page 136
D. Siegel, page 68

DEFINITION. *If* $1 \le p < n$, *the* Sobolev conjugate *of* p *is*

$$p^* := \frac{np}{n-p}. \tag{8}$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p. \tag{9}$$

The foregoing scaling analysis shows that the estimate (5) can only possibly be true for $q = p^*$. Next we prove this inequality is in fact valid.

THEOREM 1 (Gagliardo–Nirenberg–Sobolev inequality). *Assume* $1 \le p < n$. *There exists a constant* C, *depending only on* p *and* n, *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \tag{10}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Now we really do need u to have compact support for (10) to hold, as the example $u \equiv 1$ shows. But remarkably the constant here does not depend at all upon the size of the support of u.

Proof. 1. First assume $p = 1$.

Since u has compact support, for each $i = 1, \dots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i;$$

and so

$$|u(x)| \le \int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)|\, dy_i \quad (i = 1, \dots, n).$$

Consequently

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}. \tag{11}$$

Integrate this inequality with respect to x_1:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&= \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&\le \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_i \right)^{\frac{1}{n-1}},
\end{aligned} \tag{12}$$

the last inequality resulting from the general Hölder inequality (§B.2).

Now integrate (12) with respect to x_2:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}\,dx_1dx_2$$
$$\leq\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dy_2\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{\substack{i=1\\i\neq 2}}^{n}I_i^{\frac{1}{n-1}}\,dx_2,$$

for

$$I_1:=\int_{-\infty}^{\infty}|Du|\,dy_1,\quad I_i:=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dy_i\quad(i=3,\dots,n).$$

Applying once more the extended Hölder inequality, we find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}\,dx_1dx_2$$
$$\leq\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dy_2\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dy_1dx_2\right)^{\frac{1}{n-1}}$$
$$\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dx_2dy_i\right)^{\frac{1}{n-1}}.$$

We continue by integrating with respect to $x_3,\dots,x_n$, eventually to find

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}\,dx&\leq\prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|\,dx_1\dots dy_i\dots dx_n\right)^{\frac{1}{n-1}}\\&=\left(\int_{\mathbb{R}^n}|Du|\,dx\right)^{\frac{n}{n-1}}.\end{aligned}\tag{13}$$

This is estimate (10) for $p=1$.

2. Consider now the case that $1<p<n$. We apply estimate (13) to $v:=|u|^{\gamma}$, where $\gamma>1$ is to be selected. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}}&\leq\int_{\mathbb{R}^n}|D|u|^{\gamma}|\,dx=\gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|\,dx\\&\leq\gamma\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|Du|^p\,dx\right)^{\frac{1}{p}}.\end{aligned}\tag{14}$$

We choose γ so that $\frac{\gamma n}{n-1}=(\gamma-1)\frac{p}{p-1}$. That is, we set

$$\gamma:=\frac{p(n-1)}{n-p}>1,$$

in which case $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$. Thus, in view of (5), estimate (14) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^n} |Du|^p\, dx\right)^{\frac{1}{p}}. \qquad \square$$

THEOREM 2 (Estimates for $W^{1,p}, 1 \leq p < n$). *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}, \tag{15}$$

the constant C depending only on p, n, and U.

Proof. Since ∂U is C^1, there exists according to Theorem 1 in §5.4 an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$, such that

$$\begin{cases} \bar{u} = u \text{ in } U,\ \bar{u} \text{ has compact support, and} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \end{cases} \tag{16}$$

Because $\bar{u}$ has compact support, we know from Theorem 1 in §5.3 that there exist functions $u_m \in C_c^\infty(\mathbb{R}^n)$ $(m = 1, 2, \dots)$ such that

$$u_m \to \bar{u} \quad \text{in } W^{1,p}(\mathbb{R}^n). \tag{17}$$

Now according to Theorem 1, $\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$ for all $l, m \geq 1$. Thus

$$u_m \to \bar{u} \quad \text{in } L^{p^*}(\mathbb{R}^n) \tag{18}$$

as well. Since Theorem 1 also implies $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)}$, assertions (17) and (18) yield the bound

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)}.$$

This inequality and (16) complete the proof. $\square$

THEOREM 3 (Estimates for $W_0^{1,p}, 1 \leq p < n$). *Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, the constant C depending only on p, q, n and U.

Step 1 (**) $\displaystyle\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx \le \left[\int_{\mathbb{R}^n} |Du|\,dx\right]^{n/n-1}$, (case $p=1$)

Step 2
For general p, applying (**) for $v=|u|^\gamma$ where $\gamma>1$ to be selected

$$|D|u|^\gamma| = \gamma|u|^{\gamma-1}|Du|$$

Then
$$\left[\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right]^{\frac{n-1}{n}} \le \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|Du|\,dx$$
$$\le \gamma\left[\int_{\mathbb{R}^n}\left(|u|^{\gamma-1}\right)^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left[\int_{\mathbb{R}^n}|Du|^p\,dx\right]^{1/p}$$

Choose γ so that $\dfrac{\gamma n}{n-1} = (\gamma-1)\,p/(p-1)$. $\gamma = \dfrac{p(n-1)}{n-p} > 1$

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}dx\right)^{1/p^*} \le \gamma\left(\int_{\mathbb{R}^n}|Du|^p dx\right)^{1/p}.$$

$\dfrac{\gamma n}{n-1} = \dfrac{np}{n-p}$

$\dfrac{n-1}{n} - \dfrac{p-1}{p} = \dfrac{np-p-np+n}{np}$

Next, going from $C_0^\infty(\mathbb{R}^n)$ to $\overset{\circ}{W}{}_p^1(G)$

$u\in \overset{\circ}{W}{}_p^1(G)$, $\exists\ \{u_m\}\subset C_0^\infty(\mathbb{R}^n)$ $\quad \|u_m-u\|_{1,p} \xrightarrow[m\to\infty]{} 0$.

Extend $u_m=0$ on $\mathbb{R}^n-\overline{G}$. We get the $\|u\|_{L_{p^*}(G)} \le C\|Du\|_{L_p(G)}$.

$\|u\|_{L_q(G)} \le C\|u\|_{L_{p^*}(G)}$ if $1\le q\le p^*$. We get the Poincaré.

An application of Theorem 1

Let G open, bounded and ∂G is C^1. $1 \le p < n$

Then if $u \in W^{1,p}(G)$, $u \in L_{p^*}(G)$ and

$$\|u\|_{L_{p^*}(G)} \le C \|u\|_{1,p}.$$

Constant C depends on p, n, G.

Case $p > n$ Morrey's Inequality (See Evans' book)

If $u \in W^{1,p}(G)$ then $u \in C^{0,\gamma}(G)$
Hölder continuous.

Given G open, bounded.

Application The existence Theorem for the Dirichlet Problem.

Theorem 4 (Dirichlet Principle)

Let G be a nonempty bounded open set in $\mathbb{R}^n$, $n = 1, 2, \dots$

Given $f \in L_2(G)$, $g \in W_2^1(G)$.

Consider the generalized Dirichlet problem

$$\boxed{\min_{u - g \in \overset{\circ}{W}{}_2^1(G)} \frac{1}{2} \int_G \sum_{j=1}^n (\partial_j u)^2 \, dx - \int_G f u \, dx} \qquad (1^*)$$

along with the generalized boundary-value problem

$$\boxed{\int_G \sum_j \partial_j u \, \partial_j v \, dx = \int_G f v \, dx \quad \text{for all } v \in \overset{\circ}{W}{}_2^1(G), \quad u - g \in \overset{\circ}{W}{}_2^1(G).} \qquad (2^*)$$

Notation "$u-g\in \overset{\circ}{W}{}_2^1(G)$" corresponds to the boundary condition

$$u-g=0 \text{ on } \partial G$$

Conclusion ① The problem (1*) has a unique solution $u\in W_2^1(G)$

② This is also the unique solution $u\in W_2^1(G)$ for (2*)

Proof Recall Main Theorem on Quadratic variational Problem

H: Hilbert, $a: H\times H\to \mathbb{R}$ symmetric, bilinear, continuous, coercive and $b\in H^*$

Then the variational problem $\min_{v\in H} \frac{1}{2}a(v,v)-b(v)$

has a unique solution

~~[illegible]~~ for all $u,v\in W_2^1(G)$

$$a(u,v):=\int_G \sum^2 \partial_j u\,\partial_j v\,dx \quad , \quad b_1(v)=\int_G fv\,dx \, .$$

Let $w=u-g\in \overset{\circ}{W}{}_2^1(G)$. then (1*) can be rewritten as

$$\min_w \frac{1}{2}a(w+g,w+g)-b_1(w+g) \quad , \quad w\in \overset{\circ}{W}{}_2^1(G) .$$

$$\min_w \frac{1}{2}a(w,w)-\underbrace{(-a(w,g)+b_1(w))}_{b(w)}$$

$$\min_{w\in \overset{\circ}{W}{}_2^1(G)} \frac{1}{2}a(w,w)-b(w) \qquad H=\overset{\circ}{W}{}_2^1(G)$$

Verify a is bilinear, symmetric easy ✓

Verify a is bounded

$$|a(v,w)| \leq \int_G \sum |\partial_j u \, \partial_j w| \, dx$$

Cauchy-Schwarz

$$\leq \sum \left[\int_G |\partial_j u|^2 dx\right]^{1/2} \left[\int_G |\partial_j w|^2 dx\right]^{1/2}$$

$$\leq N \, \|u\|_{1,2} \, \|w\|_{1,2} .$$

Verify a is coercive

$$a(v,v) = \sum \int_G (\partial_j v)^2 dx \geq \frac{c}{c+1} \|v\|_{1,2} \quad \text{for all } v \in H .$$

Verify $b \in H^*$ b is linear ✓

b is bounded ✓