Contents

1 Preliminaries

1.1 Recall Some Important Notions and Results from Real Analysis

This section is based on *Real Analysis and Applications: Theory in Practice* by Davidson and Donsig (a textbook for AMATH/PMATH 331). The electronic version of the book can be downloaded from the UW Library.

Definition 1. *A sequence of real number $\{x_n\}$ is said to converge to $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an integer N_ε such that $|x_n - x| < \varepsilon$ for all $n > N_\varepsilon$.*

Note: Conversely, a sequence of real number $\{x_n\}$ is said to not converge to $x \in \mathbb{R}$ if there exists an $\varepsilon > 0$ such that for every N, there exists an $n > N$ such that $|x_n - x| \geq \varepsilon$.

Definition 2. *A sequence of real number $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists an N_ε such that $|x_n - x_m| < \varepsilon$ for all $n, m > N_\varepsilon$.*

Theorem 1 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Theorem 2 (Completeness of $\mathbb{R}$)**.** *Let $\{x_n\}$ be a sequence of real number. Then $\{x_n\}$ converges if and only if $\{x_n\}$ is a Cauchy sequence.*

Definition 3. *A set $S \subset \mathbb{R}$ is said to be bounded above by b if $x \leq b$ for all $x \in S$.*
A set $S \subset \mathbb{R}$ is said to be bounded below by a if $x \geq a$ for all $x \in S$.

Theorem 3 (Theorem and Definition)**.** *A set $S \subset \mathbb{R}$ which is bounded above has a least upper bound or supremum, written*

$$M = \sup_{x \in S} x,$$

with the properties

1. *If $x \in S$, then $x \leq M$.*
2. *If $c < M$, then there is an $x \in S$ such that $x > c$.*

Example 1. *Consider a real-valued $f : S \to \mathbb{R}$ and assume f has a supremum in S, $M = \sup_{x \in S} f(x)$. By definition of the supremum, there is a sequence $\{x_n\} \subset S$ such that $f(x_n) > M - \dfrac{1}{n}$.*

Theorem 4 (Theorem and Definition)**.** *A set $S \subset \mathbb{R}$ which is bounded below has a greatest lower bound or infimum, written*

$$m = \inf_{x \in S} x,$$

with the properties

1. *If $x \in S$, then $x \geq m$.*

2. *If $c > m$, then there is an $x \in S$ such that $x < c$.*

Note: For a set S, $\max_{x\in S} x$ and $\sup_{x\in S} x$ are not the same. There are sets where the supremum exists but the maximum does not. For example, $S = (0, 1)$.

Definition 4 (Continuous Functions, $\varepsilon - \delta$ Definition)**.** *A function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is continuous at a point $x_0 \in \Omega$ (this automatically means that $f(x_0)$ exists) iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x_0) - f(x)| < \varepsilon \text{ whenever } |x_0 - x| < \delta,\ x \in \Omega.$$

The function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is said to be continuous on Ω iff f is continuous at every point of Ω.

Definition 5 (Sequential Continuity)**.** *A function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is said to be sequentially continuous at a point $x_0 \in \Omega$ iff for every sequence $\{x_n\} \subset \Omega$ converging to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.*

Proposition 1. *A function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is continuous at a point $x_0 \in \Omega$ iff it is sequentially continuous at x_0.*

Proposition 2. *A real-valued function that is continuous on a closed and bounded region $\Omega \subset \mathbb{R}$ is bounded, and achieves its supremum and infimum in Ω.*

Example 2. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Since $[a, b]$ is closed and bounded, f is bounded and achieves its supremum and infimum in $[a, b]$. That is there exist $x_1, x_2 \in [a, b]$ such that*

$$f(x_1) = \max_{x\in[a,b]} f(x), \quad f(x_2) = \min_{x\in[a,b]} f(x).$$

Definition 6. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $\Omega \subset \mathbb{R}$. We say that $\{f_n\}$ converges pointwise to a function $f : \Omega \to \mathbb{R}$ if*

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \text{for all } x \in \Omega.$$

That is, for each $x \in \Omega$ and for every $\varepsilon > 0$, there exists $N_{\varepsilon,x}$ (depending on ε and x) such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{whenever} \quad n > N_{\varepsilon,x}.$$

Note:

- Pointwise limit of continuous functions can be discontinuous.
- Limit of integral may not be integral of limit.
- Pointwise limit of discontinuous functions can be continuous.

Definition 7. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $\Omega \subset \mathbb{R}$. We say that $\{f_n\}$ converges uniformly to a function $f : \Omega \to \mathbb{R}$ if given $\varepsilon > 0$, there exists an integer N_ε (depending on ε) so that*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in \Omega \text{ and for all } n > N_\varepsilon.$$

Note:

- Uniform convergence implies pointwise convergence.
- If $\{f_n\}$ converges pointwise to f, then f is the only potential limit for uniform convergence.
- Let $\{f_n : S \to \mathbb{R}\}$ be a sequence of continuous functions. If $\{f_n\}$ converges uniformly to a function f, then f is continuous.

Lemma 1 (Minkowski's Inequalities)**.** *Let $p \in \mathbb{R}$ and $1 \leq p < \infty$.*

1. *(for finite sum) Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

2. *(for infinite sequence) Consider $\ell_p = \{x = (x_1, x_2, \ldots), x_i \in \mathbb{R}, \sum\limits_{i=1}^{\infty} |x_i|^p < \infty\}$. Let $x, y \in \ell_p$. Then $x + y \in \ell_p$ and*

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

3. *(for integrable functions)*

$$\left(\int_a^b |f(t) + g(t)|^p \, dt\right)^{1/p} \leq \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} + \left(\int_a^b |g(t)|^p \, dt\right)^{1/p}$$

1.2 Recall Some Important Notions and Results from Linear Algebra

In this section, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 1. *A vector space X over $\mathbb{K}$ is a set X together with an addition, $u + v$, and a scalar multiplication, αu, satisfies the following rules for every $u, v, w \in X$ and $\alpha, \beta \in \mathbb{K}$:*

1. $u + v \in X$
2. $(u + v) + w = u + (v + w)$
3. $u + v = v + u$
4. *There is a vector $0 \in X$, called the zero vector, such that $u + 0 = 0 + u = u$*
5. *For every $u \in X$, there exists $(-u) \in X$ such that $u + (-u) = 0$*
6. $\alpha u \in X$
7. $\alpha(\beta u) = (\alpha\beta)u$

8. $(\alpha + \beta)u = \alpha u + \beta u$

9. $\alpha(u + v) = \alpha u + \alpha v$

10. $1\, u = u$

The elements of a vector space X are called vectors.

Example 1. 1. $\mathbb{R}^n$ *is a vector space over* $\mathbb{R}$; $\mathbb{C}^n$ *is a vector space over* $\mathbb{C}$

2. $X = \{\text{all functions } f : \mathbb{R} \to \mathbb{R}\}$

3. $\ell_p = \{\{x_i\} \subset \mathbb{K} \mid \sum_i |x_i|^p < \infty\}$, *where* $1 \leq p < \infty$.

4. $\ell_\infty = \{\{x_i\} \subset \mathbb{K} \mid \sup_i |x_i| < \infty, \forall n\}$

Definition 2. *Let X be a vector space over $\mathbb{K}$. If $\mathbb{S}$ is a subset of X and $\mathbb{S}$ is a vector space under the same operations as X, then $\mathbb{S}$ is called a subspace of X.*

Lemma 2. *(Subspace Test) If $\mathbb{S}$ is a nonempty set of X such that $u + v \in \mathbb{S}$ and $\alpha u \in \mathbb{S}$ for all $u, v \in \mathbb{S}$ and $c \in \mathbb{K}$ under the operation of X, then $\mathbb{S}$ is a subspace of X.*

Example 2. *Using subspace test, we can verify the following sets are vector spaces over $\mathbb{R}$.*

1. $X = P(x) = \{\text{all univariate polynomials}\}$

2. $X = P_n(x) = \{\text{all univariate polynomials of degree at most } n\}$

3. $X = C[a, b] = \{\text{all continuous functions on } [a, b]\}$

4. $X = C^1[a, b] = \{\text{all continuously differentiable functions on } [a, b]\}$

5. $X = C^\infty[a, b] = \{\text{all infinitely differentiable functions on } [a, b]\}$

6. $X = L_p[a, b] = \{\text{all Lebesgue integrable functions on } [a, b]\} = \{f : [a, b] \to \mathbb{R} \mid \int_a^b |f(x)|^p dx < \infty\}$

7. $X = L_\infty[a, b] = \{\text{all bounded almost everywhere functions on } [a, b]\}$

Definition 3. *Let X be a vector space over $\mathbb{K}$. The vectors $\{u_1, \ldots, u_k\} \subset X$ are called linearly independent if the only solution to $0 = \alpha_1 u_1 + \cdots + \alpha_k u_k$ is the trivial solution $\alpha_1 = \ldots = \alpha_k = 0$.*

If the maximal number of linearly independent vectors in X is $n < \infty$, we say X is an n-dimensional vector space and $\dim X = n$. Any set of n linearly independent vectors in X is called a basis for the vector space X.

We write $\dim X = \infty$ if for each $n = 1, 2, \ldots$, there exist n linearly independent vectors in X. In this case, X is called an infinite dimensional space.

Convention: $\dim\{\vec{0}\} = 0$.

Example 3. $\dim \mathbb{R}^n = n, \quad \dim P_n(x) = n+1, \quad \dim C[a,b] = \infty.$

Lemma 3. *Let X be an n-dimensional vector space and $\{u_1, \ldots, u_n\}$ be a basis for X. Then every vector $u \in X$ can be uniquely expressed as a linear combination of $\{u_1, \ldots, u_n\}$.*

Definition 4 (Quotient Space)**.** *Let $\mathbb{V}$ be a vector space and $\mathbb{W}$ be a subspace of $\mathbb{V}$. Consider the relation $\sim$ on $\mathbb{V}$:*

$$\text{For } x, y \in \mathbb{V}, \quad x \sim y \Leftrightarrow x - y \in \mathbb{W}.$$

It is easy to verify that relation is an equivalent relation (symmetric, reflexivity, and transitivity). Denote

$$[x] = \{y \in \mathbb{V} \mid x \sim y\}$$

Define the following set

$$\mathbb{V}/\mathbb{W} = \{[x] \mid x \in \mathbb{V}\},$$

with the following operators:

$$[x] + [y] := [x+y], \quad \alpha[x] := [\alpha x],$$

for any $x, y \in \mathbb{V}, \alpha \in \mathbb{K}$. Those operators are well-defined and $\mathbb{V}/\mathbb{W}$ is a vector space. Moreover, if $\dim \mathbb{V} < \infty$, $\dim \mathbb{V}/\mathbb{W} = \dim \mathbb{V} - \dim \mathbb{W}$.

Example 4. *Let $\mathbb{V}$ be the set of all real-valued integrable functions on $[a,b]$ and $\mathbb{W} = \{f \in \mathbb{V} \mid f = 0 a.e.\}$. We can verify that $\mathbb{W}$ is a subspace of $\mathbb{V}$, hence $\mathbb{V}/\mathbb{W}$ is also a vector space. Indeed, $L_1[a,b] = \mathbb{V}/\mathbb{W}$.*

Definition 5. *A map $T : \mathbb{V} \to \mathbb{W}$ between two vector spaces over $\mathbb{K}$ is called a linear operator if it preseves the operations of addition of vectors and multiplication by scalars, i.e.,*

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Denote $\ker T = \{u \in \mathbb{V} \mid T(u) = 0\}$ *Kernel of T, a subspace of $\mathbb{V}$.*
$\operatorname{Im} T = \{Tu \mid u \in \mathbb{V}\}$ *Image (range) of T, a subspace of $\mathbb{W}$.*

Theorem 1. *Let $T : \mathbb{V} \to \mathbb{W}$ be a linear operator between two vector spaces $\mathbb{V}$ and $\mathbb{W}$ over $\mathbb{K}$. Then*

- *T is one-to-one iff* $\ker T = \{\vec{0}\}$
- *T is onto iff* $\operatorname{Im} T = \mathbb{W}$
- *If* $\dim \mathbb{V} < \infty$, *then* $\dim \ker(T) + \dim \operatorname{Im}(T) = \dim \mathbb{V}$.

2 Normed Linear Spaces

2.1 Normed Linear Spaces: Definitions and Examples

Definition 1. *Let X be a real (or complex) vector space. A real-valued function $\|\cdot\| : X \to \mathbb{R}$ is a norm on X if*

1. *$\|x\| \geq 0$ for all $x \in X$ (positivity)*

2. *$\|x\| = 0$ if and only if $x = 0$ (strict positivity)*

3. *$\|\alpha x\| = |\alpha|\|x\|$ for any scalar α and for all $x \in X$ (homogeneity)*

4. *$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality)*

The pair $(X, \|\cdot\|)$ is called a normed linear space.

Example 1. 1. *The following functions are norms on $\mathbb{R}^n$:*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p \geq 1,$$

and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Proof. For $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, with $p \geq 1$, the first three requirements can be verified from the definition of $\|x\|_p$. The triangle inequality can be verified using the Minkowski's inequality for finite sums. □

2. *The following functions are norms on $X = C[a, b] = \{f : [a, b] \to \mathbb{R}$ continuous on $[a, b]\}$:*

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} \qquad (1 \leq p < \infty)$$

and

$$\|f\|_\infty = \max_{a \leq t \leq b} |f(t)|.$$

Proof. Let $f \in C[a, b]$. Since f is continuous on $[a, b]$, f is integrable on $[a, b]$ and achieves its maximum and minimum in $[a, b]$. Therefore, $\|f\|_p$ and $\|f\|_\infty$ are well-defined.
For $\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p}$ with $1 \leq p < \infty$,

- Positivity: By the definition of $\|f\|_p$, we have $\|f\|_p \geq 0$.
- Homogeneity: By the definition of $\|f\|_p$, we have

$$\|\alpha f\|_p = \left(\int_a^b |\alpha f(t)|^p \, dt\right)^{1/p} = |\alpha| \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} = |\alpha|\|f\|_p.$$

- Triangle inequality: Using the Minkowski's inequality for integrable functions, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p, \quad \text{for all} \quad f, g \in C[a, b].$$

- Strict positivity: Prove by contradiction. Suppose there exists $f \in C[a,b]$ with $\|f\|_p = 0$ but $f \not\equiv 0$. That is, there exists $x_0 \in [a,b]$ such that $f(x_0) \neq 0$. Since f is continuous, there exists a subinterval of width δ, $x_0 \ni I \subset [a,b]$ such that $\frac{|f(x_0)|}{2} \geq |f(x) - f(x_0)|$ for every $x \in I$. Since $|f(x) - f(x_0)| \geq |f(x_0)| - |f(x)|$, we have

$$\frac{|f(x_0)|}{2} \geq |f(x) - f(x_0)| \geq |f(x_0)| - |f(x)|, \quad |f(x)| \geq \frac{|f(x_0)|}{2},$$

for all $x \in I$. So,

$$0 = \|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right) \geq \left(\int_I |f(t)|^p \, dt\right)^{1/p} \geq \delta^{1/p} \frac{|f(x_0)|}{2} > 0,$$

a contradiction. Therefore the assumption is wrong. That means if $\|f\|_p = 0$ for some $f \in C[a,b]$, then $f \equiv 0$.

For $\|f\|_\infty = \max_{a \leq t \leq b} |f(t)|$, DIY.
Question: what is the best norm for $C[a,b]$? □

3. *For $1 \leq p < \infty$, the vector space*

$$L_p[a,b] = \{f : [a,b] \to \mathbb{R} \text{ measurable s.t. } \int_a^b |f(x)|^p dx < \infty\} / \sim$$

(where $f \sim g$ iff $f = g$ a.e.) is a normed space, with the norm defined as

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}.$$

Proof. The positive and homogenous properties are obvious. If $f \in L_p[a,b]$ and $\|f\|_p = 0$, then $f = 0$ a.e., which proves the strict positive property. The triangle inequality comes from the Minkowski's inequality for integrable functions. □

More examples:

4. *For $C^1[a,b]$,*

$$\|f\|_{1,\infty} = \max_{a \leq t \leq b} \{|f(t)|, |f'(t)|\}$$

and

$$\|f\|_{1,2} = \left(\max_{a \leq t \leq b} |f(t)|^2 + \max_{a \leq t \leq b} |f'(t)|^2\right)^{1/2}$$

are norms.

5. *For $1 \leq p < \infty$, the vector space $\ell_p = \{x = \{x_i\}, x_i \in \mathbb{R} \mid \sum_i |x_i|^p < \infty\}$ is a normed space, with the norm defined as* $\|x\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$.

6. *The vector space* $\ell_\infty = \{x = \{x_i\}, x_i \in \mathbb{R} \mid \sup_i |x_i| < \infty\}$ *is a normed space, with the norm defined as* $\|x\|_\infty = \sup_i |x_i|$.

Definition 2. *Let* $(X, \|\cdot\|)$ *be a normed linear space. A sequence* $\{x_n\} \subset X$ *is said to converge or to be convergent if there is an* $x \in X$ *such that*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

x *is called the limit of* $\{x_n\}$ *and we write* $\lim_{n\to\infty} x_n = x$.

Proposition 3 (Uniqueness of Limits)**.** *Let* $(X, \|\cdot\|)$ *be a normed linear space. A sequence in* X *converges to at most one point in* X.

Proof. Consider a sequence $\{x_n\}$ in X. If $\{x_n\}$ diverges, the proof is done. Suppose $\{x_n\}$ converges to two elements $x, y \in X$. Then

$$\|x - y\| = \|(x - x_n) + (x_n - y)\| \le \|x - x_n\| + \|x_n - y\| \to 0 \quad \text{as } n \to \infty.$$

Hence $\|x - y\| = 0$, i.e., $x = y$. □

Proposition 4. *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $\{x_n\} \subset X$*, and* $x_n \to x \in X$*. Then* $\|x_n\| \to \|x\|$.

Proof. Exercise. □

Definition 3. *Let* $(X, \|\cdot\|)$ *be a normed linear space. A sequence* $\{x_n\} \subset X$ *is called a Cauchy sequence if for every* $\varepsilon > 0$*, there exists an* N_ε *so that*

$$\|x_n - x_m\| < \varepsilon, \quad \textit{for all} \quad n, m > N_\varepsilon.$$

Proposition 5. *Every convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_n\} \subset X$ be a convergent sequence. For every $\varepsilon > 0$, there exists N so that

$$\|x_n - x\| < \frac{\varepsilon}{2}, \quad \text{for all} \;\; n > N.$$

Then for $n, m > N$, we have

$$\|x_n - x_m\| \le \|x_n - x\| + \|x_m - x\| < \varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. □

Conversely, there exist normed linear spaces such that not every Cauchy sequence converges.

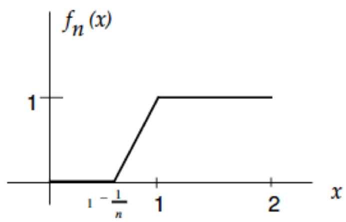
Example 2. *Consider the set of all rational numbers* $\mathbb{Q}$*. The set* $\mathbb{Q}$ *is a normed linear space under the standard addition* $u+v$*, the scalar multiplication* $\alpha\, u$*, and the absolute operator as a norm on* $\mathbb{Q}$*,* $\|u\| = |u|$ $(u, v, \alpha \in \mathbb{Q})$*. Consider the following sequence that approximates* $\sqrt{2} = 1.4142135...$

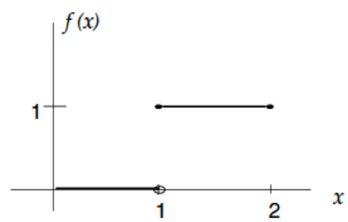
$$x_1 = 1, \quad x_2 = 1.4 = \frac{14}{10}, \quad x_3 = 1.41 = \frac{141}{100}, \dots$$

The sequence $\{x_n\}$ *converges to* $\sqrt{2}$ *and is a Cauchy sequence in* $\mathbb{Q}$*. However,* $\sqrt{2} \notin \mathbb{Q}$.

Example 3. *Consider the following sequence of (piecewise linear) functions in $C[0,2]$:*

$$f_n(x) = \begin{cases} 0 & \textit{for} \quad 0 \le x < 1 - \dfrac{1}{n} \\ 1 & \textit{for} \quad 1 < x \le 2 \\ 1 + n(x-1) & \textit{for} \quad 1 - \dfrac{1}{n} \le x \le 1. \end{cases}$$





Claim: The sequence $\{f_n\}$ is a Cauchy sequence in $(C[0,2], \|\cdot\|_1)$, but $\{f_n\}$ does not converge in $(C[0,2], \|\cdot\|_1)$.

Proof.

- Claim 1: $\{f_n\}$ is a Cauchy sequence w.r.t. $\|\cdot\|_1$. Indeed, with $m > n$, we have

$$\begin{aligned} \|f_n - f_m\| &= \int_0^2 |f_n(x) - f_m(x)| dx \\ &= \text{area of the triangle formed by } \left(1 - \frac{1}{m}, 0\right); \left(1 - \frac{1}{n}, 0\right); (1,1) \\ &= \frac{\frac{1}{n} - \frac{1}{m}}{2} < \frac{1}{2n} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Suppose $\{f_n\}$ converges to some function $f \in C[0,2]$ w.r.t. $\|\cdot\|_1$, i.e., $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

- Claim 2: The function f must be

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad 1 < x \le 2 \end{cases}$$

Part 2.1: Prove that $f(x) = 1$ for all $1 < x \le 2$. Suppose $f(x) \ne 1$ for all $1 < x \le 2$. Then there exists $x_1 \in (1,2]$ such that $f(x_1) \ne 1, f(x_1) - 1 \ne 0$. Since $f - 1 \in C[0,2]$, similar to the argument in Example 1 - part 2, there exists a subinterval I of width δ such that $x_1 \ni I \subset (1,2]$ such that $|f(x) - 1| \ge \dfrac{|f(x_1) - 1|}{2}$ for all $x \in I$. Then

$$\|f_n - f\|_1 = \int_0^2 |f_n(x) - f(x)| dx \ge \int_I |1 - f(x)| dx \ge \delta \frac{|f(x_1) - 1|}{2}.$$

Therefore, $0 = \lim_{n\to\infty} \|f_n - f\|_1 \ge \delta \dfrac{|f(x_1) - 1|}{2} > 0$, a contradiction. That means $f(x) = 1$ for all $1 < x \le 2$.

Part 2.2: Prove that $f(x) = 0$ for all $0 \le x < 1$. (hint: follow the same argument as in Part 2.1).

- Claim 3: The function f is not continuous at $x = 1$ since $\lim_{x\to 1^-} f(x) = 0$ and $\lim_{x\to 1^+} = 1$. Therefore, $f(x) \notin C[0,2]$, a contradiction.

  In conclusion, $\{f_n\}$ is a Cauchy sequence in $(C[0,2], \|\cdot\|_1)$, but $\{f_n\}$ does not converge in $(C[0,2], \|\cdot\|_1)$.

□

2.2 Banach Spaces: Definitions and Examples

Definition 1. *A normed linear space (X, d) is called a Banach space if every Cauchy sequence in X converges (that is, has a limit which is an element of X).*
Banach spaces are also called complete normed spaces.

Most proofs of completeness are based on the completeness of $\mathbb{R}$.

Theorem 1. $(\mathbb{R}^n, \|\cdot\|_p)$ *is a Banach space for* $1 \le p \le \infty$.

Proof. Exercise. □

Theorem 2. $(C[a,b], \|\cdot\|_1)$ *is not a Banach space (See Example 3 Section 2.1)*

Theorem 3. $(C[a,b], \|\cdot\|_\infty)$ *is a Banach space.*

Note: The compactness of the domain $[a,b]$ is used implicitly to ensure $\|\cdot\|_\infty$ well-defined.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C[a,b]$ w.r.t. $\|\cdot\|_\infty$. Then for every $\varepsilon > 0$, there exists N_ε such that for all $n, m > N_\varepsilon$, we have

$$\varepsilon > \|f_n - f_m\|_\infty = \max_{t\in[a,b]} |f_n(t) - f_m(t)|. \tag{1}$$

- Step 1: Show that f_n converges pointwise to some function f.

  Fixed $x \in [a,b]$. Then for every $n, m > N_\varepsilon$,

  $$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon.$$

  Therefore, $\{f_n(x)\}_{n\ge 1}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{f_n(x)\}_{n\ge 1}$ converges. Denote $f(x) := \lim_{n\to\infty} f_n(x)$. In other words, we have constructed a function $f : [a,b] \to \mathbb{R}$ such that $\{f_n\}$ converges pointwise to f.

- Step 2: Prove that $\{f_n\}$ converges uniformly to f, i.e., $\|f_n - f\|_\infty \to 0$.
  From the inequality (1), for every $\varepsilon > 0$, there exists N_ε such that

  $$|f_n(t) - f_m(t)| < \varepsilon, \quad \text{for all } n, m > N_\varepsilon, \quad \text{for all } t \in [a,b].$$

  Now letting $m \to \infty$ and keeping everything else fixed, we get

  $$|f_n(t) - f(t)| \le \varepsilon, \quad \text{for all } n > N_\varepsilon, \quad \text{for all } t \in [a,b].$$

  Therefore $\{f_n\}$ converges uniformly to f, $f_n \xrightarrow{\|\cdot\|_\infty} f$.

- Step 3: Prove that $f \in C[a,b]$.
  Since the uniform convergence of continuous functions is a continuous function, $f \in C[a,b]$. We actually can prove this claim in a few lines.
  Pick $\varepsilon > 0$. Since $f_n \xrightarrow{\|\cdot\|_\infty} f$, there exists N so that $|f_n(t) - f(t)| < \varepsilon/3$, for all $n \geq N$, for all $t \in [a,b]$.
  Since f_N is continuous, there exists $\delta > 0$ such that $|f_N(x) - f_N(y)| < \varepsilon/3$ whenever $|x - y| < \delta$.
  Then for every $x, y \in [a,b]$ with $|x - y| < \delta$, we have

  $$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \varepsilon,$$

  which means $f \in C[a,b]$.

  In conclusion, the Cauchy sequence $\{f_n\}$ converges in $C[a,b]$ w.r.t. the infty norm $\|\cdot\|_\infty$. That completes the proof.

□

Lecture 04: Riesz-Fischer Theorem

Lemma 4. *Let $(X, \|\cdot\|)$ be a normed linear space and $\{x_n\}$ be a Cauchy sequence in X. Then there exists a subsequence $\{x_{n_k}\}_k \subset \{x_n\}$ such that*

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}, \quad \textit{for all } k = 1, 2, \ldots$$

Proof. Since $\{x_n\}$ is a Cauchy sequence,

- ◇ For $\varepsilon = \frac{1}{2}$, there exists $n_1 > 0$ such that $\|x_n - x_m\| < \frac{1}{2}$ for every $n, m \geq n_1$.
- ◇ For $\varepsilon = \frac{1}{2^2}$, there exists $n_2 > n_1$ such that $\|x_n - x_m\| < \frac{1}{2^2}$ for every $n, m \geq n_2$.
- ◇ For $\varepsilon = \frac{1}{2^3}$, there exists $n_3 > n_2$ such that $\|x_n - x_m\| < \frac{1}{2^3}$ for every $n, m \geq n_3$.

...

We have constructed a subsequence $\{x_{n_k}\}_k$ with

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}, \quad \text{for every} \quad k \geq 1.$$

□

Lemma 5. *Let $(X, \|\cdot\|)$ be a normed linear space and $\{x_n\}$ be a Cauchy sequence in X. If there is a subsequence $\{x_{n_k}\}_k \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x \in X$, then $\{x_n\}$ also converges to that limit.*

Proof. Pick $\varepsilon > 0$. Since $\lim_{k\to\infty} x_{n_k} = x$, there exists N_1 such that

$$\|x_{n_k} - x\| < \frac{\varepsilon}{2}, \quad \text{for all} \quad k \geq N_1.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists $N_2 > N_1$ such that

$$\|x_n - x_m\| < \frac{\varepsilon}{2}, \quad \text{for all} \quad n, m \geq N_2.$$

Note that $n_{N_2} \geq N_2 > N_1$. For all $n \geq N_2$, we have

$$\|x_n - x\| \leq \|x_{n_{N_2}} - x\| + \|x_{n_{N_2}} - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $\lim_{n\to\infty} x_n = x$. □

Recall Some Important Results from Measure Theory

Theorem 4 (Lebesgue Monotone Convergence Theorem)**.** *Assume $\Omega \subset \mathbb{R}^d$ is measurable. If $\{f_n : \Omega \to [0, \infty]\}_n$ is a sequence of nonnegative measurable functions satisfying*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \quad \textit{for a.e.} \quad x \in \Omega,$$

then

$$\lim_{n\to\infty} \int_{\Omega} f_n(x)\, dx = \int_{\Omega} \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

Theorem 5 (Lebesgue Dominated Convergence Theorem)**.** *Assume $\Omega \subset \mathbb{R}^d$ is measurable. Let $\{f_n : \Omega \to [-\infty, \infty]\}_n$ be a sequence of measurable functions that converge pointwise for a.e. $x \in \Omega$. If there is a measurable function g such that*

$$|f_n(x)| \leq g(x) \quad \textit{for every } n \textit{ and a.e. } x \in \Omega,$$

then

$$\lim_{n\to\infty} \int_{\Omega} f_n(x)\, dx = \int_{\Omega} \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

Recall:

$$L_p[a,b] = \{f : [a,b] \to \mathbb{R} \text{ measurable s.t. } \int_a^b |f(x)|^p dx < \infty\}/W,$$

where $W = \{f : [a,b] \to \mathbb{R} \mid f = 0 \quad a.e.\}$. In practice, we consider $[f] \in L_p[a,b]$ as a function $f : [a,b] \to \mathbb{R}$ with $\int_a^b |f(x)|^p dx < \infty$ and functions that coincides μ-almost everywhere are the same.

Theorem 6 (Riesz-Fischer theorem)**.** *The set $(L_p[a,b], \|\cdot\|_p)$ with $1 \leq p < \infty$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $(L_p[a,b], \|\cdot\|_p)$.
By Lemma 4, there is a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\| < \dfrac{1}{2^k}$, for every $k = 1, 2, \ldots$
By Lemma 5, to prove $\{f_n\}$ converges, it suffices to show that $\{f_{n_k}\}$ converges in $(L_p[a,b], \|\cdot\|_p)$. Consider the following series

$$f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

and

$$|f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

The corresponding partial sums are

$$S_{1,m}(x) = f_{n_1}(x) + \sum_{k=1}^{m} (f_{n_{k+1}}(x) - f_{n_k}(x)) = f_{n_{m+1}}(x),$$

$$S_{2,m}(x) = |f_{n_1}(x)| + \sum_{k=1}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Since $\{S_{2,m}(x)\}$ is an increasing sequence, the limit

$$g(x) := \lim_{m\to\infty} S_{2,m}(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

always exists, where $g(x)$ could be $+\infty$ at some points.

- Step 1: Prove that $g \in L_p[a,b]$.
  The triangle inequality in $L_p[a,b]$ gives

$$\|S_{2,m}\|_p \le \|f_{n_1}\|_p + \sum_{k=1}^{m} \|f_{n_{k+1}} - f_{n_k}\|_p \le \|f_{n_1}\|_p + \sum_{k=1}^{m} \frac{1}{2^k} < \|f_{n_1}\|_p + 1.$$

  Therefore

$$\int_a^b (S_{2,m}(x))^p \, dx = \|S_{2,m}\|_p^p \le (\|f_{n_1}\|_p + 1)^p,$$

  and

$$\lim_{m\to\infty} \int_a^b (S_{2,m}(x))^p \, dx \le (\|f_{n_1}(x)\|_p + 1)^p < \infty.$$

  On the other hand, since $\{(S_{2,m}(x))^p\}$ is a monotone increasing sequence of nonnegative functions, the Lebesgue monotone convergence theorem implies

$$\lim_{m\to\infty} \int_a^b (S_{2,m}(x))^p \, dx = \int_a^b (\lim_{m\to\infty} (S_{2,m}(x))^p) \, dx = \int_a^b g(x)^p \, dx.$$

  Hence $\int_a^b g(x)^p \, dx < \infty$ and $g \in L_p[a,b]$. It also implies $g(x)$ is finite a.e. in $[a,b]$. In other words, $S_{2,m}(x)$ pointwise converges a.e. in $[a,b]$. Hence $S_{1,m}(x)$ pointwise converges a.e. in $[a,b]$ to a finite value $f(x)$:

$$f(x) := \lim_{m\to\infty} S_{1,m}(x) = \lim_{m\to\infty} f_{n_m}(x).$$

- Step 2: Prove that $f \in L_p[a,b]$.
  Since $|S_{1,m}(x)| \leq S_{2,m}(x) \leq g(x)$, we have $|f(x)| \leq g(x)$. Since $g \in L_p[a,b]$, we conclude that $f \in L_p[a,b]$.

- Step 3: Prove that $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$.
  We have
  $$|f_{n_m}(x) - f(x)|^p \leq (2\max\{|f(x)|, |S_{1,m-1}(x)|\})^p \leq (2g(x))^p.$$
  Since $(2g(x))^p$ is measurable, applying the Lebesgue Dominated Convergence Theorem, we obtain
  $$\lim_{m\to\infty} \int_a^b |f_{n_m}(x) - f(x)|^p \, dx = \int_a^b \left(\lim_{m\to\infty} |f_{n_m}(x) - f(x)|^p\right) dx = 0.$$
  It means $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$.
  Note: we also can prove $f \in L_p[a,b]$ after proving $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$. For $\varepsilon = 1$, there exists N so that $\|f_{n_m} - f\|_p < 1$ for all $m \geq N$. Then
  $$\|f\|_p \leq \|f_{n_N} - f\|_p + \|f_{n_N}\|_p < 1 + \|f_{n_N}\|_p < \infty.$$

In conclusion, we have proved that $(L_p[a,b], \|\cdot\|_p)$ is a Banach space. □

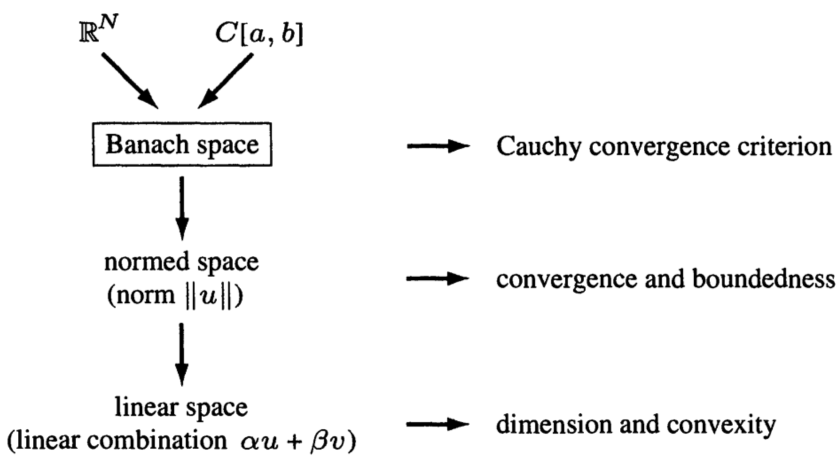


Figure 1: Source: from Zeidler's book

Recap: So far, we have been studied the completeness of the following normed linear spaces.

1. $(\mathbb{R}^n, \|\cdot\|_p)$ (with $1 \leq p \leq \infty$) is a Banach space.

2. $(C[a,b], \|\cdot\|_\infty)$ is a Banach space. (Proved in class)

3. $(L_p[a,b], \|\cdot\|_p)$ (with $1 \leq p < \infty$) is a Banach space. (Proved in class)

   More examples of Banach spaces (Exercises):

4. $(\ell_p, \|\cdot\|_p)$ (with $1 \le p \le \infty$) is a Banach space.

5. $(L_\infty[a,b], \|\cdot\|_\infty)$ is a Banach space, where

$$L_\infty[a,b] := \{f : [a,b] \to \mathbb{R} \mid \text{There exists an } M \text{ such that } |f(x)| \le M \text{ for almost every } x \in [a,b]\ \}/W,$$

$$W = \{f : [a,b] \to \mathbb{R} \mid f = 0 \quad a.e.\}$$

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} |f(x)| = \inf\{M \mid |f(x)| \le M \quad \text{for almost every} \quad x \in [a,b]\}$$

Some incomplete normed linear spaces:

6. $(C[a,b], \|\cdot\|_1)$ is not a Banach space. (Proved in class)

7. $(C[a,b], \|\cdot\|_2)$ is not a Banach space. (Exercise)

Lecture 05: Open and Closed Sets. Convexity. Banach Fixed-Point Theorem

2.3 Open and Closed Sets

Definition 1. *Let $(X, \|\cdot\|)$ be a normed linear space. Given a point $x_0 \in X$ and a real number $r > 0$, Define the following sets*

$$\begin{aligned} B(x_0,r) &= \{x \in X \mid \|x - x_0\| < r\} && (\textit{open ball}) \\ \overline{B(x_0,r)} &= \{x \in X \mid \|x - x_0\| \le r\} && (\textit{closed ball}) \\ S(x_0,r) &= \{x \in X \mid \|x - x_0\| = r\} && (\textit{sphere}) \end{aligned}$$

In all three cases, x_0 is called the center, and r the radius.

Definition 2. *A subset M of a normed linear space X is said to be open if for every $x_0 \in M$, there exists $r > 0$ such that $B(x_0, r) \subset M$.*

A subset M of a normed linear space X is said to be closed if the situation $\{x_n\} \subset M$, $x_n \to x \in X$ implies $x \in M$.

Proposition 6. *Let $(X, \|\cdot\|)$ be a normed linear space and M be a subset of X. Then M is open if and only if $M^c := X \backslash M$ is closed.*

Proof. ($\Rightarrow$) Suppose M is open, we need to show that M^c is closed. Let $\{x_n\} \subset M^c$ and $x_n \xrightarrow{\|\cdot\|} x \in X$. Assume $x \notin M^c$, then $x \in M$. Since M is open, by definition, there exists $r > 0$ such that $B(x,r) \subset M$. On the other hand, since $x_n \to x$, there exists N_r so that $\|x_n - x\| < r$ for every $n > N_r$. Choose $n_0 = [N_r + 1]$. Then $x_{n_0} \in B(x,r) \subset M$. Therefore, $x_{n_0} \in M \cap M^c = \emptyset$, a contradiction. Hence, the assumption $x \notin M^c$ is wrong, which means $x \in M^c$.

($\Leftarrow$) Suppose M^c is closed, we need to show that M is open. We will prove by contradiction. Assume M is not open. Then there exists $x_0 \in M$ so that for every $r > 0$, $B(x_0, r) \not\subset M$, that is $B(x_0, r) \cap M^c \neq \emptyset$. Let $x_n \in B\left(x_0, \frac{1}{n}\right) \cap M^c$. Since

$$\|x_n - x_0\| < \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty,$$

the sequence $x_n \to x_0$ (due to Squeeze Limit Theorem). Since M^c is closed and $\{x_n\} \subset M^c$, by definition, $x_0 \in M^c$. We have $x_0 \in M \cap M^c = \emptyset$, a contradiction. Therefore, the assumption is wrong and M is open. □

An interesting example: Consider $X = (0, 2) \subset \mathbb{R}$ with subset $A = (1, 2)$. X is a metric space (not a normed linear space). Then $A^c = (0, 1]$. But $(0, 1]$ does not look like a closed set since the sequence $x_n = \dfrac{1}{n} \in A^c$ converges to $0 \notin A^c$. The interval $(0, 1]$ is neither closed nor open if we consider it as a subset of $\mathbb{R}$. In the metric space X, the sequence $x_n = \dfrac{1}{n}$ is not a convergent sequence since $\lim x_n = 0 \notin X$.

Proposition 7. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_0 \in X$, and $r \in \mathbb{R}_+$. Then $B(x_0, r)$ is open and $\overline{B(x_0, r)}$ is closed.*

Proof. (a). Let $x_1 \in B(x_0, r)$. Then $\|x_1 - x_0\| < r$. Denote $r_1 = r - \|x_1 - x_0\|$.
Claim: $B(x_1, r_1) \subset B(x_0, r)$. Indeed, for any $y \in B(x_1, r_1)$, we have

$$\|y - x_0\| \leq \|y - x_1\| + \|x_1 - x_0\| < r_1 + \|x_1 - x_0\| = r,$$

which implies $y \in B(x_0, r)$. Hence $B(x_1, r_1) \subset B(x_0, r)$.

(b). Let $\{x_n\} \subset \overline{B(x_0, r)}$ and $x_n \to x \in X$. Then $\|x_n - x_0\| \leq r$. Using Proposition 4, we have $\|x_n - x_0\| \to \|x - x_0\|$. By Squeeze Limit Theorem, we have $\|x - x_0\| \leq r$. Therefore $x \in \overline{B(x_0, r)}$. □

Theorem 1. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{W}$ is a subspace of X. Then $(\mathbb{W}, \|\cdot\|)$ is a Banach space iff $\mathbb{W}$ is closed.*

Proof. ($\Rightarrow$) Suppose $(\mathbb{W}, \|\cdot\|)$ is a Banach space, $\{x_n\} \subset \mathbb{W}$, and $\lim_{n\to\infty} x_n = x \in X$. Since $\{x_n\}$ is a convergent sequence in X, $\{x_n\}$ is a Cauchy sequence in X. In addition, since $\{x_n\} \subset W$, $\{x_n\}$ is a Cauchy sequence in $\mathbb{W}$. Because $\mathbb{W}$ is a Banach space, there exists $y \in \mathbb{W}$ so that $x_n \to y \in \mathbb{W} \subset X$. By the uniqueness of the limits, $x = y$. Therefore $\mathbb{W}$ is closed.
($\Leftarrow$) Suppose $\mathbb{W}$ is closed. Let $\{x_n\} \subset \mathbb{W}$ is a Cauchy sequence. Since X is a Banach space, $x_n \to x \in X$. Since $\mathbb{W}$ is closed, $x \in \mathbb{W}$. Therefore, $\mathbb{W}$ is a Banach space. □

2.4 Convexity

Definition 1. *The set M in a linear space is called convex iff*

$$u, v \in M \quad \textit{and} \quad 0 \leq \alpha \leq 1 \quad \textit{imply} \quad \alpha u + (1 - \alpha)v \in M.$$

The function $f : M \to \mathbb{R}$ is called convex iff M is convex and

$$f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha)f(v),$$

for all $u, v \in M$ and all $\alpha \in [0, 1]$.

Example 1. *Let X be a normed space, and let $u_0 \in X$, $r \geq 0$ be given Then the closed ball*

$$B = \{u \in X \mid \|u - u_0\| \leq r\}$$

is convex.

Proof. If $u, v \in B$ and $0 \leq \alpha \leq 1$, we have

$$\begin{aligned}\|\alpha u + (1-\alpha)v - u_0\| &= \|\alpha(u - u_0) + (1-\alpha)(v - u_0)\| \\ &\leq \|\alpha(u - u_0)\| + \|(1-\alpha)(v - u_0)\| \\ &\leq \alpha\|u - u_0\| + (1-\alpha)\|v - u_0\| \leq \alpha r + (1-\alpha)r = r.\end{aligned}$$

□

Example 2. *Let $(X, \|\cdot\|)$ be a normed space. The function $f : X \to \mathbb{R}$, $f(u) := \|u\|$ is continuous and convex.*

Proof. Exercise. □

2.5 The Banach Fixed-Point Theorem and the Iteration Method

Definition 1. *Let M and Y be sets. An operator $A : M \to Y$ associates to each point $u \in M$ a point $v \in Y$, denoted by $v = Au$.*

Example 1. *Let $-\infty < a < b < \infty$ and let the function*

$$F : [a, b] \times \mathbb{R} \to \mathbb{R}$$

be continuous. For each $u \in C[a, b]$, define

$$Au : [a, b] \to \mathbb{R}, \ (Au)(x) := \int_a^x F(t, u(t))dt \quad \text{for all} \ \ x \in [a, b].$$

Since u and F are continuous function, $G(t) = F(t, u(t))$ is also continuous. By the Fundamental Theorem of Calculus, Au is continuous. In conclusion, we have defined an operator from $C[a, b]$ to itself:

$$A : C[a, b] \to C[a, b], \quad (Au)(x) := \int_a^x F(t, u(t))dt \quad \text{for all} \ \ x \in [a, b].$$

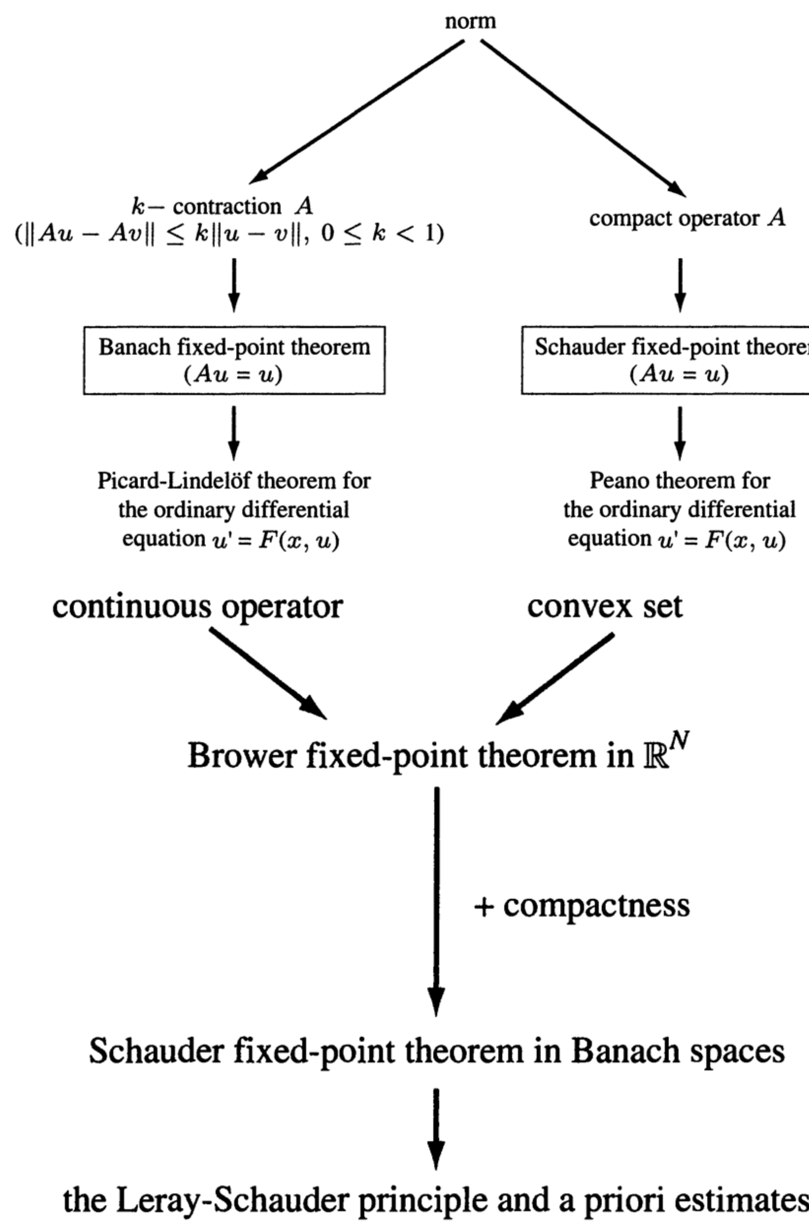


Figure 2: Source: From Zeidler's book.

Next, we will discuss about the Banach fixed-point theorem. It represents a fundamental convergence theorem for a wide class of iteration methods such as Newton's method. It is also used to prove the existence and uniqueness of solutions to certain ODEs (Picard-Lindelöf Theorem), to integral equations, and to value iteration, policy iteration, and policy evaluation of reinforcement learning.
Problem statement: Given an operator $A : M \to M$, we want to solve the operator equation

$$u = Au, \quad u \in M, \tag{2}$$

by using the iteration method:

$$u_0 \in M, \quad u_{n+1} = Au_n \quad n = 0, 1, \ldots, \tag{3}$$

Each solution of $u = Au$ is called a fixed point of the operator A.

Theorem 1 (Banach Fixed-Point Theorem)**.** *Assume that:*

(i) M is a closed, nonempty set in the Banach space X

(ii) The operator $A : M \to M$ is $k-$contractive, i.e.,

$$\|Au - Av\| \leq k\|u - v\| \quad \textit{for all } u, v \in M$$

and fixed $k \in [0, 1)$.

Then the following hold true:

1. *Existence and uniqueness. The equation $u = Au,\ u \in M$ has exactly one solution $u_* \in M$.*

2. *Convergence of the iteration method. For each given $u_0 \in M$, the sequence $\{u_n\}$ constructed by the iteration method* (3) *converges to the unique solution u_* of Equation* (2)*.*

3. *Error estimates. For all $n = 0, 1, \ldots,$ we have a priori error estimate*

$$\|u_n - u_*\| \leq \frac{k^n}{1-k}\|u_1 - u_0\|,$$

and for all $n = 1, 2, \ldots,$ we have a posteriori error estimate

$$|u_n - u_*\| \leq \frac{k}{1-k}\|u_n - u_{n-1}\|.$$

4. *Rate of convergence. For all $n = 0, 1, \ldots$ we have*

$$\|u_{n+1} - u_*\| \leq k\|u_n - u_*\|.$$

Proof. 1 & 2. Step 1: Show that $\{u_n\}$ is a Cauchy sequence in X. Then since X is Banach, $\{u_n\}$ to some $u_* \in X$. Since M is closed and $\{x_n\} \subset M$, $u_* \in M$.
Step 1.1: Evaluate

$$\|u_{n+1} - u_n\| = \|Au_n - Au_{n-1}\| \leq k\|u_n - u_{n-1}\| \leq k^2\|u_{n-1} - u_{n-2}\| \leq \cdots \leq k^n\|u_1 - u_0\|.$$

Step 1.2: Evaluate

$$\begin{aligned}
\|u_{n+m} - u_n\| &= \|(u_{n+m} - u_{n+m-1}) + \cdots + (u_{n+2} - u_{n+1}) + (u_{n+1} - u_n)\| \\
&\leq \|u_{n+m} - u_{n+m-1}\| + \cdots + \|u_{n+1} - u_n\| \\
&\leq (k^{n+m-1} + \cdots + k^n)\|u_1 - u_0\| \\
&\leq k^n(k^{m-1} + \cdots + k + 1)\|u_1 - u_0\| = k^n\frac{1-k^m}{1-k}\|u_1 - u_0\| \\
&\leq \frac{k^n}{1-k}\|u_1 - u_0\|.
\end{aligned}$$

Since $k \in [0,1)$, $k^n \to 0$ as $n \to \infty$. Therefore the sequence $\{u_n\}$ is Cauchy. Since X is Banach, $\{u_n\}$ to some $u_* \in X$. Also, because M is closed and $\{x_n\} \subset M$, $u_* \in M$.

Step 2: Show that $u_* = Au_*$.
Observe that

$$\|u_{n+1} - Au_*\| = \|Au_n - Au_*\| \leq k\|u_n - u_* \xrightarrow{n\to\infty} 0.$$

Therefore, $Au_* = \lim_{n\to\infty} u_{n+1} = u_*$.

Step 3: Uniqueness of the solution: Show that if $u_* = Au_*$ and $v_* = Av_*$ for some $v_* \in M$ then $u_* = v_*$.
We have

$$\|u_* - v_*\| = \|Au_* - Av_*\| \leq k\|u_* - v_*\|, \quad (k-1)\|u_* - v_*\| \geq 0$$

Since $k \in [0,1)$, this implies $\|u_* - v_*\| = 0$, and hence $u_* = v_*$.

3. From $\|u_{n+m} - u_n\| \leq \dfrac{k^n}{1-k}\|u_1 - u_0\|$, letting $m \to \infty$, we get

$$\|u_* - u_n\| \leq \frac{k^n}{1-k}\|u_1 - u_0\|, \quad \text{for all } n = 0, 1, \ldots$$

Notice that

$$\|u_{n+m} - u_n\| \leq \|u_{n+m} - u_{n+m-1}\| + \cdots + \|u_{n+1} - u_n\| \leq (k^m + \cdots + k)\|u_n - u_{n-1}\| \leq \frac{k}{1-k}\|u_n - u_{n-1}\|$$

Letting $m \to \infty$, we get

$$|u_n - u_*\| \leq \frac{k}{1-k}\|u_n - u_{n-1}\|.$$

4. It comes from

$$\|u_{n+1} - u_*\| = \|Au_n - Au_*\| \leq k\|u_n - u_*\|.$$

□

Comments: The priori error estimates can help to determine the maximal number of iterations required to attain a given precision. The posteriori error estimates base on u_n and u_{n+1} to determine the accuracy of the approximation u_{n+1}. Experience shows that a posteriori estimates are better than a priori estimates.

Lecture 06: Applications of the Banach Fixed Point Theorem to ODEs and Integral Equations

2.6 Applications to Ordinary Differential Equations

Given $(x_0, u_0) \in \mathbb{R}^2$, let $F(x,w)$ be a continuous function on a rectangle

$$S = \{(x,w) \in \mathbb{R}^2 : \ |x - x_0| \leq a \text{ and } |w - u_0| \leq b\},$$

and thus bounded on S,

$$|F(x,w)| \le c, \quad \text{for all } (x,w) \in S.$$

For $0 < h \le a$, consider the following initial value problem:

$$\begin{cases} u' &= F(x,u), \quad x_0 - h \le x \le x_0 + h \\ u(x_0) &= u_0. \end{cases} \tag{4}$$

We are looking for a differentiable function $u : [x_0 - h, x_0 + h] \to \mathbb{R}$ that satisfies Equation (4) and

$$(x, u(x)) \in S \quad \text{for all } x \in [x_0 - h, x_0 + h].$$

Questions: When does the IVP (4) have a solution? Is the solution unique? What is the value of h with respect to a, b, c?

Denote $X := C[x_0 - h, x_0 + h]$ and $M := \{u \in X : \|u - u_0\|_\infty \le b\}$.
Consider the following integral equation (Picard integral equation)

$$u(x) = u_0 + \int_{x_0}^{x} F(y, u(y))dy, \quad x_0 - h \le x \le x_0 + h, \quad u \in M, \tag{5}$$

along with the iteration method

$$u_0(x) = u_0, \quad u_{n+1}(x) = u_0 + \int_{x_0}^{x} F(y, u_n(y))dy, \quad x_0 - h \le x \le x_0 + h, \ n = 0, 1, \dots \tag{6}$$

Proposition 8. *Suppose $F(x,u)$ is a continuous function on S. A function $u \in C^1[x_0 - h, x_0 + h]$ is a solution to the IVP (4) on S iff u is a solution to the integral equation (5).*

Proof. ($\Rightarrow$) Suppose $u \in C^1[x_0, x_0 + h]$ is a solution to the IVP. Integrating the ODE w.r.t x shows that the function u is also the solution of (5).
($\Leftarrow$) Suppose u is a solution to the integral equation (5). Then $u(x_0) = u_0$. Also, by the Fundamental theorem of calculus, $u \in C^1[x_0 - h, x_0 + h]$ and $u' = F(x, u(x))$. So u is a solution to the IVP (4). □

Theorem 1 (The Picard-Lindelöf Theorem)**.** *Assume the following:*

1. *The function $F : S \to \mathbb{R}$ is continuous.*
2. *$F(x,u)$ satisfies a Lipschitz condition with respect to u on S, that is, there exists $L \ge 0$ such that*

   $$|F(x,u_1) - F(x,u_2)| \le L|u_1 - u_2|,$$

   for all $(x,u_1), (x,u_2) \in S$.
3. *We choose the real number h in such a way that*

   $$0 < h \le a, \quad hc \le b, \quad hL < 1.$$

Then the following hold true:

(i) The sequence $\{u_n\}$ *constructed by* (6) *converges to some* $u_* \in X$.

(ii) The IVP (4) *has a unique solution, which is* u_* *in part (i)* .

(ii) For $n = 0, 1, \ldots,$ *we have the following error estimates*

$$\|u_n - u_*\|_\infty \le k^n(1-k)^{-1}\|u_1 - u_0\|_\infty,$$

$$\|u_{n+1} - u_*\|_\infty \le k(1-k)^{-1}\|u_{n+1} - u_n\|_\infty,$$

where $k := hL$.

Proof. We know from previous lectures that $(X, \|\cdot\|_\infty)$ is a Banach space and the closed ball $M = \{u \in X : \|u - u_0\|_\infty \le b\}$ is closed and nonempty.
For each $u \in M$, consider the following operator A

$$Au(x) := u_0 + \int_{x_0}^{x} F(y, u(y))dy, \quad \text{for } x \in [x_0 - h, x_0 + h].$$

Since F and u are continuous functions, by the Fundamental Theorem of Calculus, $Au : [x_0, x_0 + h] \to \mathbb{R}$ is also continuous. Therefore, we get the operator

$$A : M \to X.$$

We will prove that

1. $A : M \to M$.

2. The operator A is k-contractive, where $k = hL$.

Proof of (1): $A : M \to M$. Indeed, let $u \in M$. Then for every $x \in [x_0 - h, x_0 + h]$, we have

$$\left|\int_{x_0}^{x} F(y, u(y))dy\right| \le |x - x_0| \max_{(y,u)\in S} |F(y, u)| \le hc \le b.$$

Therefore

$$\|Au - u_0\|_\infty = \max_{x\in[x_0-h, x_0+h]} \left|\int_{x_0}^{x} F(y, u(y))dy\right| \le b,$$

i.e., $Au \in M$.

Proof of (2): The operator A is k-contractive, where $k = hL$. Indeed, for $u, v \in M$ and for any $x \in [x_0, x_0 + h]$, we have

$$\left|\int_{x_0}^{x} (F(y, u(y)) - F(y, v(y)))\, dy\right| \le \int_{x_0}^{x} |F(y, u(y)) - F(y, v(y))|\, dy \le \int_{x_0}^{x_0+h} |F(y, u(y)) - F(y, v(y))|\, dy$$

$$\le L \int_{x_0}^{x_0+h} |u(y) - v(y)|\, dy \le L\|u - v\|_\infty \int_{x_0}^{x_0+h} dy = hL\|u - v\|_\infty.$$

The same argument holds for $x \in [x_0 - h, x_0]$. Therefore,

$$\|Au - Av\|_\infty = \max_{x \in [x_0 - h, x_0 + h]} \left| \int_{x_0}^{x} \left(F(y, u(y)) - F(y, v(y))\right) dy \right| \le hL\|u - v\|_\infty.$$

Since $0 < hL < 1$, by the Banach fixed-point theorem, the integral equation $u = Au$ has a unique solution $u_* \in M$ and the iterative method constructs a sequence $u_n \to u_*$. By Proposition 8, u_* is the unique solution to the IVP. □

Proposition 9. *If* $\dfrac{\partial F}{\partial u}$ *is continuous on* S *then* F *satisfies a Lipschitz condition with respect to* u *on* S.

Proof. Let $u_1, u_2 \in \mathbb{R}$ such that $|u - u_0| \le r$. By the Mean Value Theorem,

$$F(x, u_1) - F(x, u_2) = \frac{\partial F}{\partial u}(x, c)(u_1 - u_2),$$

for some c between u_1 and u_2. Since $\dfrac{\partial F}{\partial u}$ is continuous on S and S is compact, we have

$$L = \max_{(x,u) \in S} \left| \frac{\partial F}{\partial u} \right| < \infty,$$

and

$$|F(x, u_1) - F(x, u_2)| = \left| \frac{\partial F}{\partial u}(x, c) \right| |u_1 - u_2| \le L|u_1 - u_2|,$$

for all $(x, u_1), (x, u_2) \in S$. □

Example 1. *Consider the initial value problem*

$$u' = 1 + u^2, \quad u(0) = 0.$$

What is the maximum of h *that the P-L theory works?*

Proof. Consider $S = \{(x, w) \in \mathbb{R}^2 : |x| \le a, |w| \le b\}$. The function $F(x, u) = 1 + u^2$ is continuous on S with $c = \max_{(x,u) \in S} |F(x, u)| = 1 + b^2$, and F satisfies a Lipschitz condition w.r.t u on S with $L = \max_{(x,u) \in S} \left| \dfrac{\partial F}{\partial u} \right| = 2 \max_{(x,u) \in S} |u| = 2b$. The P-L theorem requires

$$0 < h \le a, \quad h \le \frac{b}{b^2 + 1}, \quad h < \frac{1}{2b}$$

We have $\max_b \min \left\{ \dfrac{b}{b^2 + 1}, \dfrac{1}{2b} \right\} = \dfrac{1}{2}$ when $b = 1$. Therefore the P-L theorem gives a solution on $[-h, h]$ for any $0 < h < \dfrac{1}{2}$.

On the other hand, we can find the closed form of the IVP: $u = \tan x$ on a larger interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. There is a room to improve the P-L theorem! (see Remarks 1.1. below). □

Remark 1. *(From "Supplementary Remarks to IVP" by E. Vrscay – attached here in the next 4 pages).*

1. *The restriction on h can often be softened so that the existence of a unique solution to the IVP can be established over a larger interval.*

2. *However, h might not be arbitrarily large. There are IVPs that their solutions blow up at finite time. For example $u' = u^2,\ u(0) = u_0 > 0$. The solution of this IVP is $u(x) = \dfrac{u_0}{1 - u_0 x}$ for $0 \leq x < \dfrac{1}{u_0}$, which blows up at $x = \dfrac{1}{u_0}$.*

3. *The iterative method in the P-L Theorem provide estimates u_n to the solution u_* of the IVP.*

4. *Consider $u' = u^{1/3},\ u(0) = u_0 = 0,\ x \in [0, T]$. Then*

$$\frac{du}{u^{1/3}} = dx, \quad \int_0^x \frac{du}{u^{1/3}} = \int_0^x dy, \quad u(x)^{2/3} - u_0^{2/3} = \frac{2}{3}x.$$

So $u(x) = \left(u_0^{2/3} + \frac{2}{3}x\right)^{3/2} = \left(\frac{2}{3}x\right)^{3/2}$ is a solution. There is another solution $u(x) \equiv 0$. The reason why we can not apply the Picard-Lindelöf theorem is because the function F is not Lipschitz.

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Supplementary remarks to Section 2.7, "Initial Value Problem"

Recall that the solution to the initial value problem

$$
\begin{aligned}
y' &= f(t,y), \quad t_0 \le t \le a, \\
y(t_0) &= y_0,
\end{aligned}
\tag{1}
$$

also satisfies the integral equation

$$
y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds, \qquad t_0 \le t \le a, \tag{2}
$$

which is obtained by simple integration of the DE. We see that the solution $y(t)$ to the IVP in (1) is the fixed point of the integral operator T, defined as follows: $h = Tg$, where

$$
h(t) = (Tg)(t) = y_0 + \int_{t_0}^{t} f(s, g(s))\, ds. \tag{3}
$$

The integral operator T is often called the *Picard integral operator*.

We saw that under certain conditions on f, the operator T is contractive on a complete metric space (S_a, d_∞) of functions supported on $[t_0, a]$. You will also recall that some work had to be done to obtain an estimate of a, based on the properties of f:

1. First, $a \le t_0 + \dfrac{b}{M}$, where b can be prescribed and $M = \max_R |f(t,y)|$.
2. Then $a < t_0 + \dfrac{1}{L}$, where L is the Lipschitz constant for the second argument of f.

In what follows, we show that these restrictions can often be "softened" so that the existence of a unique solution to Eq. (1) can be established over a larger interval. This is done by showing that the operator T is "eventually contractive".

Now return to the following fundamental set of identities involving the Picard integral operator:

$$
\begin{aligned}
|(Tg)(t) - (Th)(t)| &= \left| \int_{t_0}^{t} [f(s, g(s)) - f(s, h(s))]\, ds \right| \\
&\le \int_{t_0}^{t} |f(s, g(s)) - f(s, h(s))|\, ds \\
&\le L \int_{t_0}^{t} |g(s) - h(s)|\, ds \\
&\le L d_\infty(g,h) \int_{t_0}^{t} ds \\
&= L d_\infty(g,h)(t - t_0)
\end{aligned}
\tag{4}
$$

Note that we have not integrated out to the value a, but rather are keeping the right-hand side as a function of t. This will be useful below.

We replace g and h in the above relations with Tg and Th, respectively:

$$
\begin{aligned}
|(T^2 g)(t) - (T^2 h)(t)| &= \left| \int_{t_0}^{t} [f(s, Tg(s)) - f(s, Th(s))]\, ds \right| \\
&\le \int_{t_0}^{t} |f(s, Tg(s)) - f(s, Th(s))|\, ds \\
&\le L \int_{t_0}^{t} |Tg(s) - Th(s)|\, ds.
\end{aligned}
\tag{5}
$$

Now insert the final result from (4):

$$\begin{aligned} |(T^2g)(t) - (T^2h)(t)| &\leq L^2 d_\infty(g,h) \int_{t_0}^{t} (s-t_0)\, ds \\ &= \frac{1}{2} L^2 d_\infty(g,h)(t-t_0)^2 \end{aligned} \tag{6}$$

We can repeat this procedure for T^2g and T^2h, etc., to arrive at the following result, which can be proved by induction:

$$|(T^n g)(t) - (T^n h)(t)| \leq \frac{1}{n!} L^n (t-t_0)^n d_\infty(g,h), \qquad t \in [t_0, a]. \tag{7}$$

Taking the supremum over $t \in [t_0, a]$ on both sides, we obtain the important result,

$$d_\infty(T^n g, T^n h) \leq \frac{1}{n!} L^n (a-t_0)^n d_\infty(g,h). \tag{8}$$

For sufficiently large n, say $n = p$,

$$\frac{1}{p!} L^p (a-t_0)^p < 1, \tag{9}$$

which implies that the operator $U = T^p$ for some $p > 1$ is a contraction. We say that T is "eventually contractive." From Banach's Contraction Mapping Theorem, it follows that U has a unique fixed point $\bar{u}(t) \in S_a$. We now state the following result, which will be left as an exercise:

The fixed point $\bar{u}$ of T^p is also a unique fixed point of T.

This implies that $\bar{u}$ is the unique solution to the IVP in Eq. (1).

Note that the above analysis can also be extended over to the "other side" of t_0, i.e., an interval $[c, t_0]$, provided that suitable conditions on f be met.

A final comment: From Eq. (9), one might be tempted to conclude that the outer endpoint a of the interval $[t_0, a]$, over which the unique solution exists, can be made as large as possible: Given any $a > 0$, we can find a $p > 0$ which guarantees that the inequality in Eq. (9) is true. This could pose a problem, since we know that some solutions "blow up" in finite time. Consider the following IVP, which is solved in the Addendum to Page 15, posted on the Course webpage.

$$\frac{dy}{dt} = y^2, \quad y(0) = y_0 > 0, \tag{10}$$

The function $f(t,y) = y^2$ is Lipschitz in the variable y, so a unique solution exists. It is given by

$$y(t) = \frac{y_0}{1 - y_0 t}, \quad 0 \leq t < \frac{1}{y_0}. \tag{11}$$

Nevertheless, the solution $y(t)$ "blows up" at $t = \dfrac{1}{y_0}$.

If we return to the proof of the existence-uniqueness to initial value problems using the Contraction Mapping Theorem, we see that, in fact, no such problem exists. The proof rests on the assumption that the solution is an element of a closed ball of continuous functions – the space S_a on Page 15 of the Course Notes. These functions are necessarily bounded. As such, the endpoint a may not be arbitrarily large – it depends on the function $f(t,y)$ on the RHS of the IVP. It, i.e, a, probably won't have to be as small as the value determined in the proof given in the Course Notes. But finding larger values could be a tricky procedure, involving some kind of "juggling", along with the knowledge that the Picard operator T is eventually contractive.

Picard method of successive approximation

Finally, the contractivity of the T (or T^p) operator is the basis for the *Picard method of successive substitution/approximation* or, simply, "*Picard's method*", that provides estimates to the solution of the IVP in Eq. (1). Often, these estimates are in the form of power series about the point t_0 (which is often zero). Picard's method is often discussed in undergraduate courses in ODEs. As such, it is treated in many texts devoted to ODEs and will not be discussed in great detail here. A excellent discussion of both theoretical and practical aspects of this method is to be found in the book, *Differential Equations with Applications and Historical Notes*, by G.F. Simmons (McGraw-Hill).

Briefly, we start with a function $u_0(t)$ that will be the "seed" of the iteration procedure. It is often most convenient to start with the constant function $u_0(t) = y_0$. We then construct the iteration sequence

$$u_{n+1} = Tu_n \,, \tag{12}$$

which becomes

$$u_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, u_n(s))\, ds, \quad n = 0, 1, \cdots . \tag{13}$$

From the contractivity (or eventual contractivity) of the Picard integral operator T (over an appropriate interval), it follows that the sequence of functions $\{u_n\}$ will converge uniformly to the solution $y(t)$ to the IVP in Eq. (1) (over an appropriate interval).

Let us now illustrate with a simple example. Consider the following IVP,

$$\frac{dy}{dt} = ay, \qquad y(0) = y_0, \tag{14}$$

where a and y_0 are arbitrary, nonzero real numbers. For convenience we have set $t_0 = 0$. Of course, we know that the solution to this IVP is

$$y(t) = y_0 e^{at} \,, \tag{15}$$

but we'll pretend, for the moment, that we don' know it.

The solution of this IVP must satisfy the equivalent integral equation,

$$y(t) = y_0 + \int_0^t ay(s)\, ds \,, \tag{16}$$

which is the fixed point equation $y = Ty$, where T denotes the Picard integral operator associated with the IVP in Eq. (14). Just as a check, we differentiate both sides with respect to t:

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left[\int_0^t ay(s)\, ds\right] \\ &= ay(t) \,. \end{aligned} \tag{17}$$

Furthermore, if we set $t = t_0 = 0$ in Eq. (14), we obtain

$$\begin{aligned} y(0) &= y_0 + \int_0^0 ay(s)\, ds \\ &= y_0 \,. \end{aligned} \tag{18}$$

Thus, the IVP in (14) is satisfied.

Let us now perform the Picard method of successive substitution associated with this IVP. As mentioned above, it is convenient to start with the constant function,

$$u_0(t) = y_0, \tag{19}$$

as the "seed" for the iteration procedure. Then

$$\begin{aligned} u_1(t) &= y_0 + \int_0^t au_0(s)\, ds \qquad (20) \\ &= y_0 + \int_0^t ay_0\, ds \\ &= y_0 + ay_0 t \\ &= y_0[1 + at]. \end{aligned}$$

Now repeat this procedure:

$$
\begin{aligned}
u_2(t) &= y_0 + \int_0^t a u_1(s)\, ds \\
&= y_0 + \int_0^t a y_0 (1 + as)\, ds \\
&= y_0 + a y_0 t + y_0 \frac{1}{2}(at)^2 \\
&= y_0 [1 + at + \frac{1}{2}(at)^2].
\end{aligned} \tag{21}
$$

One can conjecture, and in fact prove by induction, that

$$
u_n(t) = y_0 [1 + at + \cdots + \frac{1}{n!}(at)^n], \qquad n \geq 0, \tag{22}
$$

which is the nth degree Taylor polynomial $P_n(t)$ to the solution $y(t) = y_0 e^{at}$. As you will recall from MATH 128, for each $t \in \mathbf{R}$, these Taylor polynomials are partial sums of the infinite Taylor series expansion of the function $y(t)$. As such, we see that the sequence of functions $\{u_n\}$ converges to the solution. A little more work will show that the convergence is uniform over closed subintervals that include the point $t_0 = 0$.

Earlier, we commented that it was convenient to start the Picard method with the constant function $u_0(t) = y_0$. But we don't have to. We can, in fact, start with any function that satisfies the initial condition $u_0(0) y_0$. For example, let us consider

$$
u_0(t) = y_0 \cos t\,. \tag{23}
$$

Then

$$
\begin{aligned}
u_1(t) &= y_0 + \int_0^t a u_0(s)\, ds \\
&= y_0 + \int_0^t a y_0 \cos s\, ds \\
&= y_0 + a y_0 \sin s]_0^t \\
&= y_0 [1 + a \sin t].
\end{aligned} \tag{24}
$$

Once again:

$$
\begin{aligned}
u_2(t) &= y_0 + \int_0^t a u_1(s)\, ds \\
&= y_0 + \int_0^t a y_0 [1 + a \sin s]\, ds \\
&= y_0 + a y_0 t - a^2 y_0 \cos s]_0^t \\
&= y_0 [1 + at - a^2 \cos t + a^2].
\end{aligned} \tag{25}
$$

It is perhaps not obvious that these functions are "getting closer" to the solution $y(t) = y_0 e^{at}$. But it is not too hard to show (Exercise) that the Taylor series expansions of $u_1(t)$ and $u_2(t)$ agree, respectively, to the first two and three terms of the Taylor series expansion of $y(t)$.

Lecture 07: Continuity. Compactness. Equivalent Norms.

2.7 Continuity

Definition 1. *Let X and Y be normed linear spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $f : M \subset X \to Y$.*

- *f is continuous at $x_0 \in M$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(x_0)\| < \varepsilon$ for all x so that $\|x - x_0\| < \delta$.*
- *f is continuous on M if f is continuous at all $x_0 \in M$.*
- *f is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in M$ so that $\|x - y\| < \delta$. (Note δ does not depend either on x or y).*

Proposition 10. *Let X and Y be normed linear spaces over $\mathbb{K}$ and $f : M \subset X \to Y$. Then f is continuous at $x \in M$ if and only if for every sequence $\{x_n\}$ in M,*

$$\lim_{n\to\infty} x_n = x \quad \textit{implies} \quad \lim_{n\to\infty} f(x_n) = f(x).$$

Proof. Exercise. □

Proposition 11. *Let $f : X \to Y, g : Y \to Z$, where X, Y, Z are normed linear spaces. If f is continuous at $a \in X$ and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.*

Proof. Exercise. □

2.8 Compactness

Definition 1. *Let S be a set in a normed linear space X.*

- *S is called relatively compact iff each sequence $\{u_n\}$ in S has a convergent subsequence $u_{n_k} \to u \in X$ as $k \to \infty$.*
- *S is called compact iff each sequence $\{u_n\}$ in S has a convergent subsequence $u_{n_k} \to u \in S$ as $k \to \infty$.*
- *S is called bounded iff there is a number $r \geq 0$ such that $\|u\| \leq r$ for all $u \in S$.*

Proposition 12. *Let S be a set in a normed linear space X. Then*

1. *The set S is compact iff it is relatively compact and closed.*
2. *If S is relatively compact, then S is bounded.*
3. *If S is compact, then S is closed and bounded. The reverse might not be true.*

Proof. 1. Exercise.
2. Suppose S is relatively compact but S is not bounded. Then, there exists a sequence $\{u_n\} \subset S$ such that $\|u_n\| \geq n$ for all n. Since S is relatively compact, there exists a convergent subsequence $\{u_{n_k}\}_k$. Therefore, $\{u_{n_k}\}_k$ is bounded. On the other hand, $|u_{n_k}| \geq n_k \geq k$, a contradiction.

3. Combining (1) and (2), we have the conclusion that if S is compact, then S is closed and bounded. Below is a counter example, where the reverse might not be true.

Example 1. *In $(\ell_2, \|\cdot\|_2)$, consider*

$$\overline{B_1(0)} := \{x = (x_1, x_2, \ldots) : \|x\|_2 \leq 1\}.$$

The closed ball $\overline{B_1(0)}$ is closed and bounded, but $\overline{B_1(0)}$ is not compact. Indeed, consider the following sequence in $\overline{B_1(0)}$:

$$e_k = (0, \ldots, 0, 1, 0, \ldots), \quad \text{where the } k\text{th position of } e_k \text{ is 1 and other positions are 0's}, \quad k = 1, 2, \ldots$$

Since $\|e_k - e_j\|_2 = \sqrt{2}$ for every $k \neq j$, the sequence $\{e_k\}_k \subset \overline{B_1(0)}$ has no convergent subsequences since no subsequence can be a Cauchy sequence. Therefore, $\overline{B_1(0)}$ is not compact.

□

Theorem 1. *Let X and Y be normed linear spaces and $T : X \to Y$ be a continuous mapping. Then the image of a compact subset S of X under T is compact.*

Proof. Let $\{y_n\}$ be a sequence in $T(S)$. Then $y_n = T(x_n)$ for some $x_n \in S$. Since S is compact, there is a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x_* \in S$. Since T is continuous, $y_{n_k} = T(x_{n_k}) \to T(x_*) \in T(S)$ as $k \to \infty$. Therefore $T(S)$ is compact. □

An important consequence is the following theorem, which is a generalization of the Extreme Value Theorem for continuous functions over bounded and closed intervals on $\mathbb{R}$.

Theorem 2 (The Weeirstrass Theorem)**.** *Let $T : S \to \mathbb{R}$ be a continuous function on the compact, nonempty subset S of a normed linear space. Then T has a minimum and maximum on S.*

Proof. By Theorem 1, $T(S)$ is compact in $\mathbb{R}$. Therefore, $T(S)$ is closed and bounded. Therefore, $\alpha = \inf_{x\in S} T(x)$ is finite. By the definition of the infimum, there exists a sequence $\{x_n\}$ in S such that $\lim_{n\to\infty} T(x_n) = \alpha$. Since $T(S)$ is closed, $\lim_{n\to\infty} T(x_n) \in T(S)$, i.e., $\alpha \in T(S)$. Thus T has a minimum on S. The same argument can be used to show T has a maximum on S. □

Note:

- Image of a closed set under a continuous mapping might not be closed. For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \exp^x$ and $S = (-\infty, 0] \subset \mathbb{R}$. The function f is continuous on $\mathbb{R}$, S is a closed subset in $\mathbb{R}$ but $f(S) = (0, 1]$ is not a closed subset of $\mathbb{R}$.

- Image of a bounded set under a continuous mapping might not be bounded. For example, $f : (0,1) \to \mathbb{R},\ f(x) = \dfrac{1}{x}$ and $S = (0,1)$ is a bounded set but $f(S) = (0,\infty)$ is not bounded.

Now, we will present some compactness critera for a set in a normed linear space.

2.8.1 Compactness in Finite-Dimensional Normed Linear Spaces

Next, we recall a theorem in real analysis.

Theorem 3 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Using Bolzano-Weierstrass Theorem, we have the following result.

Theorem 4. *In $(\mathbb{K}^n, \|\cdot\|_\infty)$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, a subset $S \subset \mathbb{K}^n$ is compact if and only if S is closed and bounded.*

Proof. Case $\mathbb{K} = \mathbb{R}$. It is sufficient to prove that if S is bounded in $\mathbb{R}^n$, then S is relatively compact. Consider a sequence in S:

$$\{u_m = (u_{m,1}, \dots, u_{m,n})\}_m \subset S.$$

Since S is bounded, there is a constant $M > 0$ such that

$$M \geq \|u_m\|_\infty \geq |u_{m,k}|, \quad \text{for all} \quad k = 1,2,\dots,n, \quad \text{and} \quad m = 1,2,\dots.$$

The real sequence $\{u_{m,1}\}_m$ is bounded, so by the Bolzano-Weierstrass theorem, there is a subsequence $\{u_m^{(1)}\}$ of $\{u_m\}$ such that $\{u_{m,1}^{(1)}\}$ converges.
By the Bolzano-Weierstrass theorem, there is a subsequence $\{u_m^{(2)}\}$ of $\{u_m^{(1)}\}$ such that $\{u_{m,2}^{(2)}\}$ converges. Thus $\{u_{m,1}^{(2)}\}$ and $\{u_{m,2}^{(2)}\}$ converge.
Repeating this process n times, we have constructed a subsequence $\{u_m^{(n)}\}_m$ of $\{u_m\}_m$ such that $\{u_{m,k}^{(n)}\}_m$ converges for all $k = 1,2,\dots,n$. Using the $\varepsilon - N_\varepsilon$ definition of convergent sequences, we can easily verify that $\{u_m^{(n)}\}_n$ converges in $\mathbb{R}^n$. Therefore, S is relatively compact.

Case $\mathbb{K} = \mathbb{C}$. (Sketch of the proof): Any $u \in \mathbb{C}^n$ can be written as $u = v + iw$, where $v, w \in \mathbb{R}^n$. Use $\|u\|_\infty \geq \|v\|_\infty$ and $\|u\|_\infty \geq \|w\|_\infty$.

□

Definition 2. *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X are called equivalent iff there are positive numbers $\alpha, \beta > 0$ such that*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1, \quad \textit{for all} \;\; x \in X.$$

Lecture 08: Compact Sets. Compact Operators.

Theorem 5. *Two norms on a finite-dimensional linear space X over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) are always equivalent.*

Sketch of the proof. If $\dim X = 0$, any norm on X is the zero function. Therefore, all norms on X are equivalent.

Let $0 < n = \dim X$ and $\|\cdot\|$ is a norm on X. Suppose $\{e_1, \dots, e_n\}$ is a basis for X. For each $x \in X$, there is a unique tuple $\alpha \in \mathbb{K}^n$ such that

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n.$$

Define $\|x\|_\infty := \|\alpha\|_\infty = \max\limits_{1 \le j \le n} |\alpha_j|$.

- Step 1: Prove that $\|\cdot\|_\infty : X \to \mathbb{R}$, $\left\| \sum\limits_{1 \le j \le n} \alpha_j e_j \right\|_\infty := \max\limits_{1 \le j \le n} |\alpha_j|$ is a norm on X. (Exercise).

  Set $S = \{\alpha \in \mathbb{K}^n : \|\alpha\|_\infty = 1\}$. Then S is closed and bounded in $\mathbb{K}^n$ (Exercise).
  Therefore, S is compact. Consider a function

  $$f : S \subset \mathbb{K}^n \to \mathbb{R}, \; f(\alpha) := \left\| \sum_{k=1}^n \alpha_k e_k \right\|.$$

- Step 2: Prove that f is a continuous function. (Exercise). Hint: Show that

  $$|f(\alpha) - f(\beta)| \le \|\alpha - \beta\|_\infty \sum_{k=1}^n \|e_k\|, \quad \text{for all } \alpha, \beta \in \mathbb{K}^n.$$

- Step 3: Prove $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.
  Applying the Weeirstrass theorem for continuous functions on the compact nonempty subset of S, we conclude that $f : S \to \mathbb{R}$ has a maximum and minimum on S. Let

  $$A = \min_{\alpha \in S} f(\alpha) = \min_{\alpha \in \mathbb{R}^n} \left\{ \left\| \sum_{k=1}^n \alpha_k e_k \right\| \; s.t. \; \max_{1 \le j \le n} |\alpha_j| = 1 \right\}$$

  and

  $$B = \max_{\alpha \in S} f(\alpha) = \max_{\alpha \in \mathbb{R}^n} \left\{ \left\| \sum_{k=1}^n \alpha_k e_k \right\| \; s.t. \; \max_{1 \le j \le n} |\alpha_j| = 1 \right\}$$

  Note that A and B are constants that depend on the norm $\|\cdot\|$ of X. Also, since $0 \notin S$, so $f(\alpha) > 0$ for all $\alpha \in S$. So $0 < A \le B$ and

  $$A \le f(\beta) = \left\| \sum_{k=1}^n \beta_k e_k \right\| \le B, \quad \text{for all} \;\; \beta \in S.$$

  The above inequalities can be rewritten as

  $$A \le \|y\| \le B \quad \text{for all} \quad y \in X \text{ with } \|y\|_\infty = 1.$$

For any $x \in X - \{0\}$, let $z = \dfrac{x}{\|x\|_\infty} \in X$. Then $\|z\|_\infty = 1$, so

$$A \leq \|z\| \leq B, \quad A \leq \frac{\|x\|}{\|x\|_\infty} \leq B, \quad A\|x\|_\infty \leq \|x\| \leq B\|x\|_\infty.$$

That inequality also holds for $x = 0$. So we have

$$A\|x\|_\infty \leq \|x\| \leq B\|x\|_\infty \quad \text{for all } x \in X$$

- Step 4: Prove that any two norms on X are equivalent. Let $\|\cdot\|_{(2)}$ be another norm on X. Then there exist positive constants A_2, B_2 so that

$$A_2\|x\|_\infty \leq \|x\|_{(2)} \leq B_2\|x\|_\infty \quad \text{for all } x \in X$$

So for every $x \in X$, we have

$$\frac{A}{B_2}\|x\|_{(2)} \leq \|x\| \leq \frac{B}{A_2}\|x\|_{(2)}$$

□

Corollary 1. *All norms on $\mathbb{R}^n$ are equivalent.*

Theorem 6. *In a finite dimensional normed linear space, any subset M is compact iff M is closed and bounded.*

Proof. Assignment 2. □

Note: In Assignment 2, we also prove a useful result: All finite dimensional normed spaces are Banach spaces.

2.8.2 Compactness in Infinite-Dimensional Normed Linear Spaces

Now we present without proof compactness criteria for some infinite dimensional normed spaces: $(C[a,b], \|\cdot\|_\infty)$ and $(L_1[a,b], \|\cdot\|_1)$ (see Zeidler's book page 35; See Oden and Demkowicz's book page 339-341).

Theorem 7 (The Arzela-Ascoli Theorem)**.** *Consider the normed linear space $(C[a,b], \|\cdot\|_\infty)$ where $-\infty < a < b < \infty$. Suppose we are given a set S in $C[a,b]$ such that*

1. *S is bounded.*
2. *S is equicontinuous, i.e., for each $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$|x - y| < \delta \quad \textit{and} \quad u \in S \quad \textit{imply} \quad |u(x) - u(y)| \leq \varepsilon.$$

Then S is a relatively compact subset of $C[a,b]$.

So, for $(C[a,b], \|\cdot\|_\infty)$, we have:

$$\text{compact sets} = \text{closed} + \text{bounded} + \text{equicontinuous sets.}$$

Theorem 8 (Frechét - Kolmogorov Theorem)**.** *A subset $\mathcal{F} \subset (L_p(\mathbb{R}), \|\cdot\|_p)$, $1 \le p < \infty$, is relatively compact in $L_p(\mathbb{R})$ iff the following conditions hold:*

1. *$\mathcal{F}$ is bounded, i.e., there exists an $M > 0$ such that $\|f\|_p \le M$ for every $f \in \mathcal{F}$.*

2. *For each $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$|t| < \delta \quad \textit{and} \quad f \in \mathcal{F} \quad \textit{imply} \quad \int_{\mathbb{R}} |f(t+s) - f(s)|^p ds \le \varepsilon.$$

3. *$\lim\limits_{n\to\infty} \int_{|s|>n} |f(s)|^p \, ds = 0$ for every $f \in \mathcal{F}$.*

Below is another useful compactness criteria (see Zeidler's book pages 38-39 for the proof).

Theorem 9 (Finite ε-net)**.** *Let S be a nonempty set in the Banach space X. Then the following two statements are equivalent:*

(i) S is relatively compact.

(ii) S has a finite ε-net ; that is, for each $\varepsilon > 0$, there exists a finite number of points $v_1, \ldots, v_N \in S$ such that

$$\min_{1\le k\le N} \|u - v_k\| \le \varepsilon \quad \textit{for all} \quad u \in S.$$

In other words, $S \subset \bigcup\limits_{k=1}^{N} B(v_k, \varepsilon) \subset X$.

Note: The smallest integer N such that S can be covered by N $\varepsilon-$ balls is called the covering number $\mathcal{N}(S, \|\cdot\|, \varepsilon)$. For example, when S is a subset of the unit ball in $(\mathbb{R}^n, \|\cdot\|)$,

$$\mathcal{N}(S, \|\cdot\|, \varepsilon) \le \left(1 + \frac{2}{\varepsilon}\right)^n,$$

See, for example, "A Mathematical Introduction to Compressive Sensing" by Foucart and Rauhut, page 577.

Next, we will study a useful operator, called compact operator, to generalize classical results for operator equations in finite-dimensional normed spaces to infinite-dimensional normed spaces.

2.8.3 Compact Operators

Definition 3. *Let X and Y be normed space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The operator $A : X \to Y$ is called a compact operator iff*

1. *A is continuous, and*

2. *A transforms bounded sets into relatively compact sets.*

Example 2.

Let $X = Y = C[0,1]$ endowed with the $\|\cdot\|_\infty$ norm, consider the integral operator $A : C[0,1] \to C[0,1]$, where for every $u \in C[0,1]$, define

$$Au(x) := \int_0^1 K(x,y)u(y)\,dy \quad \text{for all} \;\; x \in [0,1],$$

where $K(x,y)$ is continuous on the square $[0,1]^2$. We shall show that A is compact.

Since $K(x,y)$ is continuous on $[0,1]^2$, there exists a constant M such that

$$|K(x,y)| \le M \quad \text{for all} \quad (x,y) \in [0,1]^2.$$

- **Step 1:** *It is clear that A* **is well-defined** *(i.e., $Au \in C[0,1]$ for all $u \in C[0,1]$) since both $K(x,y)$ and $u(y)$ are continuous functions.*

- **Step 2: Show that A is continuous.** *For any $u, v \in X$, we have*

  $$\|Au - Av\|_\infty = \max_{x\in[0,1]} \left| \int_0^1 K(x,y)(u(y)-v(y))\,dy \right| \le \max_{x\in[0,1]} \int_0^1 |K(x,y)(u(y)-v(y))|\,dy \le M\|u-v\|_\infty.$$

  Therefore, for every $\varepsilon > 0$, pick $\delta = \dfrac{\varepsilon}{M}$, then whenever $u, v \in X$ with $\|u - v\|_\infty < \delta$, we have $\|Au - Av\|_\infty < \varepsilon$. Therefore, A is continuous.

  Suppose S is a bounded set of functions of $C[0,1]$. Then there is $r > 0$ such that $\|u\|_\infty \le r$ for all $u \in S$. We will show that $A(S) \subset C[0,1]$ is relatively compact.

- **Step 3: Show that $A(S)$ is bounded.** *For any $u \in S$, we have*

  $$\|Au\|_\infty = \max_{x\in[0,1]} \left| \int_0^1 K(x,y)u(y)\,dy \right| \le Mr,$$

  therefore, $A(S)$ is bounded.

- **Step 4: Show that $A(S)$ is equicontinuous.**
  Since $[0,1]^2$ is compact and K is continuous on $[0,1]^2$, $K(x,y)$ is uniformly continuous. (Prove this!) Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

  $$|K(x_1,y) - K(x_2,y)| < \frac{\varepsilon}{r}, \quad \text{whenever} \;\; |x_1 - x_2| < \delta.$$

  Then for any $x_1, x_2 \in [0,1]$ with $|x_1 - x_2| < \delta$ and for any $u \in S$, we have

  $$|Au(x_1) - Au(x_2)| = \left| \int_0^1 (K(x_1,y) - K(x_2,y))u(y) \right| \le \frac{\varepsilon}{r} r = \varepsilon.$$

  Hence $A(S)$ is equicontinuous. So by the Arzela-Ascoli Theorem, $A(S)$ is a relatively compact set in $C[0,1]$.
  In conclusion, A is a compact operator.

Example 3. *Let X be an infinite dimensional Banach space, such as $(C[0,1], \|\cdot\|_\infty)$ or $(L_1[0,1], \|\cdot\|_1)$. Consider the identity operator $A : X \to X$, $A(x) = x$. A is continuous, $\overline{B(0,1)}$ is bounded but $A(\overline{B(0,1)}) = \overline{B(0,1)}$ is not a relatively compact set in X (Assignment 2). Therefore, the identity is not a compact operator in this case.*

Lecture 09: Schauder Fixed-Point Theorem and Applications to ODEs

Theorem 10 (Approximation Theorem for Compact Operators)**.** *Let $A : S \subset X \to Y$ be a compact operator, where X and Y are Banach spaces over $\mathbb{K}$ and S is a bounded nonempty subset of X. Then for every $n = 1, 2, \ldots$, there exist a finite dimensional subspace Y_n of Y and a continuous operator $A_n : S \to Y_n$ such that*

$$\sup_{u \in S} \|Au - A_n u\| \le \frac{1}{n} \quad \text{and} \quad A_n(S) \subset co(A(S)).$$

Recall: For a set B in a linear space X, $co(B)$ is the convex hull of B, *span* B is the spanning set of B.

The idea of the proof is to use the finite ε-net for the set $A(S)$ in the Banach space Y and use those centers to define the operator A_n as a linear combination of those Au_k. The coefficients are chosen carefully to achieve the approximation of $\dfrac{1}{n}$.

Sketch of the Proof.

- Since A is compact, and S is bounded, $A(S)$ is relatively compact. Using the finite ε-net theorem, for every $n = 1, 2, \ldots$, there exists a finite $\dfrac{1}{2n}$-net for $A(S)$. That is, there are elements $Au_1, \ldots, Au_N \in A(S)$ (i.e., $u_1, \ldots, u_N \in S$) such that

$$\min_{1 \le k \le N} \|Au - Au_k\| \le \frac{1}{2n}, \quad \text{for all } u \in S. \tag{7}$$

- Define the Schauder operator $A_n : S \to Y$,

$$A_n u := \frac{\sum_{k=1}^{N} a_k(u)\, Au_k}{\sum_{k=1}^{N} a_k(u)}, \quad \text{for all } u \in S, \tag{8}$$

where

$$a_k : S \to \mathbb{R}, \quad a_k(u) := \max\left\{\frac{1}{n} - \|Au - Au_k\|, 0\right\}, \quad k = 1, \ldots, N.$$

Claim 1: $A_n : S \to Y$ is well-defined and continuous. First, a_k are nonnegative functions and because of Equation (7), for every $u \in S$, there is $k \in [1, N]$ such that $a_k(u) > 0$. Therefore, A_n is well-defined. For each k, the function a_k is continuous because a_k is the composition of continuous functions:

$$a_k : u \longmapsto (Au - Au_k) \longmapsto \|Au - Au_k\| \longmapsto \frac{1}{n} - \|Au - Au_k\| \longmapsto \max\left\{\frac{1}{n} - \|Au - Au_k\|, 0\right\}.$$

Therefore, A_n is a continuous function on S. From Equation (8), we also have

$$A_n(S) \subset co(Au_1, \ldots, Au_N) \subset Y_n = Span(Au_1, \ldots, Au_N), \quad \dim Y_n < \infty$$

$$A_n(S) \subset co(Au_1, \ldots, Au_N) \subset co(A(S)).$$

Claim 2: Show that $\|Au - A_n u\| \leq \frac{1}{n}$ for any $u \in S$. Indeed, we have

$$\|Au - A_n u\| = \frac{\left\| \sum_{k=1}^{N} a_k(u) \left(Au - Au_k\right) \right\|}{\sum_{k=1}^{N} a_k(u)} \leq \frac{\sum_{k=1}^{N} a_k(u) \|Au - Au_k\|}{\sum_{k=1}^{N} a_k(u)}$$

Due to the construction of a_k, for any $k = 1, 2, \ldots$, we have

$$a_k(u)\|Au - Au_k\| \leq \frac{1}{n} a_k(u).$$

Hence $\|Au - A_n u\| \leq \frac{1}{n}$.

□

2.9 The Brower and Schauder Fixed-Point Theorems

Rephrased from Zeidlers'book: The Brower Fixed-Point Theorem is one of the most important existence principles in mathematics. It has interesting applications to game theory, mathematical economics, and numerical mathematics. Further important existence principles in mathematics are the Hahn-Banach theorem, the Weierstrass existence theorem for minima, and the Baire category theorem. The Schauder Fixed Point Theorem is an extension of the Brower Fixed Point Theorem. We state (without proof) the Brower Fixed-Point Theorem.

Theorem 1 (Brower Fixed Point Theorem - Version 1)**.** *Any continuous map of a closed ball in $\mathbb{R}^n$ into itself must have a fixed point.*

Example 1. *A continuous function $f : [a, b] \to [a, b]$ has a fixed point $x \in [a, b]$.*

Below is another variant of the Brower Fixed-Point Theorem (in Zeidler's book).

Theorem 2 (Brower Fixed Point Theorem - Version 2)**.** *Let $(X, \|\cdot\|)$ be a finite-dimensional normed space and $S \subset X$ is compact, convex, and nonempty. Any continuous operator $A : S \to S$ has at least one fixed point.*

Example 2 (Counter Examples)**.** *The following counter examples show the essentials of each assumption in the Brower Fixed-Point Theorem (version 2).*

- *$S = [0, 1]$ compact, convex and nonempty, but $A : S \to S$ not continuous and the graph $y = A(x)$ does not cross the diagonal $y = x$. No fixed point.*

- *$S = \mathbb{R}$ and $A : S \to S, A(x) = x + 1$. A is continuous, S is convex, nonempty, but not compact. No fixed point.*

- *Let S be a closed annulus and $A : S \to S$ is a rotation of the annulus around the center. A proper rotation is fixed-point free. In this case, S is compact, nonempty but not convex.*

Theorem 3 (Schauder Fixed Point Theorem - Version 1)**.** *Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $S \subset X$ is closed, bounded, convex, and nonempty. Any compact operator $A : S \to S$ has at least one fixed point.*

The idea here is to find a fixed point for each approximation operator. Then using the compactness of the operator A to show that the limit of the convergent subsequence is the fixed point of A.

Proof. From the approximation theorem for compact operators, for every $n = 1, 2, \dots$, there exists a finite dimensional subspace X_n of X and a continuous operator $A_n : S \to X_n$ such that $A_n(S) \subset co(A(S))$ and

$$\|Au - A_n u\| \leq \frac{1}{n} \quad \text{for all} \quad u \in S.$$

Let $S_n = X_n \cap S$.

- **Step 1: Show that $A_n|_{S_n} : S_n \to S_n$ and S_n is a compact and convex set of X.** Therefore, we can apply the Brower fixed point theorem.

  Step 1.1: Show that $A_n|_{S_n} : S_n \to S_n$
  Indeed,

  $$A_n(S) \subset co(A(S)) \subset co(S) \subset S,$$

  where the first inclusion comes from the construction of A_n, the second one is because $A : S \to S$, and the third one is derived from the convexity of S.

  Therefore, $A_n|_{S_n} : S_n \to S_n$.

  Step 1.2: Show that S_n is a compact and convex set of X.

  - S is bounded, so S_n is bounded.
  - Since X_n is a finite dimensional subspace of X, X_n is a closed subset of X. Since the intersection of two closed subsets of X is a closed subset of X, S_n is closed.
  - Since X_n is a finite dimensional space and $S_n \subset X_n$ is closed and bounded, S_n must be a compact set.
  - Since S and X_n are convex, S_n is convex.

  By the Brower fixed-point theorem, the operator $A_n : S_n \to S_n$ has a fixed point u_n, i.e.,

  $$A_n u_n = u_n, \quad u_n \in S_n, \quad \text{for all} \quad n = 1, 2, \dots$$

- **Step 2: Show that $\{u_n\}$ and $\{Au_n\}$ have convergent subsequences and the limit is the fixed-point of A.**

  Since $u_n \in S_n \subset S$ and S is bounded, the sequence $\{u_n\}$ is bounded. Since A is compact, $\{Au_n\}_n$ is relatively compact in X. Therefore, there is a subsequence $\{Au_{n_k}\}_k$ of $\{Au_n\}$ such that

$$\lim_{k\to\infty} Au_{n_k} = v \in X.$$

  Since $Au_{n_k} \in S$ and S is closed, $v \in S$. Moreover,

$$\|v - u_{n_k}\| \le \|v - Au_{n_k}\| + \|Au_{n_k} - u_{n_k}\| = \|v - Au_{n_k}\| + \|Au_{n_k} - A_{n_k}u_{n_k}\| \to 0 \text{ as } k \to \infty.$$

  Hence $u_{n_k} \to v$ as $k \to \infty$. Since A is continuous, $Au_{n_k} \to Av$. Therefore, $Av = v$.

□

Since a continuous operator on a compact set is always a compact operator, the Schauder fixed point theorem - version 1 yields the Schauder fixed point theorem - version 2.

Theorem 4 (Schauder Fixed Point Theorem - Version 2)**.** *Let $(X, \|\cdot\|)$ be a Banach space and $S \subset X$ is compact, convex, and nonempty. Any continuous operator $A : S \to S$ has at least one fixed point.*

2.10 Applications to Ordinary Differential Equations

Theorem 1 (The Peano Theorem)**.** *Given $(x_0, u_0) \in \mathbb{R}^2$, let $F(x, w)$ be a real-valued continuous function on a rectangle*

$$S = \{(x, w) \in \mathbb{R}^2 : \ |x - x_0| \le a \text{ and } |w - u_0| \le b\},$$

Denote $c = \max\limits_{(x,w)\in S} |F(x, w)|$. Then for $0 < h \le a$ and $hc \le b$, the following initial value problem

$$\begin{cases} u' &= F(x, u), \quad x_0 - h \le x \le x_0 + h \\ u(x_0) &= u_0. \end{cases} \tag{9}$$

has at least one solution.

Proof. Denote $X := C[x_0 - h, x_0 + h]$ and $M := \{u \in X : \|u - u_0\|_\infty \le b\}$.
For each $u \in M$, consider the following operator A

$$Au(x) := u_0 + \int_{x_0}^{x} F(y, u(y))dy, \quad \text{for } x \in [x_0 - h, x_0 + h].$$

Similar to the part of the Picard-Lindelöf theorem, we have $A : M \to M$. Next, we will prove that A is continuous and $A(M)$ is bounded and equicontinuous. Since $A(M) \subset M$, the set $A(M)$ is bounded. The continuous of A and the equicontinuous of $A(M)$ come from the following inequality:

$$|Au(x) - Au(z)| = \left| \int_z^x F(y, u(y))\, dy \right| \le c|z - x|.$$

By the Arzela Ascoli Theorem, the set $A(M)$ is relatively compact in X. Since M is bounded, this implies $A : M \to M$ is a compact operator. Moreover, the closed ball M is closed, bounded, convex, and nonempty. By the Schauder fixed point theorem, the equation

$$Au = u, u \in M$$

has a solution $u_* \in M$. Differentiating the integral equation with respect to x, we see that u_* is also a solution of the IVP (9). □

Lecture 10: Bounded Linear Operators

2.11 Bounded Linear Operator

Recall: Let X and Y be linear spaces over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The operator $L : X \to Y$ is called linear if for every $u, v \in X$ and $\alpha, \beta \in \mathbb{K}$, we have

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv.$$

Definition 1. *Let X and Y be normed linear spaces. A linear operator $L : X \to Y$ is called a bounded linear operator if there exists a positive constant $c > 0$ such that*

$$\|Lx\|_Y \leq c\|x\|_X, \quad \text{for all } x \in X.$$

Note: We often write $\|x\|$ and $\|Lx\|$ instead of $\|x\|_X$ and $\|Lx\|_Y$.

Proposition 13. *Let $L : X \to Y$ be a linear operator where X and Y are normed spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Then the following statements are equivalent:*

1. *L is continuous at 0.*
2. *L is continuous on X.*
3. *There is a number $c > 0$ such that $\|Lx\| \leq c$ for all $x \in X$ with $\|x\| \leq 1$.*
4. *There is a number $c > 0$ such that $\|Lx\| \leq c\|x\|$ for all $x \in X$.*

Proof. ($1 \Rightarrow 2$) Let $x \in X$ and suppose $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = x$. Then $\lim_{n\to\infty}(x_n - x) = 0$. Since L is continuous at 0, we have

$$\lim_{n\to\infty} L(x_n - x) = L(0).$$

Since L is linear, $L(0) = 0$ and $L(x_n - x) = L(x_n) - L(x)$, for all $n \in \mathbb{N}$. Therefore,

$$0 = L(0) = \lim_{n\to\infty} L(x_n - x) = \lim_{n\to\infty}(L(x_n) - L(x)) = \lim_{n\to\infty} L(x_n) - L(x).$$

Hence $\lim_{n\to\infty} L(x_n) = L(x)$, which means L is continuous at $x \in X$, for any $x \in X$. That completes the proof.

($2 \Rightarrow 3$) Suppose (3) is not true. Then there exists a sequence $\{x_n\} \subset X$ such that

$$\|x_n\| \leq 1 \quad \text{and} \quad \|L(x_n)\| \geq n, \quad \text{for all } n = 1, 2, \ldots.$$

Let $w_n = n^{-1}x_n$, then

$$\|w_n\| \leq \frac{1}{n} \quad \text{and} \quad \|Lw_n\| = \left\|L\left(n^{-1}x_n\right)\right\| = n^{-1}\|L(x_n)\| \geq 1 \quad \text{for all } n = 1, 2, \ldots$$

So $\lim_{n\to\infty} \|w_n\| = 0$ and $\lim_{n\to\infty} w_n = 0$. Since L is continuous at 0, we have $\lim_{n\to\infty} L(w_n) = L(0) = 0$, a contradiction with $\|Lw_n\| \geq 1$.

($3 \to 4$) If $x = 0$, then $\|L(0)\| = 0 \leq c\|0\|$.
If $x \neq 0$, let $z = \dfrac{x}{\|x\|}$. Then $\|z\| = 1$, so $c \geq \|Lz\| = \dfrac{\|Lx\|}{\|x\|}$. Therefore, $c\|x\| \geq \|Lx\|$.
In both cases, we have $\|Lx\| \leq c\|x\|$, for all $x \in X$.

($4 \to 1$) Given $\varepsilon > 0$. Choose $\delta = \varepsilon/c$. Then when $x \in X$ with $\|x| < \delta$, we have

$$\|Lx\| \leq c\|x\| < c\delta < \varepsilon.$$

□

So for linear operators between normed linear spaces, boundedness is equivalent to continuity.

Definition 2. *For a bounded linear operator $L : X \to Y$ where X and Y are normed linear spaces, define the operator norm*

$$\|L\| := \sup_{v\in X, \|v\|\leq 1} \|Lv\| < \infty$$

Proposition 14. *Let $L : X \to Y$ be a bounded linear operator where X and Y are normed linear spaces. Then*

1. $\|Lu\| \leq \|L\|\,\|u\|$, *for all $u \in X$.*
2. *If there is a constant $C > 0$ such that $\|Lu\| \leq C\|u\|$ for all $u \in X$, then $\|L\| \leq C$.*
3. *If $X \neq \{0\}$, then*
$$\|L\| = \sup_{v\in X, \|v\|\leq 1} \|Lv\| = \sup_{v\in X, \|v\|=1} \|Lv\| = \sup_{v\in X, v\neq 0} \frac{\|Lv\|}{\|v\|}.$$

Proposition 15 (Bounded Linear Operators Between Finite Dimensional Normed Spaces)**.** *Let X and Y be finite-dimensional normed spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) with $\dim X = N$ and $\dim Y = M$ where $N, M \geq 1$. Then any linear operator $L : X \to Y$ is bounded.*

Sketch of the Proof. Let $\{e_1, \ldots, e_N\}$ and $\{f_1, \ldots, f_M\}$ be a basis in X and Y, respectively. Suppose

$$L(e_n) = \sum_{m=1}^{M} a_{mn} f_m, \quad n = 1, \ldots, N.$$

Any $x \in X$ can be written as $x = \sum\limits_{n=1}^{N} c_n e_n$, for some $c_1, \ldots, c_N \in \mathbb{K}$. Then

$$L\left(\sum_{n=1}^{N} c_n e_n\right) = \sum_{n=1}^{N} c_n L(e_n) = \sum_{n=1}^{N} c_n \sum_{m=1}^{M} a_{mn} f_m = \sum_{m=1}^{M} \left(\sum_{n=1}^{N} a_{mn} c_n\right) f_m$$

Recall that we have proved in previous lectures that

$$\left\|\sum_{n=1}^{N} c_n e_n\right\|_\infty := \max_{1\le n\le N} |c_n|$$

is a norm on the finite dimensional normed space X.

- Show that $\|Lx\|_\infty \le \|x\|_\infty \max\limits_{1\le m\le M} \sum\limits_{n=1}^{N} |a_{mn}|$.
- Using the property that any two norms on a finite dimensional normed linear spaces are equivalent, show that there is a constant $C > 0$ such that $\|Lx\| \le C\|x\|$ for all $x \in X$.

□

Example 1. *Consider a linear operator $L : \mathbb{R}^N \to \mathbb{R}^M$, $L(x) := Ax$ (matrix multiplication), where A is a matrix of real entries of size $M \times N$.*

1. *If we use the $\|\cdot\|_\infty$ norm for both $\mathbb{R}^N$ and $\mathbb{R}^M$, then $\|L\| = \max\limits_{1\le m\le M} \sum\limits_{n=1}^{N} |a_{mn}|$.*
2. *If we use the $\|\cdot\|_1$ norm for both $\mathbb{R}^N$ and $\mathbb{R}^M$, then $\|L\| = \max\limits_{1\le n\le N} \sum\limits_{m=1}^{M} |a_{mn}|$.*
3. *If we use the $\|\cdot\|_2$ norm for both $\mathbb{R}^N$ and $\mathbb{R}^M$, then $\|L\| = \sqrt{\rho(A^T A)}$, where $\rho(B)$ is the maximum of the magnitude of the eigenvalues of the square matrix B.*

Proof. (1). For any $x \in \mathbb{R}^N$, then for any $1 \le m \le M$, we have

$$|(Lx)_m| = \left|\sum_{n=1}^{N} a_{mn} x_n\right| \le \sum_{n=1}^{N} |a_{mn}||x_n| \le \|x\|_\infty \sum_{n=1}^{N} |a_{mn}| \le \|x\|_\infty \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|.$$

Therefore,

$$\|Lx\|_\infty = \max_{1\le m\le M} |(Lx)_m| \le \|x\|_\infty \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|.$$

Therefore, $\|L\| \le \max\limits_{1\le m\le M} \sum\limits_{n=1}^{N} |a_{mn}|$.

Next, we will prove that there exists $\hat{x} \in \mathbb{R}^N$ with $\|x\|_\infty = 1$ such that $\|L\hat{x}\|_\infty \ge \max\limits_{1\le m\le M} \sum\limits_{n=1}^{N} |a_{mn}|$. Then

$$\|L\| = \sup_{\|z\|=1} \|Lz\|_\infty \ge \|L\hat{x}\|_\infty \ge \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$$

Therefore, $\|L\| = \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$.

It remains to construct such $\hat{x}$. Suppose $\max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}| = \sum_{n=1}^{N} |a_{m_0 n}|$ for some $1 \le m_0 \le M$. Let

$$\hat{x}_n = \begin{cases} 1 & \text{if } a_{m_0,n} \ge 0 \\ -1 & \text{if } \ a_{m_0,n} < 0 \end{cases}.$$

Then $\|\hat{x}\|_\infty = 1$ and

$$\|L\hat{x}\|_\infty = \max_{1\le m\le M} |(L\hat{x})_m| \ge (L\hat{x})_{m_0} = \sum_{n=1}^{N} a_{m_0 n}\hat{x}_n = \sum_{n=1}^{N} |a_{m_0 n}| = \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|,$$

which completes the proof.

(2) & (3). Assignment 3 □

Lecture 11: Bounded Linear Operator (cont'd). $B(X, Y)$. Dual Spaces.

Example 2. *Let $X = C[a,b]$ with $\|\cdot\|_\infty$, where $-\infty < a < b < \infty$ and $K : [a,b]\times[a,b] \to \mathbb{R}$ be continuous. For each $u \in X$, define the integral operator*

$$Tu(x) := \int_a^b K(x,y)u(y)\,dy \quad \textit{for all} \ \ x \in [a,b].$$

From previous lectures, $T : C[a,b] \to C[a,b]$ is a continuous and a compact operator.
Moreover, T is linear (prove this!) and

$$\|T\| = \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Sketch of the proof. We will compute the operator norm of T.

- Step 1: Show that

$$\|T\| \le \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Let $u \in C[a,b]$ and $x \in [a,b]$. Then

$$|Tu(x)| = \left|\int_a^b K(x,y)u(y)\,dy\right| \le \int_a^b |K(x,y)|\,|(u(y)|\,dy \le \|u\|_\infty \int_a^b |K(x,y)|dy \le \|u\|_\infty \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

So

$$\|Tu\|_\infty = \max_{a\le x\le b} |Tu(x)| \le \|u\|_\infty \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Therefore

$$\|T\| \leq \max_{a\leq x\leq b} \int_a^b |K(x,y)|dy.$$

Suppose $\max\limits_{a\leq x\leq b} \int_a^b |K(x,y)|dy = \int_a^b |K(x_0,y)|dy$ for some $x_0 \in [a,b]$. Since $K(x_0,y)$ is continuous on a compact set $[a,b]$, $K(x_0,y)$ is uniformly continuous on $[a,b]$. Therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|K(x_0,y_2) - K(x_0,y_1| \leq \varepsilon \quad \text{for all} \quad |y_1 - y_2| < \delta, \quad y_1, y_2 \in [a,b].$$

- Step 2: Now we will construct an $u_\varepsilon \in C[a,b]$ with $\|u_\varepsilon\|_\infty \leq 1$ such that

$$Tu_\varepsilon(x_0) \geq \max_{a\leq x\leq b} \int_a^b |K(x,y)|dy - 4\varepsilon.$$

Let $A_\varepsilon = \{y \in [a,b] : |K(x_0,y)| \leq \varepsilon\}$. Then A_ε is a closed and bounded subset in $\mathbb{R}$ (prove this). Therefore, there exists $y_1, \ldots, y_N \in A_\varepsilon$ such that

$$A_\varepsilon \subset \bigcup_{i=1}^{N} [y_i - \delta, y_i + \delta].$$

Let $V_\varepsilon = \left(\bigcup\limits_{i=1}^{N} [y_i - \delta, y_i + \delta]\right) \cap [a,b]$ and $U_\varepsilon = [a,b] - V_\varepsilon$. Define a function on U_ε,

$$u_\varepsilon : U_\varepsilon \to \mathbb{R}, \quad u_\varepsilon(y) := \frac{K(x_0,y)}{|K(x_0,y)|}.$$

The function is well-defined and continuous since $|K(x_0,y)| > \varepsilon$ for all $y \in U_\varepsilon$ and $K(x_0,y)$ is continuous on $[a,b]$. Moreover, $|u_\varepsilon(y)| = 1$ for all $y \in U_\varepsilon$. Extend u_ε linearly, $u_\varepsilon : [a,b] \to \mathbb{R}$ so that $|u_\varepsilon(y)| \leq 1$ for all $y \in [a,b]$.

Next, we will evaluate $\int_{V_\varepsilon} K(x_0,y)u_\varepsilon(y)dy$. For each $y \in V_\varepsilon$, we have $|y - y_i| \leq \delta$ for some $y_i \in A_\varepsilon$, $i \in \{1, \ldots, N\}$. Therefore,

$$|K(x_0,y)| \leq |K(x_0,y_i)| + |K(x_0,y) - |K(x_0,y_i)|| \leq 2\varepsilon, \quad \text{for all } y \in V_\varepsilon,$$

and

$$\int_{V_\varepsilon} |K(x_0,y)|dy \leq \int_{V_\varepsilon} 2\varepsilon dy \leq 2(b-a)\varepsilon.$$

Also, since $|u_\varepsilon(y)| \leq 1$ for all $y \in V_\varepsilon$, we have

$$K(x_0,y)u_\varepsilon(y) \geq -|K(x_0,y)|\,|u_\varepsilon(y)| = -|K(x_0,y)u_\varepsilon(y)| \geq -|K(x_0,y)|, \quad \text{for all} \quad y \in V_\varepsilon.$$

Therefore,

$$\begin{aligned}
Tu_\varepsilon(x_0) &= \int_{U_\varepsilon} K(x_0,y)u_\varepsilon(y)dy + \int_{V_\varepsilon} K(x_0,y)u_\varepsilon(y)dy \\
&= \int_{U_\varepsilon} |K(x_0,y)|dy + \int_{V_\varepsilon} K(x_0,y)u_\varepsilon(y)dy \\
&\geq \int_{U_\varepsilon} |K(x_0,y)|dy - \int_{V_\varepsilon} |K(x_0,y)|dy \\
&\geq \int_a^b |K(x_0,y)|dy - 2\int_{V_\varepsilon} |K(x_0,y)|dy \\
&\geq \int_a^b |K(x_0,y)|dy - 4(b-a)\varepsilon \\
&\geq \max_{a\leq x\leq b} \int\limits_a^b |K(x,y)|dy - 4(b-a)\varepsilon.
\end{aligned}$$

Then

$$\|T\| = \sup_{u\in C[a,b],\|u\|_\infty\leq 1} \|Tu\|_\infty \geq \|Tu_\varepsilon\|_\infty = \max_{x\in[a,b]} |Tu(x)| \geq Tu_\varepsilon(x_0) \geq \max_{a\leq x\leq b} \int\limits_a^b |K(x,y)|dy - 4(b-a)\varepsilon.$$

Let $\varepsilon \to 0$, we have $\|T\| \geq \int\limits_a^b |K(x,y)|dy$, which completes the proof.

□

Example 3. *Here we will show an example of a discontinuous linear operator (hence the operator is not bounded).*

Consider the differentiation operator $D = \frac{d}{dt} : X = C^1[0,1] \to Y = C[0,1]$, *where* $\|\cdot\|_\infty$ *are used for both spaces. The operator D is not continuous at 0. Here is a counter example. Consider a sequence* $\{f_n(t) = \frac{1}{n}\sin n\pi t\}_n \subset X$. *Then* $\|f_n\|_\infty = \frac{1}{n}$. *So* $\lim_{n\to\infty} \|f_n\|_\infty = 0$. *Therefore* $\lim_{n\to\infty} f_n = 0$. *On the other hand,* $Df_n = f_n' = \pi\cos n\pi t$. *So* $\|Df_n\| = \pi$ *for all* $n = 1, 2, \ldots$, *which means* $Df_n \not\to 0$.

Note: The differentiation operator is continuous (prove this) when Y is equipped with the $\|\cdot\|_\infty$ *norm and X is equipped with the following norm*

$$\|f\| := \max\{\|f\|_\infty,\ \|f'\|_\infty\}.$$

2.12 B(X,Y) and Dual Spaces

Definition 1. *Let X and Y be normed linear spaces. Define the following set*

$$B(X,Y) := \{L : X \to Y \quad \text{bounded linear operator}\}.$$

Denote $X^* = B(X,\mathbb{R})$ *(the dual space of X) and* $B(X) = B(X,X)$.

Theorem 1. *The set* $B(X,Y)$ *is a normed linear space with the operator norm.*

Proof. Exercise. □

Proposition 16. *Let X, Y and Z be normed linear spaces. If $T \in B(X,Y)$ and $S \in B(Y,Z)$ then $ST \in B(X,Z)$ and $\|ST\| \leq \|S\|\|T\|$.*

Proof. For each $x \in X$, we have

$$\|ST(x)\| \leq \|S\|\|T(x)\| \leq \|S\|\|T\|\|x\|.$$

Therefore, ST is bounded and

$$\|ST\| = \sup_{\|x\|=1} \|ST(x)\| \leq \sup_{\|x\|=1} \|S\|\|T\|\|x\| = \|S\|\|T\|.$$

□

Corollary 2. *Let X be a normed linear space. If $T \in B(X)$, then $T^n \in B(X)$ and $\|T^n\| \leq \|T\|^n$ for all $n = 1, 2, \ldots$.*

Definition 2 (Convergence in Operator Norm)**.** *Let X and Y be normed linear spaces. A sequence $\{T_n\} \subset B(X,Y)$ is said to converge in operator norm to $T \in B(X,Y)$ if $\|T_n - T\| \to 0$ as $n \to 0$.*

Proposition 17. *Let X, Y and Z be normed linear spaces. If $T_n, T \in B(X,Y)$ and $S_n, S \in B(Y,Z)$ with $T_n \to T$ and $S_n \to S$ as $n \to \infty$, then $S_n T_n \to ST \in B(X,Z)$.*

Proof. We have

$$\|S_nT_n - ST\| \leq \|S_nT_n - S_nT\| + \|S_nT - ST\| \leq \|S_n\|\|T_n - T\| + \|S_n - S\|\|T\|.$$

Since $\lim_{n\to\infty} S_n = S$ and the norm is a continuous function, $\lim_{n\to\infty} \|S_n\| = \|S\|$. We also have $\lim_{n\to\infty} \|S_n - S\| = 0$ and $\lim_{n\to\infty} \|T_n - T\| = 0$. Therefore,

$$0 \leq \lim_{n\to\infty} \|S_nT_n - ST\| \leq \|S\|0 + 0\|T\| = 0,$$

So $\lim_{n\to\infty} \|S_nT_n - ST\| = 0$ and $\lim_{n\to\infty} S_nT_n = ST$. □

Theorem 2. *Let X be a normed linear space and Y be a Banach space. Then $B(X,Y)$ is a Banach space. In particular, X^* is a Banach space.*

Proof. Let $\{T_n\} \subset B(X,Y)$ be a Cauchy sequence in $B(X,Y)$. Given $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $\|T_n - T_m\| < \varepsilon$ for all $n, m \geq N_\varepsilon$.

- Step 1: Construct the limit pointwise. Indeed, for each $x \in X$ and $n, m > N_\varepsilon$, we have

$$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\|x\| < \varepsilon\|x\|. \tag{10}$$

  Therefore, for each $x \in X$, the sequence $\{T_n(x)\}_n$ is a Cauchy sequence in Y. Since Y is Banach, the sequence $\{T_n(x)\}_n$ converges. Denote $T(x) := \lim_{n\to\infty} T_n(x)$. We have defined a function $T : X \to Y$ such that for each $x \in X$, $T(x) := \lim_{n\to\infty} T_n(x)$.

- Step 2: Show that T is linear. Indeed, let $c_1, c_2 \in \mathbb{K}$ and $x_1, x_2 \in X$. For each $n = 1, 2, \ldots$, T_n is linear, so

$$T_n(c_1x_1 + c_2x_2) = c_1T_n(x_1) + c_2T_n(x_2).$$

  Letting $n \to \infty$, we have

$$T(c_1x_1 + c_2x_2) = c_1T(x_1) + c_2T(x_2),$$

  because of the construction of T.

- Step 3: We will show that $T_n - T \in B(X, Y)$ for all n sufficiently large, $T \in B(X, Y)$, and $\|T_n - T\| \to 0$ as $n \to \infty$. From (10), letting $m \to \infty$ and keeping everything else, we get

$$\|T_n(x) - T(x)\| \le \varepsilon\|x\|, \quad \text{for all } \ x \in X \quad \text{and for all } n \ge N_\varepsilon.$$

  Therefore for every $n \ge N_\varepsilon$, $T_n - T \in B(X, Y)$ and $\|T_n - T\| \le \varepsilon$. Therefore,

$$T = T_{N_\varepsilon} - (T_{N_\varepsilon} - T) \in B(X, Y),$$

  and $T_n \to T$ as $n \to \infty$.

$\square$

Lecture 12: Infinite Series. Operator Functions. Neumann Series.

Definition 3. *Let X be a normed space over $\mathbb{K}$ and let $u_j \in X$ for all j. If $\lim\limits_{m\to\infty} \sum\limits_{j=0}^{m} u_j$ exists, denote*

$$X \ni \sum_{j=0}^{\infty} u_j := \lim_{m\to\infty} \sum_{j=0}^{m} u_j,$$

and the infinite series $\sum\limits_{j=0}^{\infty} u_j$ is called convergent. This infinite series is called absolutely convergent iff

$$\sum_{j=0}^{\infty} \|u_j\| < \infty.$$

Proposition 18. *A normed linear space X is a Banach space if and only if every absolutely convergent infinite series with terms in X is convergent.*

Proof. ($\Rightarrow$) Suppose X is a Banach space. Let $\sum\limits_{j=0}^{\infty} u_j$ be an absolutely convergent infinite series in X. Then for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that for every $n > N_\varepsilon$, $k \ge 0$, we have

$$\sum_{j=n+1}^{n+k} \|u_j\| < \varepsilon.$$

Denote $s_m = \sum\limits_{j=0}^{m} u_j \in X$, then for every $n > N_\varepsilon$, $k \ge 0$, we have

$$\|s_{n+k} - s_n\| = \|\sum_{j=n+1}^{n+k} u_j\| \le \sum_{j=n+1}^{n+k} \|u_j\| < \varepsilon.$$

Hence the sequence $\{s_n\}$ is a Cauchy sequence in X. Since X is a Banach space, the limit $\lim_{n\to\infty} s_n$ exists. Therefore, the infinite series $\sum_{j=0}^{\infty} u_j$ converges.

($\Leftarrow$) Suppose every absolutely convergent infinite series with terms in X is convergent. We need to prove that X is a Banach space. Here is the sketch of the proof. Let $\{x_n\} \subset X$ be a Cauchy sequence.

- Construct a subsequence $\{x_{n_k}\}$ so that $\|x_{n_k} - x_{n_{k-1}}\| \le \frac{1}{2^k}$ for all $k \ge 1$.
- Prove that the series $\left(x_{n_0} + \sum_{k=0}^{\infty}(x_{n_{k+1}} - x_{n_k})\right)$ is absolutely convergent, hence it is convergent.
- Therefore $\lim_{m\to\infty} x_{n_{m+1}} = \lim_{m\to\infty}\left(x_{n_1} + \sum_{k=0}^{m}(x_{n_{k+1}} - x_{n_k})\right)$ exists. Denote $x = \lim_{m\to\infty} x_{n_m}$.
- Combining with the assumption that the sequence $\{x_n\} \subset X$ is a Cauchy sequence, prove that $\lim_{j\to\infty} x_j = x$.

□

Theorem 3 (Theorem and Definition)**.** *Let X be a Banach space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and a series*

$$F(z) := \sum_{j=0}^{\infty} a_j z^j, \ z \in \mathbb{K}, \quad a_j \in \mathbb{K} \text{ for all } j$$

such that

$$\sum_{j=0}^{\infty} |a_j|\,|z|^j < \infty \quad \text{for all } \ z \in \mathbb{C} \text{ with } \ |z| < r \quad \text{and some fixed } r > 0.$$

Then for each $A \in B(X)$ with $\|A\| < r$, the series $\sum_{j=0}^{\infty} a_j A^j$ is also an element in $B(X)$.

Proof. Let $A \in B(X)$ with $\|A\| < r$. From the assumption on the series, we have $\sum_{j=0}^{\infty} |a_j|\,\|A\|^j < \infty$. For every $j \ge 1$, we have

$$\|a_j A^j\| \le |a_j| \|A\|^j.$$

Therefore, by the comparison test, the series $\sum_{j=0}^{\infty} a_j A^j$ is absolutely convergent. Since $B(X)$ is a Banach space, the series $\sum_{j=0}^{\infty} a_j A^j$ is a convergent series. That is $\sum_{j=0}^{\infty} a_j A^j \in B(X)$. □

Definition 4. *Let $A : X \to Y$ and $B : Y \to X$ be linear operators, where X and Y are linear vector spaces over $\mathbb{K}$. If $AB = I_Y$ and $BA = I_X$, A is said to be bijective and denote $A^{-1} = B$.*

Example 1. *Let $X \neq \{0\}$ be a Banach space over $\mathbb{K}$.*

1. *Exponential Function. For each $A \in B(X)$, the infinite series $\sum_{j=0}^{\infty} \frac{1}{j!} A^j$ is also an element in $B(X)$. Denote*

$$e^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j \in B(X).$$

Moreover, for all $t, s \in \mathbb{K}$, we have

$$e^{tA}\, e^{sA} = e^{(t+s)A}.$$

2. *Neumann Series. Let $A \in B(X)$ with $\|A\| < 1$. Then the following statements hold.*
   - *(a) The infinite series $\sum_{j=0}^{\infty} A^j$ is also an element in $B(X)$. The series $\sum_{j=0}^{\infty} A^j$ is called the Neumann series.*
   - *(b) The operator $(I - A) \in B(X)$ is bijective and $(I-A)^{-1} = \sum_{j=0}^{\infty} A^j$.*
   - *(c) $\|(I-A)^{-1}\| \le \dfrac{1}{1 - \|A\|}$.*
   - *(d) Given $g \in X$, the equation $(I - A)u = g$ with the unknown $u \in X$ has a unique solution*

     $$u = (I-A)^{-1} g = \sum_{j=0}^{\infty} A^j g.$$

     Moreover, u can be approximated by

     $$u_n = g + Ag + A^2 g + \ldots + A^{n-1} g$$

     with the error

     $$\|u - u_n\| \le \frac{\|A\|^n}{1 - \|A\|} \|g\|, \quad \textit{for all} \;\; n = 1, 2, \ldots$$

Proof. Hint: Using Theorem 3 for $F(z) = \sum_{j=0}^{\infty} \frac{1}{j!} z^j = e^z$ and $F(z) = \sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$, respectively.

For Example 2, let's verify $\sum_{j=0}^{\infty} A^j = (I-A)^{-1}$ and $\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}$. Obviously,

$$(I - A)\left(\sum_{j=0}^{\infty} A^j\right) = I \quad \text{and} \quad \left(\sum_{j=0}^{\infty} A^j\right)(I-A) = I.$$

Hence $\sum_{j=0}^{\infty} A^j = (I-A)^{-1}$. From the inequality

$$\left\|\sum_{j=0}^{m} A^j\right\| \le \sum_{j=0}^{m} \|A\|^j \le \sum_{j=0}^{\infty} \|A\|^j = \frac{1}{1 - \|A\|},$$

letting $m \to \infty$, we have

$$\|(I-A)^{-1}\| = \left\|\sum_{j=0}^{\infty} A^j\right\| \le \frac{1}{1 - \|A\|}.$$

(d). Suppose u is a solution for $(I - A)u = g$. Then

$$(I-A)^{-1} g = (I-A)^{-1}(I-A)u = Iu = u.$$

Finally, since

$$u - u_n = A^n g + A^{n+1} g + \cdots = A^n (I + A + \cdots) g = A^n (I-A)^{-1} g,$$

we have

$$\|u - u_n\| \leq \|A^n\|\|(I - A)^{-1}\|\|g\| \leq \frac{\|A\|^n}{1 - \|A\|}\|g\|.$$

Note: u_n is the iteration generated from the Banach fixed point theorem for $u = Tu$ with $Tu = Au + g$ and the Lipschitz constant is $\|A\|$:

$$\|Tu - Tv\| = \|Au - Av\| = \|A(u - v)\| \leq \|A\|\|u - v\|.$$

□

2.13 Fréchet Derivative

Definition 1 (Definition and Theorem)**.** *Let X and Y be normed linear spaces over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). An operator $F : X \to Y$ is Fréchet differentiable (F-differentiable) at $a \in X$ if and only if there exists a bounded linear operator $DF(a) : X \to Y$ such that*

$$\lim_{h \to 0} \frac{\|F(a + h) - F(a) - DF(a)(h)\|}{\|h\|} = 0. \tag{11}$$

An operator $DF(a)$ (if exists, i.e., $DF(a) \in B(X, Y)$ and $DF(a)$ satisfies Equation (11)) is unique and is called the Fréchet-derivative of F at a.

Proof. Suppose $F : X \to Y$ is Fréchet differentiable at $a \in X$ and there are two bounded linear operators $L_1, L_2 : X \to Y$ such that

$$\lim_{h \to 0} \frac{\|F(a + h) - F(a) - L_1(h)\|}{\|h\|} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\|F(a + h) - F(a) - L_2(h)\|}{\|h\|} = 0.$$

Combining with

$$0 \leq \frac{\|L_1(h) - L_2(h)\|}{\|h\|} \leq \frac{\|F(a + h) - F(a) - L_2(h)\| + \| - (F(a + h) - F(a) - L_1(h))\|}{\|h\|},$$

we have

$$\lim_{h \to 0} \frac{\|L_1(h) - L_2(h)\|}{\|h\|} = 0.$$

Let $L = L_1 - L_2 \in B(X, Y)$, then $\lim_{h \to 0} \frac{\|L(h)\|}{\|h\|} = 0$. We will show that $L(x) = 0$ for all $x \in X$. Since L is linear, $L(0) = 0$. Fix $x \in X, x \neq 0$. For $t \in \mathbb{K}$, if $t \to 0$ then $tx \in X$ and $tx \to 0$. Therefore,

$$0 = \lim_{t \to 0} \frac{\|L(tx)\|}{\|tx\|} = \lim_{t \to 0} \frac{|t|\|L(x)\|}{|t|\|x\|} = \frac{\|L(x)\|}{\|x\|}.$$

The second equality holds because L is linear. Therefore, $\|L(x)\| = 0\|x\| = 0$, so $L(x) = 0$ for all $x \in X \setminus \{0\}$. Hence $L(x) = x$ for all $x \in X$. In other words, an operator $DF(a)$ (if exists) is unique. □

Example 1. *1. If $F \in B(X, Y)$ then F is F-differentiable everywhere and $DF(a) = F$ for all $a \in X$.*

2. *Let $F : \mathbb{R}^n \to \mathbb{R}$ and suppose $F \in C^1(\mathbb{R}^n)$ (i.e., $\partial_i f$ exists and is continuous on $\mathbb{R}^n$, $1 \le i \le n$). Then $DF(a) \in B(\mathbb{R}^n, \mathbb{R})$ is defined by*

$$DF(a)(h) := \nabla f(a) \cdot h. \quad \textit{(dot product)}$$

3. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and suppose $F \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ (i.e., $\dfrac{\partial f_j}{x_i}$ exists and is continuous, $1 \le i \le n$, $1 \le j \le m$). Then F is F-differentiable and*

$$DF(a)(h) := \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_n}(a) \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(a) & \cdots & \dfrac{\partial f_m}{\partial x_n}(a) \end{bmatrix} h. \quad \textit{(matrix multiplication)}$$

The matrix itself is the usual Jacobian matrix.

Lecture 13: Fréchet Derivative. Hahn-Banach Theorems and Applications.

Recall: An operator $F : X \to Y$ (between normed linear spaces X and Y) is Fréchet differentiable at $a \in X$ if and only if there exists a bounded linear operator $DF(a) : X \to Y$ such that

$$\lim_{h \to 0} \frac{\|F(a+h) - F(a) - DF(a)h\|}{\|h\|} = 0.$$

If F is Fréchet differentiable, we can write

$$F(a+h) = F(a) + DF(a)h + R(a,h), \quad \text{where} \quad \lim_{h \to 0} \frac{\|R(a,h)\|}{\|h\|} = 0,$$

or

$$F(a+h) = F(a) + DF(a)h + \|h\| r(h), \quad \text{where} \quad \lim_{h \to 0} r(h) = 0,$$

or

$$F(a+h) = F(a) + DF(a)h + o(\|h\|).$$

Remark 2. 1. *The Fréchet derivative is a generalization of derivative in $\mathbb{R}$. That is, if $F : \mathbb{R} \to \mathbb{R}$ is Fréchet differentiable at $a \in \mathbb{R}$, then F is differentiable at a and*

$$DF(a)(x) = F'(a)x, \quad \forall\, x \in \mathbb{R}.$$

Proof. Since F is Fréchet differentiable at $a \in \mathbb{R}$, there exists $DF(a) \in B(\mathbb{R}, \mathbb{R})$ such that

$$0 = \lim_{h \to 0} \frac{|F(a+h) - F(a) - DF(a)(h)|}{|h|}$$

Since

$$B(\mathbb{R}, \mathbb{R}) = \{L : \mathbb{R} \to \mathbb{R} \ s.t.\ L(x) = cx,\ \forall x \in \mathbb{R}, \text{ for some } \ c \in \mathbb{R}\},$$

there exists some constant $c \in \mathbb{R}$ such that

$$DF(a)(x) = cx, \forall\, x \in \mathbb{R}.$$

So

$$0 = \lim_{h\to 0} \frac{|F(a+h) - F(a) - DF(a)(h)|}{|h|} = \lim_{h\to 0} \frac{|F(a+h) - F(a) - ch|}{|h|} = \lim_{h\to 0} \left| \frac{F(a+h) - F(a)}{h} - c \right|.$$

So

$$c = \lim_{h\to 0} \frac{F(a+h) - F(a)}{h},$$

which implies F is differentiable at a and $c = F'(a)$. Therefore,

$$DF(a) : \mathbb{R} \to \mathbb{R}, \quad DF(a)(x) = F'(a)x, \quad \forall\, x \in \mathbb{R}.$$

□

2. *To compute the Fréchet-derivative of an operator $F : X \to Y$ at $a \in X$, where X and Y are normed linear spaces, we write $F(a+h) - F(a)$ as a summation of a linear operator (w.r.t. h) and a remainder (which is nonlinear in h)*

$$F(a+h) - F(a) = Lh + R(a,h),$$

and prove that L is bounded and

$$\lim_{h\to 0} \frac{\|R(a,h)\|}{\|h\|} = 0.$$

The linear operator $L \in B(X,Y)$ is the $DF(a)$ in the definition.

3. *If $F : X \to Y$ is Fréchet-derivative at $a \in X$, where X and Y are normed linear spaces, then for any $x \in X$, we have*

$$DF(a)(x) = \lim_{t\to 0} \frac{F(a+tx) - F(a)}{t}, \quad t \in \mathbb{R}. \quad \textit{(Prove this)}$$

Note: This formula is used to compute the Fréchet-derivative of an operator F. After this, we need to check that $DF(a)$ is a bounded linear operator and $R(a,h) = F(a+h) - F(a) - DF(a)h$ satisfies

$$\lim_{h\to 0} \frac{\|R(a,h)\|}{\|h\|} = 0.$$

Here is an example of an operator $F : X \to Y$, where $\lim_{t\to 0} \frac{F(a+tx) - F(a)}{t}$, $t \in \mathbb{R}$ exists for an $a \in X$ but F is not Fréchet differentiable at a. Consider $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_2 = 0, \\ \dfrac{x_1^3}{x_2} & \text{if} \quad x_2 \neq 0. \end{cases}$$

The operator F is not continuous at $(0,0)$, for example, $F(t,t^3) \to 1$ as $t \to 0$, but $F(0,0) = 0$. On the other hand,

$$\frac{F(tx) - F(0,0)}{t} = \begin{cases} 0 & \text{if} \quad x_2 = 0, \\ t\dfrac{x_1^3}{x_2} & \text{if} \quad x_2 \neq 0. \end{cases}$$

So $\lim_{t\to 0} \dfrac{F(tx) - F(0,0)}{t} = 0$ *for any* $(x_1, x_2) \in \mathbb{R}^2$.

Example 2. *1. Let $X = C_0^1[0,1]$ be the space of all C^1 functions on $[0,1]$ which vanish at the endpoints with norm*

$$\|u\| = \left[\int_0^1 \left[u^2 + (u')^2 \right] dx \right]^{1/2}.$$

Consider an operator $K : X \to \mathbb{R}$ defined by

$$K(u) = \int_0^1 \left[u^3 + (u')^2 \right] dx.$$

Compute the Fréchet derivative of K.

2. Let $X = C[a,b]$ with $\|\cdot\|_\infty$ norm. Let $T : X \to X$ be the nonlinear integral operator defined by

$$(Tu)(x) = u(x) \int_a^b K(x,s) u(s)\, ds,$$

where $K(x,s)$ is continuous on $[a,b] \times [a,b]$. Compute the Fréchet derivative of K.

Proof. Exercise and See Dr. Vrscay's notes (attached in the next pages). □

Proposition 19. *Let X and Y be normed linear spaces over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). If F is Fréchet differentiable at $a \in X$ then F is continuous at a.*

Proof. Since F is Fréchet differentiable at $a \in X$, there exists $\delta > 0$ such that when $\|h\| < \delta$, we have

$$\|F(a+h) - F(a) - DF(a)(h)\| \leq \|h\|.$$

So for all $h \in X$ with $\|h\| < \delta$, we have

$$\|F(a+h) - F(a)\| \leq \|F(a+h) - F(a) - DF(a)(h)\| + \|DF(a)(h)\| \leq \|h\| + \|DF(a)\|\|h\| = (1 + \|DF(a)\|)\|h\|$$

As $h \to 0$, $(1 + \|DF(a)\|)\|h\| \to 0$. Therefore,

$$\lim_{h\to 0} \|F(a+h) - F(a)\| = 0,$$

which means $\lim_{h\to 0} F(a+h) = F(a)$. Therefore F is continuous at a. □

Proposition 20. *Let X, Y and Z be normed linear spaces over $\mathbb{K}$.*

1. *Let $f, g : X \to Y$ be Fréchet derivative at $a \in X$. Then for any $\alpha, \beta \in \mathbb{K}$, we have*

$$D(\alpha f + \beta g)(a) = \alpha Df(a) + \beta Dg(a).$$

2. *(Chain Rule) Suppose $F : X \to Y$ is Fréchet differentiable at $a \in X$, $G : Y \to Z$ is Fréchet differentiable at $F(a)$. Then $G \circ F : X \to Z$ is Fréchet differentiable at a and*

$$D(G \circ F)(a) = DG(F(a))\, DF(a).$$

Proof. Set $b = F(a)$. By the assumptions,

$$F(a+h) - F(a) = DF(a)h + \|h\| r_1(h)$$
$$G(b+k) - G(b) = DG(b)k + \|k\| r_2(k),$$

where $\|r_1(h)\| \to 0$ as $h \to 0$ and $\|r_2(k)\| \to 0$ as $k \to 0$, for $h \in X, k \in Y$.
For $h \in X$, denote $k = DF(a)h + \|h\| r_1(h)$. Now we compute

$$\begin{aligned} G(F(a+h)) - G(F(a)) &= G\Big(b + DF(a)h + \|h\| r_1(h)\Big) - G(b) \\ &= DG(b)\Big(DF(a)h + \|h\| r_1(h)\Big) + \|k\| r_2(k) \\ &= DG(b)DF(a)h + \|h\| DG(b)\Big(r_1(h)\Big) + \|k\| r_2(k) \end{aligned}$$

The operator $DG(F(a))DF(a) : X \to Z$ is a bounded linear operator since it is the composition of two bounded linear operators $DF(a)$ and $DG(F(a))$. Observe that

$$\|k\| \le \Big(\|DF(a)\| + \|r_1(h)\|\Big)\|h\|, \quad \forall h \in X, \tag{12}$$

and

$$\frac{\|r(h)\|}{\|h\|} \le \|DG(b)\|\|r_1(h)\| + \frac{\|k\|}{\|h\|}\|r_2(k)\| \le \|DG(b)\|\|r_1(h)\| + \Big(\|DF(a)\| + \|r_1(h)\|\Big)\|r_2(k)\| \tag{13}$$

Now letting $h \to 0$. Since $\|r_1(h)\| \to 0$ as $h \to 0$, from (12), we have $k \to 0$ and hence $r_2(k) \to 0$. Therefore, from (13), we have

$$\frac{\|r(h)\|}{\|h\|} \to 0 \quad \text{as} \quad h \to 0.$$

□

2.14 Hahn-Banach Theorems. Generalized Mean Value Theorem. Separation Theorems.

Definition 1. *Let X be a vector space over $\mathbb{K}$. We say that $p : X \to [0, \infty)$ is sublinear if it satisfies*

$$p(\lambda x) = \lambda p(x) \quad \textit{(positive homogeneous)},$$
$$p(x+y) \le p(x) + p(y) \quad \textit{(triangle inequality)},$$

for any $x, y \in X$ and real $\lambda \ge 0$.

Lemma 6 (Zorn's Lemma)**.** *Suppose S is a nonempty, partially ordered set (reflexivity, antisymmetry, and transitivity). If every totally ordered subset C (that is, every two elements in C are comparable) of S has an upper bound; that is, there is some $u \in S$ such that*

$$x \le u \quad \textit{for all} \;\; x \in C.$$

Then S has at least one maximal element; that is there is some $m \in S$ such that for any $x \in S$,

$$\textit{if} \quad m \le x \quad \textit{then} \quad m = x.$$

Theorem 1 (The Hahn-Banach Theorem for linear spaces)**.** *Let X_0 be a subspace of a real vector space X and p be a sublinear on X. If $f_0 : X_0 \to \mathbb{R}$ is a linear functional such that*

$$f_0(x) \le p(x), \quad \forall\, x \in X_0,$$

then there is a linear functional $f : X \to \mathbb{R}$ such that

$$f|_{X_0} = f_0, \quad \textit{(i.e., } f \textit{ is a linear extension of } f_0\textit{)}$$

and

$$f(x) \le p(x), \quad \forall\, x \in X.$$

Sketch of the proof. **Step 1:** We first prove the statement in the special case when $X = X_0 + \operatorname{span}(v)$ with a fixed $v \notin X_0$. Set

$$f(\lambda v + x_0) = f_0(x_0) + \lambda c, \quad \forall\, x_0 \in X_0,\ \forall \lambda \in \mathbb{R},$$

where $c \in \mathbb{R}$ is a fixed number satisfying

$$\sup_{u \in X_0} (f_0(u) - p(u - v)) \le c \le \inf_{w \in X_0} (p(w + v) - f_0(w)).$$

We first show that such c exists. Indeed, for all $u, w \in X_0$, we have

$$f_0(u) + f_0(w) = f_0(u + w) \le p(u + w) \le p(u - v) + p(w + v).$$

Therefore,

$$f_0(u) - p(u - v) \le p(w + v) - f_0(w), \quad \forall u, w \in X_0,$$

which means such c exists. Next, we will verify that the defined f is a linear functional on X , $f|_{X_0} = f_0$ (leave it as an exercise).

Finally, we will prove that $f(\lambda v + x_0) \le p(\lambda v + x_0)$ for all $x_0 \in X_0$ and for all $\lambda \in \mathbb{R}$. The statement is true for $\lambda = 0$. For $\lambda > 0$, from the requirement on c,

$$c \le p(w + v) - f_0(w), \quad \forall w \in X_0$$

we have

$$c \le p(\lambda^{-1}x_0 + v) - f_0(\lambda^{-1}x_0) = \lambda^{-1}\Big(p(x_0 + \lambda v) - f_0(x_0)\Big),$$

$$f(x_0 + \lambda v) = \lambda c + f_0(x_0) \le p(x_0 + \lambda v).$$

Similarly, for $\lambda < 0$, from the requirement on c,

$$c \ge f_0(u) - p(u - v), \quad \forall u \in X_0,$$

we have

$$c \ge f_0(-\lambda^{-1}x_0) - p(-\lambda^{-1}x_0 - v) = -\lambda^{-1}(f_0(x_0) - p(x_0 + \lambda v)),$$

$$f(x_0 + \lambda v) = \lambda c + f_0(x_0) \le p(x_0 + \lambda v).$$

Step 2: Let $\mathcal{S}$ be the set of all linear extensions g of f_0 defined on a vector space $X_g \subset X$ and satisfying the property $g(x) \le p(x)$ for all $x \in X_g$. Since $f_0 \in \mathcal{S}$, $\mathcal{S}$ is not empty. Define a partial ordering on S by $g \le h$ means that h is a linear extension of g. For any totally ordered subset $\mathcal{C} \subset \mathcal{S}$, let

$$Y = \bigcup_{g \in \mathcal{C}} X_g, \quad g_{\mathcal{C}}(x) = g(x) \quad \text{for any } g \in \mathcal{C} \text{ such that } x \in X_g.$$

Since $\mathcal{C}$ is totally ordered, $g_{\mathcal{C}}$ is well-defined. Moreover $g_{\mathcal{C}} \in \mathcal{S}$ and is an upper bound for $\mathcal{C}$. Applying the Zorn's lemma, $\mathcal{S}$ has at least one maximal element f. By definition, f is a linear extension of f_0 and $f(x) \le p(x)$ for all $x \in X_f$. It remains to show that $X_f = X$. If not, there exists $v \in X \setminus X_f$. Applying results from Step 1, we can construct a linear extension of f to $\widetilde{f}$ on $X_f + \mathbb{R}v$. This contradicts the maximality of f. Therefore, $X_f = X$, which completes the proof. □

Theorem 2 (The Hahn-Banach Theorem for normed spaces)**.** *Let X_0 be a subspace of a normed space X over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $f_0 : X_0 \to \mathbb{R}$ be a linear functional such that*

$$|f_0(x)| \le \alpha \|x\| \quad \forall\, x \in X_0 \text{ and fixed } \ \alpha \ge 0.$$

Then there is a linear functional $f : X \to \mathbb{R}$ such that

$$f|_{X_0} = f_0, \quad (\text{i.e., } f \text{ is a linear extension of } f_0)$$

and

$$|f(x)| \le \alpha \|x\| \quad \forall\, x \in X.$$

Sketch of the proof. We prove the case $\mathbb{K} = \mathbb{R}$. Define

$$p(x) := \alpha \|x\| \quad \forall x \in X.$$

We can verify that $p(x)$ is sublinear (Prove this). Since $f_0(x) \le |f_0(x)| \le p(x)$, by the Hahn-Banach theorem for linear spaces, there is a linear functional $f : X \to \mathbb{R}$ such that $f|_{X_0} = f_0$ and $f(x) \le \alpha\|x\|$. Since

$$-f(x) = f(-x) \le \alpha \| - x\| = \alpha \|x\|,$$

we also have $f(x) \ge -\alpha\|x\|$ for all $x \in X$. Therefore, $|f(x)| \le \alpha\|x\|$ for all $x \in X$. □

Lecture 14: Applications of Hahn-Banach Theorems. Hilbert Spaces.

Below is another version of the Hahn-Banach theorem for normed spaces.

Theorem 3 (The Hahn-Banach Theorem for normed spaces)**.** *Let X_0 be a subspace of a normed space X over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $f_0 \in X_0^*$. Then there is a linear functional $f : X \to \mathbb{R}$ such that*

$$f|_{X_0} = f_0 \quad \text{and} \quad \|f\| = \|f_0\|.$$

Proof. In class (use previous theorem with $|f_0(x_0)| \leq \|f_0\|\|x_0\|$ for all $x_0 \in X_0$. Prove that the linear functional extension also preserves the norm, that is, $\|f\| = \|f_0\|$. □

Proposition 21 (Supporting Functional)**.** *Let X be a normed space. For every $a \in X$, $a \neq 0$, there exists $f \in X^*$ such that*

$$\|f\| = 1, f(a) = \|a\|.$$

The function f is called the supporting functional of a.

Proof. Define $f_0 : X_0 = span(a) \to \mathbb{R}$, $f_0(ta) = t\|a\|$. Obviously, f_0 is a linear functional on X_0 and $|f_0(u)| = \|u\|$ for all $u \in X_0$. Applying the Hahn-Banach theorem, there exists $f \in X^*$ such that $f(a) = f_0(a) = \|a\|$ and $|f(u)| \leq \|u\|$ for all $u \in X$. So $\|f\| \leq 1$.

On the other hand, since $|f(a)| = \|a\|$, $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(a)|}{\|a\|} = 1$. Therefore, $\|f\| = 1$. □

Example 1. *For $(\mathbb{R}^n, \|\cdot\|_2), a \in \mathbb{R}^n, a \neq 0$, a supporting functional is $f(x) = \frac{x \cdot a}{\|a\|}$.*

Proof. Exercise: Verify that $f \in (\mathbb{R}^n)^*$, $\|f\| = 1$ and $f(a) = \|a\|$. □

Recall: For a linear functional $f \in X^*$, where X is a normed linear space, the operator norm of f is

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|z\| \leq 1} |f(z)|.$$

In general, we may not be able to replace the supremum above by the maximal. That is, there exist a normed linear space $X$and a linear functional $f \in X^*$ such that $|f(x)| < \|f\|\|x\|$ for all $x \in X$, $x \neq 0$. Find an example. Note that the normed linear space in this example should be infinite dimensional. (Explain!) However, every vector $x \in X$ does attain its norm on some functional $f \in X^*$.

Corollary 3. *Let X be a normed linear space over $\mathbb{K}$. Then for all $a \in X$, we have*

$$\|a\| = \max_{g \in X^*, \|g\| \leq 1} |g(a)|.$$

Proof. If $a = 0$, $g(a) = 0$ for all $g \in X^*$. The statement holds.
If $a \neq 0$, from Proposition 21, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(a) = \|a\|$. So

$$\sup_{g \in X^*, \|g\| \leq 1} |g(a)| \geq f(a) = \|a\|.$$

On the other hand, we have

$$\sup_{g\in X^*,\|g\|\le 1} |g(a)| \le \sup_{g\in X^*,\|g\|\le 1} \|g\|\|a\| \le \|a\|.$$

Therefore,

$$f(a) = \|a\| = \sup_{g\in X^*,\|g\|\le 1} |g(a)|.$$

Note that $f \in X^*$ and $\|f\| = 1$. That means

$$\|a\| = \max_{g\in X^*,\|g\|\le 1} |g(a)|.$$

□

The following theorem is useful to prove certain operator is a contraction mapping.

Theorem 4 (Generalized Mean Value Theorem)**.** *Let $F : X \to Y$ be an operator between normed linear spaces X and Y and $a, b \in X$, $a \neq b$. Suppose F is continuous on the closed segment $\{a+t(b-a), 0 \le t \le 1\}$ and Fréchet differentiable on the open segment $\{a + t(b - a), 0 < t < 1\}$. Then*

$$\|F(b) - F(a)\| \le \sup_{0<t<1} \|DF(a + t(b - a))\| \, \|b - a\|.$$

Sketch of the proof. Let $g \in Y^*$ such that $g(F(b) - F(a)) = \|F(b) - F(a)\|$ and $\|g\| = 1$. Consider $\Phi : [0, 1] \to \mathbb{R}$

$$\Phi(t) = g(F(a + t(b - a)), \ t \in [0, 1].$$

Since $g \in Y^*$, $Dg(y) = g$ for all $y \in Y$. By the chain rule, the Fréchet derivative of Φ at $t \in (0, 1)$ is

$$D\Phi(t) : [0, 1] \to X \xrightarrow{DF} Y \xrightarrow{Dg} \mathbb{R}$$

$$D\Phi(t) = Dg(F(a + t(b - a)))\Big[DF(a + t(b - a))(b - a)\Big] = g\Big[DF(a + t(b - a)) \, (b - a)\Big].$$

By the mean value theorem,

$$\begin{aligned} \Phi(1) - \Phi(0) &= D\Phi(t_0) \quad \text{for some} \quad t_0 \in (0, 1). \\ g(F(b)) - g(F(a)) &= g\Big[DF(a + t_0(b - a)) \, (b - a) \\ g(F(b) - F(a)) &\le \|g\|\Big\|DF(a + t_0(b - a))\Big\|\|b - a\| \\ \|F(b) - F(a)\| &\le \sup_{0<t<1} \|DF(a + t(b - a))\|\|b - a\|. \end{aligned}$$

□

Theorem 5 (Separating a point from a convex set)**.** *Let K be an open convex subset of a normed space X and consider a point $x_0 \notin K$. Then there exists a linear functional $f \in X^*$, $f \neq 0$ such that*

$$f(x) \le f(x_0) \quad \forall x \in K.$$

Proof. Assignment 3. □

Theorem 6 (Separating Hyperplane Theorem). *Let A and B be disjoint, nonempty, convex subsets of a normed linear space X.*

1. *If A is open, then there exists a functional $f \in X^*$ and $c \in \mathbb{R}$ such that $f(a) \leq c \leq f(b)$ for all $a \in A$, $b \in B$.*

2. *If both A and B are open, then there exists a functional $f \in X^*$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$ for all $a \in A$, $b \in B$.*

3. *If A is compact and B is closed, then there is $f \in X^*$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$ for all $a \in A$, $b \in B$.*

Proof. Assignment 3. □

3 Inner Product Spaces

Hilbert spaces are an important and simplest class of Banach spaces, where the concept of orthogonality is defined. With a view to applications, the most important Hilbert spaces are the real and complex Lebesgue spaces $L_2(G)$ and the related Sobolev spaces $W_2^1(G)$ and $\mathring{W}_2^1(G)$, where $G \subset \mathbb{K}^N$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

3.1 Inner Product Spaces

In this chapter, the scalar field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 1 (Inner Product). *Let X be a vector space over $\mathbb{K}$. An inner product on X is a function $\langle , \rangle X \times X \to \mathbb{K}$ that satisfies*

- $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \quad \forall\, x, y, z \in X$
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \quad \forall\, x, y \in X, \quad \alpha \in \mathbb{K}$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}, \quad \forall\, x, y \in X$
- $\langle x, x\rangle \geq 0, \quad \langle x, x\rangle = 0 \quad \textit{iff} \quad x = 0$

Then $(X, \langle , \rangle)$ is an inner product space.

Note: If $(X, \langle , \rangle)$ is an inner product space, then

$$\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle, \quad \forall\, x, y \in X, \quad \alpha \in \mathbb{K}$$

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle, \quad \forall\, x, y, z \in X, \forall \alpha, \beta \in \mathbb{K}.$$

$$\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle, \quad \forall\, x, y, z \in X, \forall \alpha, \beta \in \mathbb{K}.$$

Definition 2 (Orthogonality)**.** *Let $(X, \langle , \rangle)$ be an inner product space and $x, y \in X$. Then x is called orthogonal to y if $\langle x, y\rangle = 0$.*

Theorem 1 (Cauchy-Schwarz Inequality)**.** *Let X be an inner product space. Then every two vectors $x, y \in X$ satisfy*

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

Sketch of the proof. The inequality is true if $x = 0$ or $y = 0$. For fixed $x \neq 0$ and $y \neq 0$, we have

$$\langle x - \alpha y, x - \alpha y\rangle \geq 0 \quad \forall \alpha \in \mathbb{K}$$

$$\langle x, x\rangle - \overline{\alpha}\langle x, y\rangle - \alpha\left(\langle y, x\rangle - \overline{\alpha}\langle y, y\rangle\right) \geq 0$$

Choose $\overline{\alpha} = \dfrac{\langle y, x\rangle}{\langle y, y\rangle}$. Then simplifying the left hand side, we get the result. □

Corollary 4. *Let X be an inner product space. Then X is a normed space with the norm defined as*

$$\|x\| := \langle x, x\rangle^{1/2}.$$

Proof. Exercise. □

Definition 3. *Let $(X, \langle , \rangle)$ be an inner product space. X is called a Hilbert space if X is a Banach space with the normed induced by the inner product.*

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ is generated by an inner product if and only if the parallelogram equality holds:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad \forall x, y \in X$$

Sketch of the proof. $(\Rightarrow)$ Suppose $\|x\| = \sqrt{\langle x, x\rangle}$ for some inner product $\langle , \rangle$ on X. Then verify that

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad \forall x, y \in X$$

$(\Leftarrow)$ Suppose the norm $\|\cdot\|$ satisfies the parallelogram equality. For $x, y \in X$, define

$$\langle x, y\rangle := \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right).$$

We will verify that $\langle , \rangle$ is an inner product on X (prove this). □

Remark 3. *Not all normed spaces are inner product spaces. For example, the space ℓ^p with $p \neq 2$ and the space $(C[a, b], \|\cdot\|_\infty)$.*

Lecture 15 & 16 : Examples of Hilbert Spaces. Projection Theorem. Riesz Representation Theorem. Adjoint Operators.

Example 1. *1. The space $\mathbb{R}^n$ is a Hilbert space over $\mathbb{R}$ with the standard inner product defined by*

$$\langle x, y\rangle := \sum_{k=1}^{n} x_k y_k \quad \text{for } x, y \in \mathbb{R}^n.$$

2. The space $\mathbb{C}^n$ is a Hilbert space over $\mathbb{C}$ with inner product defined by

$$\langle x, y\rangle := \sum_{k=1}^{n} x_k \overline{y_k} \quad \text{for } x, y \in \mathbb{C}^n.$$

3. The space $L_2[a,b] = \left\{ f : [a,b] \to \mathbb{K} \quad s.t \quad \int_a^b |f(t)|^2 dt < \infty \right\}$ is a Hilbert space over $\mathbb{K}$ with inner product defined by $\langle f, g\rangle := \int_a^b f(t)\overline{g(t)}\, dt$.

4. The space $\ell_2 = \left\{ x = (x_1, x_2, \ldots) \quad s.t \quad \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$ is a Hilbert space with inner product defined by

$$\langle x, y\rangle := \sum_{k=1}^{\infty} x_k \overline{y_k} \quad \text{for } x = (x_k) \in \ell_2, y = (y_k) \in \ell_2.$$

5. The space $(\ell^p, \|\cdot\|_p)$ with $p \neq 2$ is not an inner product space.

Proof. We will show that the norm does not satisfy the parallelogram equality.
Take $x = (1, 1, 0, 0, \cdots) \in \ell_p$ and $y = (1, -1, 0, 0, \cdots) \in \ell_p$. Then

$$\|x\| = \|y\| = 2^{1/p}, \|x + y\| = \|x - y\| = 2$$

So the parallelogram equality is not satisfied. □

6. The space $(C[a,b], \|\cdot\|_\infty)$ is not an inner product space, hence not a Hilbert space.

Proof. Take $f(t) = 1$ and $g(t) = \dfrac{t-a}{b-a}$. We have $\|f\| = \|g\| = 1$ and $\|f + g\| = 2, \|f - g\| = 1$. So the parallelogram equality is not satisfied. □

Proposition 22. *If in an inner product space, $x_n \to x$ and $y_n \to y$ then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Proof. We have

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \leq$$
$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \leq \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\|$$

Let $n \to \infty$, we get

$$0 \leq \lim_{n\to\infty} |\langle x_n, y_n\rangle - \langle x, y\rangle| \leq \|x\|0 + 0\|y\| = 0.$$

Hence, $\lim_{n\to\infty} |\langle x_n, y_n\rangle - \langle x, y\rangle| = 0$, $\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle$, which completes the proof. □

3.2 Orthogonal Projection

Definition 1. *Let A be a subset of an inner product space X. The orthogonal complement of A is defined as*

$$A^\perp := \{x \in X : \langle x, a\rangle = 0 \quad \text{for all} \ \ a \in A\}.$$

Proposition 23. *Let A be a subset of an inner product space X. Then $A^\perp$ is a closed linear subspace of X and $A \cap A^\perp \subset \{0\}$.*

Proof. Exercise. □

Theorem 1. *Let Y be a closed linear subspace of the real or complex Hilbert space X and $x \in X$ be given. Then the following holds*

(i) There exists a unique $y \in Y$ such that

$$\|x - y\| = \min_{z \in Y} \|x - z\|$$

(ii) The point y in part (i) is the unique vector in Y such that $x - y \in Y^\perp$.

The point y is called the orthogonal projection of x onto the subspace Y.

Proof. (i) *Existence.* Denote $d = \inf_{z \in Y} \|x - z\|$. By the definition of the infimum, there exists a sequence $\{y_n\} \subset Y$ such that $\|x - y_n\| \to d$ as $n \to \infty$. We will prove that $\{y_n\}$ is a Cauchy sequence.
Using parallelogram law, we have

$$\|y_n - y_m\|^2 + 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 = 2(\|x - y_n\|^2 + \|x - y_m\|^2) \quad \text{for} \ \ n, m \geq 1.$$

Since Y is a linear subspace of X and $y_n, y_m \in Y$, we have $\frac{1}{2}(y_n + y_m) \in Y$. Therefore, $\left\|x - \frac{1}{2}(y_n + y_m)\right\| \geq d$. Hence

$$\|y_n - y_m\|^2 = 2(\|x - y_n\|^2 + \|x - y_m\|^2) - 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 \leq 2(\|x - y_n\|^2 + \|x - y_m\|^2) - 4d^2.$$

Let $n, m \to \infty$, we have $\|x - y_n\| \to d$, $\|x - y_m\| \to d$ and

$$0 \leq \lim_{n,m\to\infty} \|y_n - y_m\|^2 \leq 4d^2 - 4d^2 = 0.$$

Therefore, $\lim_{n,m\to\infty} \|y_n - y_m\| = 0$ and $\{y_n\}$ is a Cauchy sequence. Since X is a Hilbert space, there exists $y \in X$ such that $\lim_{n\to\infty} y_n = y$. Since $y_n \in Y$ and Y is closed, $y \in Y$. In conclusion, we have

$$\|x - y\| = \min_{z \in Y} \|x - z\|.$$

Uniqueness. Suppose there is also $\hat{y} \in Y$ such that

$$\|x - \hat{y}\| = \min_{z \in Y} \|x - z\|.$$

Applying the parallelogram law and using $\left\|x - \frac{1}{2}(y+\hat{y})\right\| \geq d$ (since $\frac{1}{2}(y+\hat{y}) \in Y$), we have

$$\|y-\hat{y}\|^2 + 4\left\|x - \frac{1}{2}(y+\hat{y})\right\|^2 = 2(\|x-y\|^2 + \|x-\hat{y}\|^2) = 4d^2,$$

$$0 \leq \|y-\hat{y}\|^2 = 4d^2 - 4\left\|x - \frac{1}{2}(y+\hat{y})\right\|^2 \leq 0.$$

So $\|y-\hat{y}\|^2 = 0,\ y = \hat{y}$.

(ii). *Orthogonality.* Clearly, $\langle x-y, 0\rangle = 0$. Take $z \in Y, z \neq 0$. We will prove that $\langle x-y, z\rangle = 0$. By the construction of y in part (i), we have

$$\|x-y\|^2 \leq \|x-(y+\lambda z)\|^2 = \|x-y\|^2 + |\lambda|^2\|z\|^2 - \lambda\langle z, x-y\rangle - \overline{\lambda}\langle x-y, z\rangle,$$

$$0 \leq |\lambda|^2\|z\|^2 - \lambda\overline{\langle x-y, z\rangle} - \overline{\lambda}\langle x-y, z\rangle.$$

Plugging $\lambda = \dfrac{\langle x-y, z\rangle}{\|z\|^2}$ into the above inequality, we conclude

$$\frac{|\langle x-y, z\rangle|^2}{\|z\|^2} \leq 0,$$

which only happens when $\langle x-y, z\rangle = 0$.

Uniqueness. Assume there is also $y^* \in Y$ such that $x - y^* \in Y^\perp$. Then $y - y^* = (x-y^*) - (x-y) \in Y^\perp$. On the other hand, $y - y^* \in Y$ since $y, y^* \in Y$. So $y - y^* \in Y \cap Y^\perp \subset \{0\}$. Therefore, $y - y^* = 0$ and $y = y^*$. □

Note in part (i), we only need the condition that if $y_n, y_m \in Y$ then its average is also in Y. Therefore, we have a more general result for part (i).

Theorem 2 (Hilbert's Projection Theorem)**.** *Given a closed convex set Y in a Hilbert space X and $x \in X$. There exists a unique $y \in Y$ such that*

$$\|x-y\| = \min_{z\in Y} \|x-z\|.$$

Corollary 5 (Orthogonal Decomposition)**.** *Let Y be a closed linear subspace of the real or complex Hilbert space X. Then every vector $x \in X$ can be uniquely represented as*

$$x = y + w, \quad y \in Y,\ w \in Y^\perp.$$

The orthogonal decomposition is usually written as $X = Y \oplus Y^\perp$.

Note: We can prove that $X/Y \cong Y^\perp$ (linear isomorphism).

Definition 2 (Orthogonal Projection)**.** *Let Y be a closed linear subspace of the real or complex Hilbert space X. The map $P_Y : X \to X, P_Y(x) = y$, where $x = y + w$ and $(y, w) \in Y \times Y^\perp$, is called the orthogonal projection in X onto Y.*

3.3 Riesz Representation Theorem

Lemma 7. *Let $(X, \langle , \rangle)$ be an inner product space. Then*

1. $\langle x, 0\rangle = \langle 0, x\rangle = 0, \quad \forall x \in X$

2. *If there are $y_1, y_2 \in X$ such that $\langle x, y_1\rangle = \langle x, y_2\rangle$ for all $x \in X$, then $y_1 = y_2$.*

Proof. Exercise. □

Theorem 1 (Riesz Representation Theorem)**.** *Let X be a Hilbert space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

1. *For every $y \in X$, the functional $f : X \to \mathbb{K}$, $f(x) = \langle x, y\rangle$ is an element in X^* and $\|f\| = \|y\|$.*

2. *Conversely, for every $f \in X^*$, there exists a unique $y \in X$ such that $f(x) = \langle x, y\rangle$ for every $x \in X$. Moreover, $\|f\| = \|y\|$.*

Proof. (1). If $y = 0$, then the function $f(x) = \langle x, 0\rangle = 0$, for every $x \in X$, is an element in X^* and $\|f\| = 0 = \|y\|$.
If $y \neq 0$, we first verify that $f(x) = \langle x, y\rangle$ is linear. (prove this).
Using Cauchy-Schwarz inequality, we have

$$|f(x)| = |\langle x, y\rangle| \leq \|x\|\|y\|.$$

So f is bounded and $\|f\| \leq \|y\|$.
On the other hand, $\|f\| = \sup\limits_{x\neq 0} \dfrac{|f(x)|}{\|x\|} \geq \dfrac{|f(y)|}{\|y\|} = \|y\|$. Therefore, $\|f\| = \|y\|$.
(2). *Existence.* Consider $f \in X^*$. If $f = 0$, we can take $y = 0$ and $f(x) = \langle x, 0\rangle$ for every $x \in X$.
Consider $f \in X^*, f \neq 0$.

- Step 1: Prove that $\dim(X/\ker f) = 1$.
  Since $f \neq 0$, $\mathrm{Im} f \neq \{0\}$. Moreover, since $\mathrm{Im} f$ is a subspace of $\mathbb{K}$ and $\dim_{\mathbb{K}} \mathbb{K} = 1$, we have $\dim \mathrm{Im} f \leq 1$. So $\dim \mathrm{Im} f = 1$. From the linear isomorphism $X/\ker f \cong \mathrm{Im}\, f$, we have $\dim(X/\ker f) = 1$.

- Step 2: Verify $\ker f$ is a closed subspace of X. Indeed, we know that $\ker f$ is a linear subspace of X. Also, since f is continuous and $\{0\} \subset \mathbb{K}$ is a closed set of $\mathbb{K}$, $\ker f = f^{-1}\{0\}$ is a closed set of X.

- Step 3: By the orthogonal decomposition theorem, we have $X = \ker f \oplus (\ker f)^\perp$ and $\dim(\ker f)^\perp = \dim(X/\ker f) = 1$. Without loss of generality (by scaling), we can assume that $(\ker f)^{=} Span(y_0)$ for some $y_0 \in X$ and $f(y_0) = 1$ (note that $y_0 \notin \ker f$ so $f(y_0) \neq 0$, therefore we do the scaling).

- Step 4: Find y.
  Take $x \in X$. Then

$$f(x - f(x)y_0) = f(x) - f(x)f(y_0) = f(x) - f(x) = 0.$$

Therefore, $w = x - f(x)y_0 \in \ker f$. So $\langle w, y_0\rangle = 0$. Now compute

$$\langle x, y_0\rangle = \langle w + f(x)y_0, y_0\rangle = \langle w, y_0\rangle + f(x)\langle y_0, y_0\rangle = f(x)\langle y_0, y_0\rangle,$$

$$f(x) = \langle x, \frac{y_0}{\|y_0\|^2}\rangle.$$

Set $y = \dfrac{y_0}{\|y_0\|^2}$.

Uniqueness If there is alo $\hat{y}$ such that $f(x) = \langle x, \hat{y}\rangle$ for all $x \in X$, then

$$\langle x, y\rangle = \langle x, \hat{y}\rangle \quad \text{for every} \quad x \in X.$$

By above lemma, $y = \hat{y}$.

□

3.4 Hilbert Adjoint Operator

Definition 1. *Let $T : H \to H$ be a bounded linear operator, where H is a Hilbert space. Then the Hilbert adjoint operator T^* of T is an operator $T^* : H \to H$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \textit{for all } x, y \in H.$$

Theorem 1. *Let $T : H \to H$ be a bounded linear operator, where H is a Hilbert space. Then the Hilbert adjoint operator T^* of T exists and moreover, T^* is also a bounded linear operator and $\|T^*\| = \|T\|$.*

Proof.

- Step 1: Construct T^*. Let $y \in H$. Define $l : H \to \mathbb{K}$, $l(x) := \langle Tx, y\rangle$. We can verify that $l \in H^*$ (check this). Moreover, using Cauchy-Schwarz inequality and the boundedness of T, we have

$$|l(x)| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\| = |T\|\|y\|\|x\|.$$

Therefore l is bounded. By the Riesz representation theorem, there exists a unique $y^* \in H$ such that

$$l(x) = \langle x, y^*\rangle \quad \text{for every} \quad x \in X.$$

and $\|l\| = \|y^*\|$. We define $T^* : H \to H, T^*(y) = y^*$. Then clearly,

$$\langle Tx, y\rangle = l(x) = \langle x, y^*\rangle = \langle x, T^*y\rangle.$$

- Step 2: Verify T^* is linear. Indeed, let $\alpha, \beta \in \mathbb{K}$ and $y, z \in H$. We have

$$\begin{aligned}\langle x, T^*(\alpha y + \beta z)\rangle &= \langle Tx, \alpha y + \beta z\rangle = \langle Tx, \alpha y\rangle + \langle Tx, \beta z\rangle = \overline{\alpha}\langle Tx, y\rangle + \overline{\beta}\langle Tx, z\rangle \\ &= \overline{\alpha}\langle x, T^*y\rangle + \overline{\beta}\langle x, T^*z\rangle = \langle x, \alpha T^*y + \beta T^*z\rangle.\end{aligned}$$

Since this equality holds for every $x \in H$, from Lemma 7, we have

$$T^*(\alpha y + \beta z) = \alpha T^*y + \beta T^*z \quad \text{for all } y, z \in H, \alpha, \beta \in \mathbb{K}.$$

- Verify T^* is bounded. Indeed, we have

$$\|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \leq \|TT^*y\|\|y\| \leq \|T\|\|T^*y\|\|y\|.$$

  So $\|T^*y\| \leq \|T\|\|y\|$ for every $y \in Y$. Therefore T^* is bounded and $\|T^*\| \leq \|T\|$.

- Show that $\|T^*\| = \|T\|$.
  Since $T^* \in B(H)$, we can apply steps 1,2,3 and have $T^{**} \in B(H)$ and $\|T^{**}\| \leq \|T^*\|$.
  On the other hand, for every $x, y \in H$, we get

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, T^{**}x\rangle} = \langle T^{**}x, y\rangle,$$

  where the first and the third equalities come from the definition of T^* and T^{**}. Since $\langle Tx, y\rangle = \langle T^{**}x, y\rangle$ for all $y \in H$, we have $Tx = T^{**}x$ for all $x \in H$. Therefore, $T = T^{**}$. In conclusion, we have

$$\|T\| \geq \|T^*\| \geq \|T^{**}\| = \|T\|,$$

  which implies $\|T\| = \|T^*\|$.

□

Lemma 8. *Let X and Y be inner product spaces and $T \in B(X, Y)$. Then*

1. *$T = 0$ if and only if $\langle Tx, y\rangle = 0$ for all $x \in X$, $y \in Y$.*

2. *If $T : X \to X$ and X is a complex inner product space and $\langle Tx, x\rangle = 0$ for all $x \in X$, then $T = 0$.*

Proof. Exercise. □

Definition 2. *A bounded linear operator $T : H \to H$ on a Hilbert space H is said to be*

- *self-adjoint (Hermitian) if $T^* = T$, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$.*

- *unitary if T is bijective and $T^* = T^{-1}$.*

- *normal if $TT^* = T^*T$.*

Clearly, if T is self-adjoint or unitary, T is normal.

Proposition 24. *Let $T : H \to H$ be a bounded linear operator on a Hilbert space H and T be onto. Then T is unitary if and only if $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$.*

Proof. In class. □

3.5 Lecture 18: Generalized Fourier Series

Generalized Fourier Series

Definition (1) A sequence $\{x_k\}$ in a Hilbert space H is called an orthogonal system if $\langle x_k, x_l \rangle = 0$ for all $k \neq l$.

(2) A sequence $\{x_k\}$ in a Hilbert space H is called an orthonormal system if $\langle x_k, x_l \rangle = \delta_{kl}$

Example (orthonormal system)

(1) In $(\ell_2, \|\cdot\|_2)$, $x_k = (0, \ldots, 0, \underset{k^{th}}{1}, 0, \ldots)$

(2) In $(L_2^{\mathbb{C}}[-\pi, \pi], \|\cdot\|_2)$ $x_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$, $t \in [-\pi, \pi]$

(3) $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos t, \frac{1}{\sqrt{\pi}}\sin t, \frac{1}{\sqrt{\pi}}\cos 2t, \frac{1}{\sqrt{\pi}}\sin 2t, \ldots\right\}$

Lemma 1 An orthonormal set is linearly independent.

Proof: Exercise.

~~Theorem Bessel's inequality~~

Definition (3) Consider an orthonormal system $\{x_k\}_{k=1}^{\infty}$ in a Hilbert space H and $x \in H$. The Fourier series of x w.r.t $\{x_k\}$ is the formal series

$$\sum_{k=1}^{\infty} \langle x, x_k \rangle x_k .$$

The coefficients $\langle x, x_k \rangle$ are called the Fourier coefficients of x

Lemma 2 $\sum_{k=1}^{n} \langle x, x_k \rangle x_k$ is the orthogonal projection of x onto $\text{Span}(x_1, \ldots, x_n)$ where $\{x_k\}_{k=1}^{\infty}$ is an orthonormal system

Sketch of the proof: $\text{Span}(x_1, \ldots, x_n)$ is a linear subspace of H.

(1) Verify $\text{Span}(x_1, \ldots, x_n)$ is a closed set in H.

(2) Verify $\left(x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k\right) \in \text{Span}(x_1, \ldots, x_n)^{\perp}$

Bessel's Inequality Let $\{x_k\}$ be an orthonormal system in a Hilbert space H

Then for every $x \in H$, we have

$$\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \le \|x\|^2 .$$

Proof Denote $S_n = \sum_{k=1}^{n} \langle x, x_k \rangle x_k$

From lemma 2, $x - S_n \in \text{Span}(x_1, \dots, x_n)^{\perp}$

In particular, $x - S_n \perp S_n$.

$$\|x\|^2 = \|x - S_n\|^2 + \|S_n\|^2 \ge \|S_n\|^2 = \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 , \quad \forall n$$

let $n \to \infty$, $\|x\|^2 \ge \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2$.

Theorem (convergence) Let $\{x_k\}$ be an orthonormal sequence in a Hilbert space H. Then

ⓐ The series $\sum_{k=1}^{\infty} a_k x_k$ converges (in the norm in H) iff $\sum_{k=1}^{\infty} |a_k|^2$ converges. iff $\sum a_k x_k$ converges unconditionally in H, i.e., for every reordering of terms

ⓑ If $\sum_{k=1}^{\infty} a_k x_k$ converges then $a_k = \langle x, x_k \rangle$ where $x = \sum_{k=1}^{\infty} a_k x_k$

ⓒ Conversely, for every $x \in H$, the Fourier series $\sum_k \langle x, x_k \rangle x_k$ converges in H

Before proving the convergence theorem, we need the following lemma:

Lemma 3 If $\sum_{n=1}^{\infty} a_n w_n$ converges in H then for each $y \in H$

$$\left\langle \sum_{n=1}^{\infty} a_n w_n , y \right\rangle = \sum_{n=1}^{\infty} a_n \langle w_n, y \rangle$$

Sketch of the
Proof Given $y \in H$, $L_y(x) = \langle x, y \rangle$ is in H^*.

$$|L_y(x)| \leq \|y\| \|x\|, \text{ so } \|L_y\| \leq \|y\|.$$

$$\left| L_y(x) - \sum_{k=1}^{m} L_y(a_k w_k) \right| = \left| L_y\left(x - \sum_{k=1}^{m} a_k w_k\right) \right| \leq \|L_y\| \left\| x - \sum_{k=1}^{m} a_k w_k \right\| \xrightarrow{m \to \infty} 0$$

So $L_y(x) = \sum_{k=1}^{\infty} L_y(a_k w_k)$

Proof of the Convergence theorem

(a) $s_n = \sum_{k=1}^{n} a_k x_k$, $\sigma_n = \sum_{k=1}^{n} |a_k|^2$

For $n > m$, $\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^{n} a_k x_k \right\|^2 = \sum_{k=m+1}^{n} |a_k|^2 = \sigma_n - \sigma_m$.

So $\{s_n\}$ is Cauchy in H iff σ_n is Cauchy in $\mathbb{R}$

(b) $x = \sum_{k=1}^{\infty} a_k x_k$. From lemma 3

$$\langle x, x_l \rangle = \left\langle \sum_{k=1}^{\infty} a_k x_k, x_l \right\rangle = \sum_{k=1}^{\infty} a_k \langle x_k, x_l \rangle = a_l$$

(c) Since $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2$ (Bessel Inequality)

from part a), we conclude that the Fourier series $\sum \langle x, x_k \rangle x_k$

converges.

(d) From Bessel Inequality, $\sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \leq \|x\|^2 \; \forall n$, we have the following.

Lemma 4 Fourier coefficients Let X be an inner product space and $\{x_k\}_{k \in I}$ (I could be uncountable) is an orthonormal family in X. Then any $x \in X$ can have at most countably many nonzero Fourier coefficients

$\langle x, x_k \rangle$

Proof Since $\sum_{k=1}^{n} |\langle x, x_k\rangle|^2 \le \|x\|^2$, $\forall n$

For each m, $\#\{k\in I : |\langle x, x_k\rangle| > \frac{1}{m}\}$ is finite.

$\{k\in I : |\langle x, x_k\rangle| \ne 0\} = \bigcup_{m\ge 1} \{k\in I : |\langle x, x_k\rangle| > \frac{1}{m}\}$ has at most countable cardinal.

From lemma 4 with fixed $x\in H$, we can associate a series $\sum_{k\in I} \langle x, x_k\rangle x_k$

and we can arrange the x_k with $\langle x, x_k\rangle \ne 0$ in a sequence so that

$\sum_{k\in I} \langle x, x_k\rangle x_k$ takes the form $\sum_{k=1}^{\infty} \langle x, x_k\rangle x_k$

Note that from the convergence theorem part ⓐ, the sum $\sum_{k=1}^{\infty} \langle x, x_k\rangle x_k$ does not depend on the order of $\{x_k\}$

Next question Let H be a Hilbert space.

$\{x_k\}_{k=1}^{\infty}$ is an orthonormal set.

$$\text{Is } x = \sum_{k=1}^{\infty} \langle x, x_k\rangle x_k \ ?$$

Lemma 5

Answer In general, it is not. Indeed, $\sum_{k=1}^{\infty} \langle x, x_k\rangle x_k$ is the orthogonal projection of x onto $\overline{\text{Span}(x_1, x_2, \dots)}$

Proof Verify $\left(x - \sum_{k=1}^{\infty} \langle x, x_k\rangle x_k\right) \perp \overline{\text{Span}(x_1, x_2, \dots)}$

Note $\text{Span}(x_1, x_2, \dots) := \left\{ \sum_{k=1}^{m} a_k x_{i_k},\ a_k \in \mathbb{C},\ x_{i_k} \in \{x_1, x_2, \dots\} \right\}$

Definition 5 Let H be a Hilbert space. An orthonormal sequence $\{x_k\}_{k=1}^{\infty}$ in H is maximal if $\langle x, x_k\rangle = 0$ $\forall k$ implies $x = 0$.

Lemma 6 If $\{x_k\}_{k=1}^{\infty}$ is a maximal orthonormal sequence in H ~~then~~

iff $\overline{\text{Span}\{x_k\}} = H$.

Proof ($\Leftarrow$) Assume $H = \overline{\text{Span}\{x_k\}}$ and $\langle x, x_k\rangle = 0$ $\forall k$.

$x \in H = \overline{\text{Span}\{x_k\}}$ So there exists a sequence $\{y_n\} \subset \text{Span}\{x_k\}$

s.t $y_n \to x$.

Since $y_n \in \text{Span}\{x_k\}$ and $\langle x, x_k\rangle = 0$ $\forall k$, we have

$$\langle x, y_n\rangle = 0 \longrightarrow \langle x, x\rangle = \|x\|^2$$

So $x = 0$

($\Rightarrow$) Assume If $\exists x \in H$: $\langle x, x_k\rangle = 0$ $\forall k$ then $x = 0$.

We need to prove $H = \overline{\text{Span}\{x_k\}}$

$\{x \in H : \langle x, x_k\rangle = 0 \ \forall k\} = \{0\}$. So $(\text{Span}\{x_k\})^{\perp} = \{0\}$

$\Downarrow$

$(\overline{\text{Span}\{x_k\}})^{\perp} = \{0\}$

$\overline{\text{Span}\{x_k\}}$ is a closed subspace of H.

$$H = \overline{\text{Span}\{x_k\}} \oplus \left(\overline{\text{Span}\{x_k\}}\right)^{\perp} = \overline{\text{Span}\{x_k\}}$$

Combining Lemma 5 and Lemma 6, we have the following theorem

Theorem (Fourier expansion) ~~Let $\{x_k\}$ be an orthonormal sequence in~~

Let $\{x_k\}$ be a maximal orthonormal sequence in a Hilbert space H. Then every vector $x \in X$ can be expanded in its Fourier series

$$x = \sum_k \langle x, x_k\rangle x_k, \quad \|x\|^2 = \sum_k |\langle x, x_k\rangle|^2 \text{ (Parseval Identity)}$$

3.6 Lecture 19 – Notes from Ararat

19/Nov/19

Note:

Hilbert space H, $T: H \to H$ operator, $\langle Tu, u \rangle = 0$ $\forall u \in H$, Is $T = 0$? No.

(T has to have a condition, e.g., bnded & linear)

Continuation of the proof last time:

($\Leftarrow$) If $H = \overline{\text{span}\{x_k\}}$, we need to show $\{x_k\}_{k=1}^{\infty}$ is maximal. Take $x \in H$ s.t. $\langle x, x_k \rangle = 0$ $\forall k$. Since $H = \overline{\text{span}\{x_k\}_{k=1}^{\infty}}$, there exists a sequence $\{y_n\} \subset \text{span}\{x_k\}$ s.t. $\lim_{n\to\infty} y_n = x$.

For each n, $y_n \in \text{span}\{x_k\}$, $\langle x, x_k \rangle = 0$, $\forall k$. So, $\langle x, y_n \rangle = 0$, $\forall n$. On the other hand,

↳ $y_n = \sum \alpha_k x_k$

$y_n \to x$, $\overset{0}{\langle x, y_n \rangle} \to \langle x, x \rangle$. So, $\langle x, x \rangle = 0$. It means $x = 0$. □

Thm (Fourier Expansion)

Let H be a Hilbert, $\{x_k\}_{k=1}^{\infty}$ be a maximal orthonormal seq. in H.

Then, $\forall x \in H$ can be written as $x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k$.

Note: ① The Thm relies on the existence of a maximal orthonormal seq.

② Combining w/ the previous lemmas, with the same assumptions as the Fourier expansions Thm, $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2$ (Parseval's Identity)

↳ use the fact that $\|x\|^2 = \langle x, x \rangle$ & $x_k x_m = 0$, $\forall k \neq m$.

Thm Let H be a Hilbert space. Then H has a maximal orthonormal seq. iff H is seperable.

↳ (It means H has a dense countable subset)

$H = \overline{\text{span}\{x_k\}}$, $\{x_k\}$ orthonorm.

$$H = \overline{\bigcup_{k=1}^{\infty} z_k}, \quad z_k \in H$$

$\|x\|^2 = \langle x, x \rangle$

$= \langle \sum_k \langle x, x_k \rangle x_k, \ldots \rangle$

$= \sum_k \langle \langle x, x_k \rangle x_k, \ldots$

$= \sum_k x_k \langle \langle x, x_k \rangle, \ldots \rangle$

$= \sum_k \langle \langle x, x_k \rangle, x_k \sum_k \ldots$

$= \sum_k |\langle x, x_k \rangle|^2$

All 0's except for one k

Pf. $H = \overline{\bigcup_{k=1}^{\infty} z_k}$.

$\{z_k\}$ vectors in H $\xrightarrow{\text{construct}}$ a maximal linearly independent set $\{z_{k_i}\}_{i=1}^{\infty}$

↓ Gram-Schmit procedure

at most countable orthonorm. set $\{u_k\}_{k=1}^{\infty}$

So, $H = \overline{\bigcup_{k=1}^{\infty} z_k} = \overline{\text{span}\{\bigcup_{k=1}^{\infty} u_k\}}$. □

Example Seperable Hilbert space

$L_2[a,b] = \overline{C[a,b]} = \overline{P_{\mathbb{Q}}(x)}$

where $P_{\mathbb{Q}}(x) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_k \in \mathbb{Q}\}$

a maximal orthonorm. basis

Thm $\{\Phi_n\} = \{1, \sqrt{2}\cos 2\pi t, \sqrt{2}\sin 2\pi t, \dots, \sqrt{2}\cos 2\pi k t, \sqrt{2}\sin 2\pi k t, \dots\}$

is a maximal orthonorm. seq. for $L_2[0,1]$.
↓ orthonormal basis

Pf. See Siegel.

$H = \overline{\text{span}\{\Phi_k\}_{k=1}^{\infty}}$

Spectral theory (for compact self-adjoint operators)

$\langle Lx, y\rangle = \langle x, Ly\rangle$ implies L bnded & linear, $L^* = L$.

For the remainder of today, we only work w/ $L: H \to H$ compact, self-adj. and H Hilbert.

↳ If $\{z_n\} \subset H$ bnded, then $\exists$ a subseq $\{Lz_{n_k}\}_k$ converges.

Definitions λ is called an eigenvalue of L, if $\exists v \in H$, $v \neq 0$ s.t. $Lv = \lambda v$.

Example $\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}$

After integration, bla bla, $y(x) = \lambda \int_0^1 g(x, s) y(s)\, ds$ where $g(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \le 1 \\ x(1-s) & 0 \le x \le s \le 1 \end{cases}$.

Define $L: L_2[0,1] \to L_2[0,1]$, L compact, $L^* u(x) = \int_0^1 \overline{g(s,x)}\, u(s)\, ds$, $\overline{g(s,x)} = g(s,x) = g(x,s)$.

$Lu(x) := \int_0^1 g(x,s) u(s)\, ds$

↖ $L^* = L$ self-adjoint

If y is a solution of the ODE, $y = \lambda L y$ (y is an eigenvector of L w/ eigvalue $1/\lambda$).

Prop 1: Eigenvectors belonging to distinct eigvalues are orthogonal. All eigenvalues of L are real.

Pf. $Lv_1 = \lambda_1 v_1$, $Lv_2 = \lambda_2 v_2$, $v_1, v_2 \neq 0$, $\lambda_1 \neq \lambda_2$.

$\langle Lv_1, v_1\rangle = \langle \lambda_1 v_1, v_1\rangle = \lambda_1 \|v_1\|^2$

self-adj. ‖

$\langle v_1, Lv_1\rangle = \langle v_1, \lambda_1 v_1\rangle = \overline{\lambda_1} \|v_1\|^2$

$\Rightarrow \lambda_1 = \overline{\lambda_1} \Rightarrow \lambda_1 \in \mathbb{R}$

$\int_0^s \cdot\, ds$ $\quad$ $s\Big|_0^{\min(x,0)} = \min(x,0) - 0$ $\quad$ $s \min(x,s) - 0$ $\quad x$

Next, $\langle Lv_1, v_2\rangle = \lambda_1 \langle v_1, v_2\rangle$

self-adj. ‖

$\langle v_1, Lv_2\rangle = \underbrace{\lambda_2}_{\in \mathbb{R}} \langle v_1, v_2\rangle$

$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle v_1, v_2\rangle = 0 \Rightarrow \langle v_1, v_2\rangle = 0 \Rightarrow$ orth. □

Prop 2 $\|L\| = \sup_{\substack{v\in H\\ \|v\|\le 1}} |\langle Lv,v\rangle| = \sup_{\substack{\|v\|=1\\ v\in H}} |\langle Lv,v\rangle|$ (Prove this)

(L: self-adj, compact)

Prop 3 $\exists\, \Phi_1 \in H$, $\|\Phi_1\|=1$, $L\Phi_1 = \mu_1 \Phi_1$ for some $\mu_1 \in \mathbb{R}$. Indeed, $|\mu_1| = \|L\|$, and Φ_1 is a solution of $\max_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle|$.

$\|L\| =$

Here

(Pf). Construct Φ_1

Idea: Since $\|L\| = \sup_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle|$, $\exists \{v_n\} \subset H$ s.t. $|\langle Lv_n, v_n\rangle| \to \|L\|$. ($\in \mathbb{R}$ or $\mathbb{C}$)

So, $\{\langle Lv_n, v_n\rangle\}_n$ also converges to μ_1 (some real number) $\in \mathbb{R}$, $|\mu_1| = \|L\|$.

From the uniqueness of limits,

Step 1 Show that $Lv_n - \mu_1 v_n \to 0$. Recall Lemma If L is self-adj, $\langle Lx,x\rangle \in \mathbb{R}$, $\forall x\in H$.

$$\|\underbrace{Lv_n}_{\in\mathbb{R}} - \underbrace{\mu_1 v_n}_{\in\mathbb{R}}\| = \|Lv_n\|^2 - 2\mu_1\langle Lv_n, v_n\rangle + \mu_1^2 \|v_n\|^2$$

$$\le \|L\|^2 \|v_n\|^2 - 2\mu_1 \langle Lv_n, v_n\rangle + \mu_1^2 \qquad |\mu_1| = \|L\|$$

($\|v_n\|^2 = 1$)

$$\le 2\mu_1^2 - 2\mu_1\langle Lv_n, v_n\rangle$$

$$n\to\infty \quad 2\mu_1^2 - 2\mu_1 \langle Lv_n, v_n\rangle \to 0$$

Therefore, $Lv_n - \mu_1 v_n \to 0$ as $n\to\infty$.

$\|v_n\|=1$. $\{v_n\}$ bdded. Since L is compact, $\exists \{v_{n_k}\}$ s.t. $\{Lv_{n_k}\}_k$ converges.

$\{Lv_{n_k} - \mu_1 v_{n_k}\}_k$ also converges to 0! So, $\{\mu_1 v_{n_k}\}_k$ converges.

$|\mu_1| = \|L\| \ne 0$, $\mu_1 \ne 0$. therefore, $\{v_{n_k}\}_k$ converges.

Denote $\Phi_1 = \lim_{k\to\infty} v_{n_k}$. Since L is continuous, $L\Phi_1 = \lim_{k\to\infty} Lv_{n_k} = \lim_{k\to\infty} \mu_1 v_{n_k} = \mu_1\Phi_1$.

Since $\|v_{n_k}\|=1$, $\|\Phi_1\| = \lim_{k\to\infty}\|v_{n_k}\| = 1$. Φ_1 is an eigenvector of L.

$\langle L\Phi_1, \Phi_1\rangle = \mu_1\langle \Phi_1,\Phi_1\rangle = \mu_1$. So, Φ_1 is a solution of $\max_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle| = \|L\|$

(μ_1 "= $\|L\|$")

Summary, Φ_1: $\|\Phi_1\|=1$, $\Phi_1\in H$, $|\langle L\Phi_1,\Phi_1\rangle| = \max_{\substack{v\in H\\ \|v\|=1}} |\langle Lv,v\rangle|$

$L\Phi_1 = \mu_1\Phi_1$

$|\mu_1| = \|L\|$.

Next step

Φ_2: $H_1 = \{x\in H : \langle x, \Phi_1\rangle = 0\}$ ($= \{\Phi_1\}^\perp = \text{span}\{\Phi_1\}^\perp$) subspace closed. $L|_{H_1} : H_1 \to H$.

Claim $Im(L|_{H_1}) \subseteq H_1$.

Take $x \in H_1$. Show that $Lx \in H_1$.

$\langle Lx, \Phi_1 \rangle \overset{\text{selfadj.}}{=} \langle x, L\Phi_1 \rangle = \lambda_1 \langle x, \Phi_1 \rangle = \lambda_1 0 = 0$

Therefore, $Lx \in H_1$.

So, $L_1 = L|_{H_1} : H_1 \to H_1$. H_1 closed subspace.

L_1 is also compact, self-adj. $\|L_1\| \leq \|L\|$

$\exists \Phi_2 \in H_1$, $\|\Phi_2\| = 1$, $L_1\Phi_2 = \mu_2 \Phi_2$, $|\mu_2| = \|L_2\| \leq |\mu_1|$. ($L_1\Phi_2 = L\Phi_2$)

$|\langle L\Phi_2, \Phi_2 \rangle| = \max_{\substack{v \in H_1 \\ \|v\|=1}} |\langle Lv, v \rangle|$

$\langle \Phi_2, \Phi_1 \rangle = 0$

⋮

$H_n = \{x \in H : \langle x, \Phi_k \rangle = 0 \text{ for all } k = 1, \ldots, n\}$

$\Phi_{n+1} \in H_n$. eigenvector, $\langle \Phi_{n+1}, \Phi_k \rangle = 0$, $\forall k = 1, \ldots, n$

$|\langle L\Phi_{n_{k+1}}, \Phi_{n_{k+1}} \rangle| = \max_{\substack{v \in H_n \\ \|v\|=1}} |\langle Lv, v \rangle|$

Observation $L\Phi_k = \mu_k \Phi_k$; $|\mu_k| = \|L_k\|$ where $L_k = L|_{H_k}$

$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n| \geq \cdots$ $\mu_k \in \mathbb{R}$. → Can an operator have inf. many eigvalues?

Lemma $\lim_{n \to \infty} \mu_n = 0$.

Pf. Assume $\lim_{n \to \infty} \mu_n \neq 0$. $\exists \varepsilon, \{n_k\} : |\mu_{n_k}| > \varepsilon$, $\forall k = 1, 2, \ldots$

$\left\| \frac{\Phi_{n_k}}{\mu_{n_k}} \right\| < \frac{1}{\varepsilon}$ $\underbrace{\|\Phi_{n_k}\|}_{=1} < \underbrace{\frac{|\mu_{n_k}|}{\varepsilon}}_{1<}$

$\{ \Phi_{n_k}/\mu_{n_k} \}_k$ is bnded in H.

L compact $L\left(\frac{\Phi_{m_k}}{\mu_{m_k}}\right) \to$ converges where $\left\{\frac{\Phi_{m_k}}{\mu_{m_k}}\right\} \subset \left\{\frac{\Phi_{n_k}}{\mu_{n_k}}\right\}$ ($= \Phi_{m_k}$ ← why?)

On the other hand, $\|\Phi_{m_k} - \Phi_{m_\ell}\|^2 = 2$. So, $\{\Phi_{m_k}\}$ could not converge. Contradiction. □

Spectral Theorem (cont'd) Lecture 20

<u>Recall</u> Let H be a Hilbert space, $L: H \to H$ a compact, self-adjoint operator.

① all eigenvalues of L are real (using the self-adjoint property)

② Eigenvectors belonging to different eigenvalues are orthogonal.

③ Eigenvalues and Eigenvectors Construction.

$\exists \phi_1 \in H$ s.t $\|\phi_1\| = 1$, $L\phi_1 = \mu_1 \phi_1$ where $|\mu_1| = \|L\|$

Moreover, $\underbrace{|\langle L\phi_1, \phi_1\rangle|}_{|\mu_1|} = \max_{\substack{v \in H \\ \|v\|=1}} |\langle Lv, v\rangle|$.

$H_1 = \{x \in H : \langle x, \phi_1\rangle = 0\}$ closed subspace of H

$\exists \phi_2 \in H_1$ s.t $\|\phi_2\| = 1$, $L\phi_2 = \mu_2 \phi_2$ where

$$|\mu_2| = |\langle L\phi_2, \phi_2\rangle| = \max_{\substack{v \in H_1 \\ \|v\| = 1}} |\langle Lv, v\rangle| = \|L|_{H_1}\| = \sup_{\substack{v \in H_1 \\ v \neq 0}} \frac{\|Lv\|}{\|v\|}$$

$\vdots$

$\exists \phi_{n+1} \in H_n = \{x \in H : \langle x, \phi_1\rangle = \dots = \langle x, \phi_n\rangle = 0\}$

s.t $\|\phi_{n+1}\| = 1$, $L\phi_{n+1} = \mu_{n+1}\phi_{n+1}$ where

$$|\mu_{n+1}| = |\langle L\phi_{n+1}, \phi_{n+1}\rangle| = \max_{\substack{v \in H_n \\ \|v\| = 1}} |\langle Lv, v\rangle| = \|L|_{H_n}\| = \sup_{\substack{v \in H_n \\ v \neq 0}} \frac{\|Lv\|}{\|v\|}$$

<u>Lemma</u> ① $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n| \geq \dots$

② $\lim_{n \to \infty} \mu_n = 0$

③ The $\{\Phi_n\}_{n=1}^{\infty}$ is an orthonormal set of eigenvectors

④ For any $v \in H$, $\quad Lv = \sum_{k=1}^{\infty} \mu_k \langle v, \Phi_k \rangle \Phi_k = \sum_{k=1}^{\infty} \langle Lv, \Phi_k \rangle \Phi_k$

⑤ The $\{\mu_n\}_{n=1}^{\infty}$ are all nonzero eigenvalues of L

⑦ The $\{\Phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for H iff 0 is not an eigenvalue of $L \iff \text{Ker} L = \{0\}$

⑥ { ~~eigenvectors associated with eigenvalue μ_n~~ } If $L\Phi = \mu_n \Phi$, $\Phi \neq 0$ Then $\Phi \in \text{Span}\{\Phi_j\}$ $\mu_j = \mu_n$

Proof ③ From the construction of Φ_n

④ Take $v \in H$. Let $g_n = v - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \Phi_k$

For $1 \leq j \leq n$

$$\langle g_n, \Phi_j \rangle = \langle v, \Phi_j \rangle - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \langle \Phi_k, \Phi_j \rangle = \langle v, \Phi_j \rangle - \langle v, \Phi_j \rangle = 0$$

So $g_n \in H_n$.

Therefore, $\|Lg_n\| \leq |\mu_{n+1}| \, \|g_n\| \leq |\mu_{n+1}| \, \|v\|$ since

$$\|g_n\|^2 = \langle v - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \Phi_k \, , \, v - \sum_{j=1}^{n} \langle v, \Phi_j \rangle \Phi_j \rangle$$

$$= \|v\|^2 - \sum_{k=1}^{n} |\langle v, \Phi_k \rangle|^2 \leq \|v\|^2$$

So $\quad 0 \leq \|Lg_n\| \leq |\mu_{n+1}| \, \|v\|$

$n \to \infty$: both sides $\to 0$

Therefore $0 = \lim_{n\to\infty} Lg_n = \lim_{n\to\infty} \left(Lv - \sum_{k=1}^{n} \langle v, \Phi_k \rangle L\Phi_k \right)$

$$Lv = \sum_{k=1}^{\infty} \langle v, \phi_k \rangle L\phi_k = \sum_{k=1}^{\infty} \langle v, \phi_k \rangle \mu_k \phi_k = \sum_{k=1}^{\infty} \langle v, \mu_k \phi_k \rangle \phi_k$$

$$= \sum_{k=1}^{\infty} \langle v, L\phi_k \rangle \phi_k \overset{\text{Since } L=L^*}{=} \sum_{k=1}^{\infty} \langle Lv, \phi_k \rangle \phi_k$$

(5) $\{\mu_n\} = \{$ nonzero eigenvalues of $L\}$

"$\supset$" If $L\phi = \mu\phi$ for some $\mu \neq 0$, $\mu \notin \{\mu_n\}_{n=1}^{\infty}$. Then $\langle \phi, \phi_k \rangle = 0$
eigenvector $\phi \neq 0$ $\forall k = 1, 2, \ldots$

Applying (4), $L\phi = \sum_{k=1}^{\infty} \mu_k \langle \phi, \phi_k \rangle \phi_k = 0$

$\mu\phi = 0$ ~~but~~

$\phi = 0$, a contradiction

"$\subset$" Obviously since $|\mu_n| = \|L|_{H_n}\|$ and $L \neq 0$.

---

(6) Suppose $L\phi = \mu_n \phi$, $\phi \neq 0$. Then

$$\mu_n \phi = L\phi = \sum_{k=1}^{\infty} \mu_k \langle \phi, \phi_k \rangle \phi_k \quad \text{Since } \langle \phi, \phi_k \rangle = 0 \text{ if } \mu_k \neq \mu_n$$

$$= \sum_{k \text{ s.t. } \mu_k = \mu_n} \mu_k \langle \phi, \phi_k \rangle \phi_k$$

So $$\phi = \sum_{k \text{ s.t. } \mu_k = \mu_n} \langle \phi, \phi_k \rangle \phi_k$$

6a There exists a finite number of linearly independent eigenvectors corresponding to eigenvalue $\mu = \mu_n$ for any $n = 1, 2, \ldots$

Proof Suppose there is an infinite sequence of linearly independent vectors $\{x_k\}_{k=1}^{\infty}$ s.t $Lx_k = \mu x_k$, $k = 1, 2, \ldots$

Consider subspaces $E_n = \text{span}\{x_k\}_{k=1}^{n}$

Then $E_1 \subsetneq E_2 \subsetneq \ldots \subsetneq E_n \subsetneq \ldots$

Therefore, $\exists\, y_n \in E_n$, $\|y_n\| = 1$ and $\text{dist}(y_n, E_{n-1}) \geq \frac{1}{2}$

Recall a lemma: Let Y be a finite dimensional subspace of X, $Y \subsetneq X$. Then $\exists\, x_0 \in X$ s.t $\|x_0\| = 1$ and $\text{dist}(x_0, Y) \geq \frac{1}{2}$. (See Assignment 2 Solution Question 5)

We will show that $\{Ty_n\}$ contains no Cauchy sequences.

$y_n = a_n x_n + u_{n-1}$, where $u_{n-1} \in E_{n-1}$

$Ty_n = a_n \mu x_n + Tu_{n-1}$, where $Tu_{n-1} \in E_{n-1}$

$n > m$ $\quad Ty_m \in E_m \subseteq E_{n-1}$

$\|Ty_n - Ty_m\| = \|a_n \mu x_n + w_{n-1}\|$ where $w_{n-1} \in E_{n-1}$

$= \mu \|y_n + \frac{w_{n-1}}{\mu}\|$ $\quad y_n \in E_n \setminus E_{n-1}$, $\frac{w_{n-1}}{\mu} \in E_{n-1}$

$\geq \frac{\mu}{2} > 0$

So $\{Ty_n\}$ contains no Cauchy sequence, contradiction

(6b) If $L\Phi = \mu_n \Phi$, $\Phi \neq 0$ for some $n = 1, 2, \ldots$

Then $\Phi \in \text{Span}\{\Phi_J\}$ (See page 3)

$\mu_J = \mu_n$.

(6c) Each eigenvalue μ_n of T has finite multiplicity

That is $\dim \ker(T - \mu_n I) < \infty$

(7) $\{\Phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for H iff $\mu = 0$ is not an eigenvalue of L.

<u>Proof</u> Since $\sum_{k=1}^{\infty} |\langle v, \Phi_k \rangle|^2 \leq \|v\|^2$

the series $\sum_{k=1}^{\infty} \langle v, \Phi_k \rangle \Phi_k$ converges to $w = \sum_{k=1}^{\infty} \langle v, \Phi_k \rangle \Phi_k \in H$

~~From (4), $Lw = \sum_{k=1}^{\infty} \mu_k \langle w, \Phi_k \rangle \Phi_k$~~

<u>Claim</u> $Lw = \sum_{k=1}^{\infty} \langle v, \Phi_k \rangle L\Phi_k$

<u>Proof</u> $\| Lw - \sum_{k=1}^{n} \langle v, \Phi_k \rangle L\Phi_k \| = \| L(w - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \Phi_k) \|$

$\leq \|L\| \, \| w - \sum_{k=1}^{n} \langle v, \Phi_k \rangle \Phi_k \|$

$\downarrow n \to \infty$

0

Also, $Lv = \sum_{k=1}^{\infty} \langle v, \Phi_k \rangle L\Phi_k$

Let $h = v - w$. Then $Lh = 0$.

($\Leftarrow$) If $\mu = 0$ is not an eigenvalue, $Lh = 0$ implies $h = 0$

So $v = w$. Done

($\Rightarrow$) If $\{\phi_k\}$ is a maximal orthonormal set ~~Then $Lv=0$~~

Consider $v \in H$ s.t. $Lv = 0$

$$Lv = \sum_{k=1}^{\infty} \mu_k \langle v, \phi_k \rangle \phi_k = 0$$

So $\langle v, \phi_k \rangle = 0 \quad \forall k$

So $v = 0$ since $\{\phi_k\}$ is maximal.

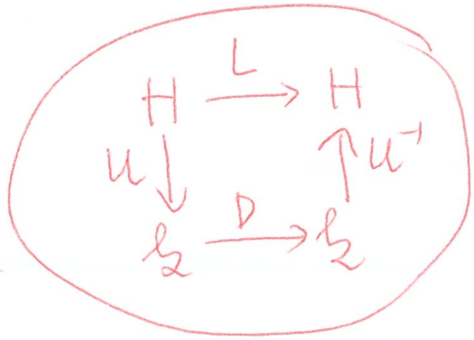
Spectral Theorem Let $L: H \to H$ compact, self-adjoint, $\dim H = +\infty$.

There exists orthonormal eigenvectors $\{\phi_k\}$ and nonzero eigenvectors $\{\mu_k\}$

$L\phi_k = \mu_k \phi_k$, $k = 1, 2, \ldots$, $|\mu_1| \geq |\mu_2| \geq \ldots$

$\lim \mu_n = 0$

and $Lv = \sum_{k=1}^{\infty} \mu_k \langle v, \phi_k \rangle \phi_k \quad \forall v \in H$



In addition, $\{\phi_k\}$ is an orthonormal basis iff 0 is not an eigenvalue of H.

Spectral Decomposition, let $\{e_n\}$ be an orthonormal basis of all eigenvectors with eigenvalues $\{\mu_n\}_{n=1}^{\infty} \cup \{0\}$.

Define $U: H \to l_2$ $\qquad$ $U^{-1}: l_2 \to H$

$Ux = (\langle x, e_1 \rangle, \langle x, e_2 \rangle, \ldots)$ $\qquad$ $U^{-1}(y_1, y_2, \ldots, y_n, \ldots) = \sum_{n=1}^{\infty} y_n e_n$

$D: l_2 \to l_2$

3.7 Sturm-Liouville Problem

The next three pages are taken from Dr. Siegel's notes for AMATH 731.

4.10 Sturm-Liouville Problem

Consider the Sturm-Liouville problem:

$$\begin{cases} Lu + \lambda r u = 0 \quad \text{on} \quad [a,b],\ Lu = (pu')' + qu \\ R_1 u = 0,\ R_2 u = 0,\ R_1 u = \alpha_1 u(a) + \alpha_2 u'(a), R_2 u = \beta_1 u(b) + \beta_2 u'(b)\,, \end{cases}$$

$|\alpha_1| + |\alpha_2| > 0$ and $|\beta_1| + |\beta_2| > 0$, $p, p', q, r \in C[a,b]$, $p(x),\ r(x) > 0$ on $[a,b]$, $D(L) = \{f \in C^2[a,b] :\ R_1 f = 0,\ R_2 f = 0\}$, $L : D(L) \to C[a,b]$.

Theorem 4.16: *If $\lambda = 0$ is not an eigenvalue of L then*

$$L^{-1}v(x) = \int_a^b g(x,y)v(y)dy \quad \textit{where} \quad g \in C([a,b]^2) \quad \textit{and} \quad g(y,x) = g(x,y)\,.$$

Proof: $L[u] = pu'' + p'u' + qu = 0$ has nonzero solutions

$$u_1(x),\ u_2(x) \quad \text{on} \quad [a,b] \quad \text{so that} \quad R_1 u_1 = 0,\ R_2 u_2 = 0$$

Consider the Wronskian $w(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$.

$$w'(x) = -\frac{p'(x)}{p(x)} w(x) \Rightarrow \ln\left|\frac{w(x)}{w(a)}\right| = -\ln\left|\frac{p(x)}{p(a)}\right| \Rightarrow w(x)p(x) = w(a)p(a)$$

$w(a) \neq 0$, since if $w(a) = 0$ then $R_1 u_2 = 0$ and then u_2 is a nonzero solution to $Lu_2 = 0,\ R_1 u_2 = 0,\ R_2 u_2 = 0$, a contradiction. Thus $w(x) \neq 0 \quad \forall x \in [a,b]$ so that u_1 and u_2 are independent solutions. Given $v \in C[a,b]$ we solve $\begin{cases} L[u] = v \\ R_1 u = R_2 u = 0 \end{cases}$ by variation of parameters:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + z(x),\ z(x) = u_1(x)v_1(x) + u_2(x)v_2(x)$$

$$\begin{cases} u_1 v_1' + u_2 v_2' &= 0 \\ u_1' v_1' + u_2' v_2' &= \frac{v}{p} \end{cases} \Rightarrow \begin{cases} v_1' &= -\dfrac{u_2 v}{pw} = \dfrac{-u_2 v}{p(a)w(a)} \\ v_2' &= \dfrac{u_1 v}{pw} = \dfrac{u_1 v}{p(a)w(a)} \end{cases}$$

Choose c_1, c_2 to satisfy the BC's:

$$R_1 u = 0 \Rightarrow c_2 = 0$$

$$R_2 u = 0 \Rightarrow c_1 = \int_a^b \frac{u_2(y)v(y)dy}{p(a)w(a)}$$

$$\Rightarrow u(x) = u_1(x)\int_a^b \frac{u_2(y)v(y)}{p(a)w(a)}dy + \int_a^x \frac{[-u_1(x)u_2(y) + u_2(x)u_1(y)]}{p(a)w(a)}v(y)dy$$

$$= \int_x^b \frac{u_1(x)u_2(y)}{p(a)w(a)}\,v(y)dy + \int_a^x \frac{u_2(x)u_1(y)}{p(a)w(a)}v(y)dy$$

$$= \int_a^b g(x,y)v(y)dy,\; g(x,y) = \frac{1}{p(a)w(a)}\cdot\begin{cases} u_1(x)u_2(y), & a \le x \le y \le b \\ u_2(x)u_1(y), & a \le y \le x \le b \end{cases}$$

Example 4.11: $Lu = u'', u(0) = u(1) = 0 \Rightarrow p = 1$

$$u_1(x) = x,\; u_2(x) = x - 1 \Rightarrow w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 1$$

$$\Rightarrow g(x,y) = \begin{cases} x(y-1), & 0 \le x \le y \le 1 \\ (x-1)y, & 0 \le y \le x \le 1 \end{cases}$$

We now return to the general Sturm-Liouville problem. Suppose that $\lambda = 0$ is not an eigenvalue of L. $Lu + \lambda r u = 0 \Rightarrow u = -\lambda L^{-1}(ru)$.

Let $Tu(x) = -L^{-1}(ru) = -\int_a^b g(x,y)r(y)u(y)dy.$ $\Rightarrow u = \lambda T(u)$.

Let $\mu = \dfrac{1}{\lambda}$ (for $\lambda \neq 0$) $\to T(u) = \mu u$.

Let $\langle u, v\rangle_r = \int_a^b u(x)v(x)r(x)dx$, $\|u\|_r = \sqrt{\int_a^b u^2(x)r(x)dx}$ since $0 < r_0 \le r(x) \le r_1$, on $[a,b]$, r_0, r_1 constants,

$\|u\|_r$ is equivalent to $\|u\| = \sqrt{\int\limits_a^b u^2(x)dx}$.

$$\langle Tv, w\rangle_r = -\int\limits_a^b r(x)w(x)\int\limits_a^b g(x,y)r(y)v(y)dydx$$
$$= -\int\limits_a^b r(y)v(y)\int\limits_a^b g(y,x)r(x)w(x)dxdy = \langle v, Tw\rangle_r$$

Thus T is self-adjoint and compact on $(L_2[a,b], \langle\,,\rangle_r)$. By the Spectral Theorem T has eigenfunctions ϕ_i with eigenvalues μ_i, $\mu_i \neq 0 : T\phi_i = \mu_i\phi_i$ or $-L^{-1}(r\phi_i) = \mu_i\phi_i$

$$\phi_i \in L_2[a,b] \Rightarrow \phi_i \in C[a,b] \Rightarrow \phi_i \in D(L) = \{f \in C^2[a,b] : R_1 f = R_2 f = 0\}$$
$$= T(C[a,b])$$

Claim: $\{\phi_i\}$ is an orthonormal basis for $L_2[a,b]$ (equivalently $\mu = 0$ is not an eigenvalue for T).

$$Tf = \sum_{i=1}^{\infty}\langle Tf, \phi_i\rangle\phi_i\,.$$

Let $S = D(L)$. S is dense in $L_2[a,b]$ (with respect to the L_2 norm). Given $h \in L_2[a,b]$ there exists $\{h_n\} \subset S$ so that $h_n \overset{L_2}{\to} h$.

$$h_n = \sum_{i=1}^{\infty}\langle h_n, \phi_i\rangle\phi_i \quad \text{and} \quad \sum_{i=1}^{\infty}\langle h, \phi_i\rangle\phi_i \quad \text{converges to} \quad \overline{h}$$

$$\|h_n - \overline{h}\| = \|\sum_{i=1}^{\infty}\langle h_n - h, \phi_i\rangle\phi_i\| \leq \|h_n - h\|$$

Thus $h_n \to \overline{h}$ so $\overline{h} = h \Rightarrow h = \sum_{i=1}^{\infty}\langle h, \phi_i\rangle\phi_i$.

Therefore $\{\phi_i\}$ is an orthonormal basis for $L_2[a,b]$.

If $\lambda = 0$ is an eigenvalue for L with regard to $R_1 u = R_2 u = 0$ take λ^* not an eigenvalue. Replace $q(x)$ by $\hat{q}(x) = q(x) + \lambda^* r(x)$, $\hat{L}u = (pu')' + \hat{q}u$. $\hat{L}$ has eigenvalues $\hat{\lambda}_n = \lambda_n - \lambda^*$ which are never zero.

Theorem 4.17: *The Sturm-Liouville problem has a set of eigenfunctions $\{\phi_n\}$ which form an orthonormal basis for $L_2[a,b]$.*

More is known: $\lambda_1 < \lambda_2 < \cdots$, $\lim\limits_{n\to\infty}\lambda_n = \infty$, each eigenvalue has multiplicity one.

3.8 Sobolev Spaces

Definition 1. *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Then*

1. $C^k(G) = \{u : G \to \mathbb{R} \quad \text{s.t. } u \text{ has continuous partial derivatives of orders } m = 0, 1, \ldots, k\}$.
2. $C^\infty(G) = \{u : G \to \mathbb{R} \quad \text{s.t. } u \text{ has continuous partial derivatives of orders } m = 0, 1, \ldots\}$.
3. $C_0^\infty(G) = \{u \in C^\infty(G)$ *s.t. u vanishes outside a compact subset C of G that depends on u, i.e., $u(x) = 0$ for all $x \in G - C\}$.*

Proposition 25. *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Then $L_2(G) = \overline{C^\infty(G)} = \overline{C_0^\infty(G)}$. That is, for every $u \in L_2(G)$, there exists $\{u_n\} \subset C_0^\infty(G)$ such that $u_n \to u$ in $L_2(G)$.*

Sketch of the proof. Main idea: using mollifier, an important smoothing technique. The details can be found in Zeidler's book, pages 186-189.

- Consider

$$\Phi(x) = \begin{cases} c\, e^{\frac{1}{|x|^2 - 1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

  The constant c is chosen so that $\int_{\mathbb{R}^n} \Phi(x) dx = 1$. Verify that $\Phi \in C_0^\infty(\mathbb{R}^n)$.

- For each $\varepsilon > 0$, define

$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \Phi\left(\frac{x}{\varepsilon}\right), \quad G_\varepsilon = \{x \in G : \operatorname{dist}(x, \partial G) > \varepsilon\}.$$

  Verify that $\Phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ and $\Phi_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$ for all $\varepsilon > 0$.

- For each $u \in L_2(G)$, set $u = 0$ outside G. Define

$$u_\varepsilon(x) := \int_{\mathbb{R}^n} \Phi_\varepsilon(x - y) u(y) dy.$$

  Verify that $u_\varepsilon \in C^\infty(G_\varepsilon),\ u_\varepsilon \in L_2(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $L_2(G)$ as $\varepsilon \to 0$.

□

Lemma 9 (Variational Lemma)**.** *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$ and $u \in L_2(G)$ such that*

$$\int_G uv dx = 0 \quad \forall v \in C_0^\infty(G).$$

Then $u(x) = 0$ for almost all $x \in G$. In addition, if $u \in C(G)$ then $u(x) = 0$ for all $x \in G$.

Proof. Since $L_2(G) = \overline{C_0^\infty(G)}$, there exists $\{u_n\} \subset C_0^\infty(G)$ such that $u_n \to u$. Then

$$\langle u, u \rangle = \langle u, \lim_{n\to\infty} u_n \rangle = \lim_{n\to\infty} \langle u, u_n \rangle = 0.$$

So $u(x) = 0$ for almost all $x \in G$. $\square$

Recall Integration by Parts

1. In 1D, $u, v \in C^1[a,b]$, then $\int_a^b u'v dx = uv \mid_a^b - \int_a^b uv' dx$.
   In addition, if $v(a) = v(b) = 0$, then $\int_a^b u'v dx = -\int_a^b uv' dx$.

2. In $\mathbb{R}^n$, let G be an open set in $\mathbb{R}^n$. Then

$$\int_G u D^\alpha \Phi \, dx = (-1)^{|\alpha|} \int_G D^\alpha u \Phi \, dx \quad \text{for } u \in C^k(G),\ \Phi \in C_0^\infty(G),$$

   where $\quad \alpha = (\alpha_1, \ldots, \alpha_n)$ and $D^\alpha \Phi = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \Phi.$

Below is the definition of weak derivatives from Zeidler's book.

Definition 2 (Weak Derivatives)**.** *Let G be a nonempty open set in $R^n, n \geq 1$. Let $u, w \in L_2(G)$ and suppose*

$$\int_G u \partial_j \Phi \, dx = -\int_G w \Phi \, dx, \quad \text{for all } \ \Phi \in C_0^\infty(G).$$

Then w is called an $\alpha^{th}-$weak partial derivative of u, where $\alpha = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $1's$ is at the j^{th}-position.

Here is the general definition of weak derivatives.

Definition 3. *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Let $u, w \in L^1_{loc}(G)$ where*

$$L^1_{loc}(G) = \{v : G \to \mathbb{R} \quad s.t. \quad v \in L_1(V) \text{ for each } V \subset \overline{V}_{compact} \subset G\}.$$

Suppose

$$\int_G u \, D^\alpha \Phi \, dx = (-1)^{|\alpha|} \int_G w \, \Phi \, dx \quad \text{for all } \ \Phi \in C_0^\infty(G).$$

Then w is called an α^{th}-weak partial derivative of u.

Lemma 10. *A weak α^{th}-partial derivative of u if exists, is uniquely defined up to a set of measure zero.*

Proof. Assume $w, \widetilde{w} \in L^1_{loc}(G)$ satisfying the formula. Then

$$\int_G (w - \widetilde{w}) \, \Phi \, dx = 0.$$

By the variational lemma, $w - \widetilde{w} = 0$ a.e. $\square$

Example 1. *Consider* $u : (-1,1) \to \mathbb{R},\ u(x) := |x|$ *for all* $x \in (-1,1)$. *Then the following function is the weak derivative of* u *in the weak sense.*

$$w(x) = \begin{cases} -1 & \text{if} \quad -1 < x < 0 \\ c & \text{if} \quad x = 0 \\ 1 & \text{if} \quad 0 < x < 1 \end{cases}$$

where c *is fixed, but otherwise arbitrary real number.*

Proof. Let $\Phi \in C_0^\infty(-1,1)$. Then

$$\int_{-1}^{1} u\Phi'\,dx = \int_{-1}^{0} u\Phi'\,dx + \int_{0}^{1} u\Phi'\,dx = -\int_{-1}^{0} x\Phi'\,dx + \int_{0}^{1} x\Phi'\,dx.$$

Using integration by parts, we have

$$-\int_{-1}^{0} x\Phi'\,dx + \int_{0}^{1} x\Phi'\,dx = \int_{-1}^{0} \Phi\,dx - \int_{0}^{1} \Phi\,dx = -\int_{-1}^{1} w\Phi\,dx,$$

which implies w is the derivative of u in the weak sense. □

Example 2. *Consider*

$$u(x) = \begin{cases} x & \text{if} \quad 0 < x \le 1 \\ 2 & \text{if} \quad 1 < x < 2 \end{cases}$$

The function u *does not have a weak derivative.*

Proof. Let $\Phi \in C_0^\infty(0,2)$. Suppose there exists $w \in L_2(0,2)$ such that

$$\begin{aligned} -\int_{0}^{2} w\Phi\,dx &= \int_{0}^{2} u\Phi' dx \\ &= \int_{0}^{1} u\Phi' dx + \int_{1}^{2} u\Phi' dx \\ &= -\int_{0}^{1} \Phi dx - \Phi(1), \end{aligned}$$

where the third line is obtained by integration by parts on the right hand side. Therefore,

$$\Phi(1) = \int_{0}^{2} w\Phi\,dx - \int_{0}^{1} \Phi dx \quad \forall \Phi \in C_0^\infty(0,2).$$

Consider $\{\Phi_n\} \subset C_0^\infty(0,2)$ such that $\Phi_n(1) = 1$, $0 \le \Phi_n(x) \le 1$ for all $x \in (0,2)$ and $\Phi_n(x) \to 0$ as $n \to \infty$ for all $x \in (0,2) \setminus \{1\}$. Then

$$\lim_{n\to\infty} \left(\int_{0}^{2} w\Phi_n\,dx - \int_{0}^{1} \Phi_n dx \right) = 0,$$

but

$$\lim_{n\to\infty} \Phi_n(1) = 1,$$

a contradiction. This completes the proof. $\square$

Definition 4 (Sobolev space $W_p^k(G)$)**.** *Let G be a nonempty open set in $\mathbb{R}^n, n \geq 1$. Denote*

$$W_2^1(G) = \{u \in L_2(G) \quad s.t. \quad \partial_j u \text{ exists in the weak sense and } \partial_j u \in L_2(G) \quad \forall j = 1, \ldots, n\}.$$

On $W_2^1(G)$, define

$$\langle u, v\rangle_{1,2} := \int_G \left(uv + \sum_{j=1}^{n} \partial_j u \, \partial_j v \right) dx,$$

$$\|u\|_{1,2} := \left(\int_G u^2 \, dx + \sum_{j=1}^{n} \int_G (\partial_j u)^2 \, dx \right)^{1/2}.$$

In general, fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. Define

$$W_p^k(G) = \{u \in L^1_{loc}(G) \;\; s.t. \;\; D^\alpha u \text{ exists in the weak sense and } D^\alpha u \in L_p(G) \text{ for all } |\alpha| \leq k\}.$$

On $W_p^k(G)$, define

$$\|u\|_{k,p} := \left(\sum_{|\alpha| \leq k} \int_G |D^\alpha u|^p \, dx \right)^{1/p}.$$

Denote $H^k(G) = W_2^k(G)$.

Theorem 1. *For each $k = 1, 2 \ldots$ and $1 \leq p < \infty$, the Sobolev space $W_p^k(G)$ is a Banach space and $W_2^k(G)$ is a Hilbert space, provided we identify two functions whose values differ only on a set of measure zero.*

Sketch of the proof. We will sketch the proof for $W_2^1(G)$.

- Verify that $W_2^1(G)$ is an inner product space.
- Verify that $W_2^1(G)$ is a Banach space.
  - Let $\{u_n\} \subset W_2^1(G)$ be a Cauchy sequence. For every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

    $$\|u_n - u_m\|_{1,2} \leq \varepsilon \quad \forall n, m \geq N_\varepsilon.$$

    Since $\|v\|_{1,2} \geq \|v\|_2$ and $\|v\|_{1,2} \geq \|\partial_j v\|_2$ for all $v \in W_2^1(G)$, the sequences $\{\partial_j u\}_n$ for every $j = 1, \ldots, n$, and $\{u_n\}_n$ are Cauchy sequences in $L_2(G)$. Since $L_2(G)$ is a Banach space, there exists $w_j, u \in L_2(G)$ such that

    $$\lim_{n\to\infty} \|\partial_j u_n - w_j\|_2 = 0, \quad \forall j = 1, \ldots, n \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u\|_2 = 0.$$

- Show that $w_j = \partial_j u$ in the weak sense. Indeed, from

$$\int_G u_n \partial_j \Phi dx = -\int_G \partial_j u_n \, \Phi dx,$$

letting $n \to \infty$, we have

$$\int_G u \partial_j \Phi dx = -\int_G w_j \Phi \, dx,$$

which implies $w_j = \partial_j u$ in the weak sense.

- Finally, show that $\|u_n - u\|_{1,2} \to 0$ as $n \to \infty$.

□

Definition 5. *Let G be a nonempty open subset in $\mathbb{R}^n, n \geq 1$. Let $\overset{\circ}{W}{}_2^{1}(G)$ be the closure of $C_0^\infty(G)$ in the Hilbert space $W_2^1(G)$. That is, $u \in \overset{\circ}{W}{}_2^{1}(G)$ if and only if there exists $\{u_m\} \subset C_0^\infty(G)$ such that $\|u_m - u\|_{1,2} \to 0$ as $m \to \infty$.*

Proposition 26. *The space $\overset{\circ}{W}{}_2^{1}(G)$ is a real Hilbert space.*

Proof. Hint: $C_0^\infty(G)$ is a linear subspace of the Hilbert space $W_2^1(G)$ and the closure of a linear subspace of a Hilbert space is also a Hilbert space. □

Proposition 27. *Let $G = (a, b) \subset \mathbb{R}$, where $-\infty < a < b < \infty$. If $u \in \overset{\circ}{W}{}_2^{1}(G)$, there exists a unique continuous function $v : [a, b] \to \mathbb{R}$ such that $u(x) = v(x)$ for almost all $x \in (a, b)$ and $v(a) = v(b) = 0$. In addition*

$$\|v\|_\infty \leq (b-a)^{1/2} \left(\int_a^b (u')^2 \, dx \right)^{1/2} \leq (b-a)^{1/2} \|u\|_{1,2}.$$

Recall Some notations.

Let G open $\subset \mathbb{R}^n$, $n \geq 1$.

$C^\infty(G) = \{u : G \to \mathbb{R} : u \text{ is infinitely differentiable}\}$

$C_0^\infty(G) = \{u \in C^\infty(G)$ s.t $u(x) = 0$ for all $x \in G - C$, C: a compact subset of $G\}$

$L_p(G) = \{u : G \to \mathbb{R}$ s.t $\int_G |u|^p dx < \infty\}$ for $1 \leq p < \infty$

$L^1_{loc}(G) = \{u : G \to \mathbb{R}$ s.t $u \in L_1(V)$ for all $V \subset \overline{V}_{compact} \subset G\}$
$= \{$locally integrable functions$\}$

$W_p^k(G) = \{u \in L^1_{loc}(G)$ s.t $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L_p(G)$ for all $|\alpha| \leq k\}$

↑ Banach spaces

$H^k = W_2^k(G) = \{u \in L^1_{loc}(G)$ s.t. $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L_2(G)$ for all $|\alpha| \leq k\}$

↑ Hilbert spaces

$H^0 = L_2(G)$

Norm on $W_p^k(G)$: $\|u\|_{k,p} = \left[\sum_{|\alpha| \leq k} \int |D^\alpha u|^p dx\right]^{1/p}$, $1 \leq p \leq \infty$, $k = 0, 1, 2, \ldots$

Weak-derivatives: Let $u \in L^1_{loc}(G)$. Then $w \in L^1_{loc}(G)$ is the α^{th}-weak partial derivative of u if

$$\int_G u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_G w \phi \, dx \quad \text{for all } \phi \in C_0^\infty(G).$$

$\overset{\circ}{W}{}_p^k(G) = \overline{C_0^\infty(G)}$ in $(W_p^k(G), \|\cdot\|_{k,p})$

$L_2(G) = \overline{C_0^\infty(G)}$ in $(L_2(G), \|\cdot\|_2)$

Note: $\overset{\circ}{W}{}_p^k(G) = \{u \in W_p^k(G)$ s.t $D^\alpha u = 0$ on ∂G for all $|\alpha| \leq k-1\}$

Some Useful Inequalities

(1) **Hölder's Inequality**: Assume $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Then if $u \in L_p(G)$ and $v \in L_q(G)$, we have

$$\int_G |uv|\,dx \le \|u\|_{L_p(G)} \|v\|_{L_q(G)}$$

(2) **Minkowski's Inequality** Assume $1 \le p \le \infty$ and $u, v \in L_p(G)$. Then

$$\|u+v\|_{L_p(G)} \le \|u\|_{L_p(G)} + \|v\|_{L_p(G)}$$

(3) **Cauchy-Schwarz Inequality** Let X be an inner product space and $x, y \in X$

Then $|\langle x, y\rangle| \le |\langle x, x\rangle|^{1/2} \langle y, y\rangle^{1/2}$

For example, $X = L_2(G)$, $f, g \in L_2(G)$ (real spaces)

$$\left|\int_G f g\,dx\right| \le \left(\int_G f^2 dx\right)^{1/2} \left(\int_G g^2 dx\right)^{1/2}.$$

---

(4) **Generalized Hölder's Inequality**

$$\int_G |u_1 \ldots u_m|\,dx \le \|u_1\|_{L_{p_1}(G)} \cdots \|u_m\|_{L_{p_m}(G)}$$

where $u_k \in L_{p_k}(G)$, $\infty > p_1, \ldots, p_m \ge 1$

$$\sum_{k=1}^{m} \frac{1}{p_k} = 1.$$

Sobolev's Inequalities – to ~~discover~~ discover embeddings of various Sobolev spaces, If $u \in W_p^1(G)$, does u belong to certain other spaces

1. Gagliardo – Nirenberg – Sobolev Inequality for $W_p^1(G)$, $G_{open} \subset \mathbb{R}^n$, $n \geq 1$.

Theorem 1 Assume $1 \leq p < n$. There exists a constant C, depending only on p and n such that

$$\boxed{\|u\|_{L_{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L_p(\mathbb{R}^n)} \quad \text{for all } u \in C_0^1(\mathbb{R}^n)}$$

where $p^* = \dfrac{np}{n-p}$.

Note: The constant C does not depend on the size of the support of u.

2. Poincaré – Friedrichs Inequality for $\mathring{W}_p^1(G)$, $G_{open,\,bounded} \subset \mathbb{R}^n$, $n \geq 1$.

Theorem 2 Assume $1 \leq p < n$. There exists a constant C, depending only on p, q, n, and G such that

$$\boxed{\|u\|_{L_q(G)} \leq C \|Du\|_{L_p(G)} \quad \text{for all } u \in \mathring{W}_p^1(G), \ \text{for all } q \in [1, p^*]}$$

Theorem 3 $\mathring{W}_2^1(G)$, $G_{open,\,bounded} \subset \mathbb{R}^n$, $n \geq 1$, $p = 2$

Then there exists a constant C such that

$$\boxed{\|u\|_{L_2(G)} \leq C \|Du\|_{L_2(G)} \quad \text{for all } u \in \mathring{W}_2^1(G)}$$

References: L.C. Evans, Partial Differential Equations, page 277

Zeidler, page 136

D. Siegel, page 68

DEFINITION. *If $1 \le p < n$, the* Sobolev conjugate *of p is*

$$p^* := \frac{np}{n-p}. \tag{8}$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p. \tag{9}$$

The foregoing scaling analysis shows that the estimate (5) can only possibly be true for $q = p^*$. Next we prove this inequality is in fact valid.

THEOREM 1 (Gagliardo–Nirenberg–Sobolev inequality). *Assume $1 \le p < n$. There exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \tag{10}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Now we really do need u to have compact support for (10) to hold, as the example $u \equiv 1$ shows. But remarkably the constant here does not depend at all upon the size of the support of u.

Proof. 1. First assume $p = 1$.

Since u has compact support, for each $i = 1, \dots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i;$$

and so

$$|u(x)| \le \int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)|\, dy_i \quad (i = 1, \dots, n).$$

Consequently

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}. \tag{11}$$

Integrate this inequality with respect to x_1:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&= \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&\le \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_i \right)^{\frac{1}{n-1}},
\end{aligned} \tag{12}$$

the last inequality resulting from the general Hölder inequality (§B.2).

Now integrate (12) with respect to x_2:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}\,dx_1dx_2 \leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dy_2\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{\substack{i=1\\ i\neq 2}}^{n} I_i^{\frac{1}{n-1}}\,dx_2,$$

for

$$I_1 := \int_{-\infty}^{\infty}|Du|\,dy_1, \quad I_i := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dy_i \quad (i = 3,\dots,n).$$

Applying once more the extended Hölder inequality, we find

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}\,dx_1dx_2 &\leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dy_2\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dy_1dx_2\right)^{\frac{1}{n-1}}\\ &\quad \prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1dx_2dy_i\right)^{\frac{1}{n-1}}.\end{aligned}$$

We continue by integrating with respect to $x_3,\dots,x_n$, eventually to find

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}\,dx &\leq \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|\,dx_1\dots dy_i\dots dx_n\right)^{\frac{1}{n-1}}\\ &= \left(\int_{\mathbb{R}^n}|Du|\,dx\right)^{\frac{n}{n-1}}.\end{aligned} \tag{13}$$

This is estimate (10) for $p = 1$.

2. Consider now the case that $1 < p < n$. We apply estimate (13) to $v := |u|^{\gamma}$, where $\gamma > 1$ is to be selected. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n}\left|D|u|^{\gamma}\right|dx = \gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|\,dx\\ &\leq \gamma\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|Du|^p\,dx\right)^{\frac{1}{p}}.\end{aligned} \tag{14}$$

We choose γ so that $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1}$. That is, we set

$$\gamma := \frac{p(n-1)}{n-p} > 1,$$

in which case $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$. Thus, in view of (5), estimate (14) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}. \qquad \square$$

THEOREM 2 (Estimates for $W^{1,p}, 1 \le p < n$). *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \le p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)}, \tag{15}$$

the constant C depending only on p, n, and U.

Proof. Since ∂U is C^1, there exists according to Theorem 1 in §5.4 an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$, such that

$$\begin{cases} \bar{u} = u \text{ in } U,\ \bar{u} \text{ has compact support, and} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}. \end{cases} \tag{16}$$

Because $\bar{u}$ has compact support, we know from Theorem 1 in §5.3 that there exist functions $u_m \in C_c^\infty(\mathbb{R}^n)$ $(m = 1, 2, \dots)$ such that

$$u_m \to \bar{u} \quad \text{in } W^{1,p}(\mathbb{R}^n). \tag{17}$$

Now according to Theorem 1, $\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$ for all $l, m \ge 1$. Thus

$$u_m \to \bar{u} \quad \text{in } L^{p^*}(\mathbb{R}^n) \tag{18}$$

as well. Since Theorem 1 also implies $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du_m\|_{L^p(\mathbb{R}^n)}$, assertions (17) and (18) yield the bound

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le C\|D\bar{u}\|_{L^p(\mathbb{R}^n)}.$$

This inequality and (16) complete the proof. $\square$

THEOREM 3 (Estimates for $W_0^{1,p}, 1 \le p < n$). *Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \le p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, the constant C depending only on p, q, n and U.

Step 1 (**) $\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \le \left[\int_{\mathbb{R}^n} |Du| dx\right]^{n/n-1}$, (case $p=1$)

Step 2

For general p, applying (**) for $v = |u|^\gamma$ where $\gamma > 1$ to be selected

$$|D|u|^\gamma| = \gamma |u|^{\gamma-1} |Du|$$

Then
$$\left[\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx\right]^{\frac{n-1}{n}} \le \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| dx$$
$$\le \gamma \left[\int_{\mathbb{R}^n} \left(|u|^{\gamma-1}\right)^{\frac{p}{p-1}} dx\right]^{\frac{p-1}{p}} \left[\int_{\mathbb{R}^n} |Du|^p dx\right]^{1/p}$$

Choose γ so that $\frac{\gamma n}{n-1} = (\gamma-1)p/(p-1)$. $\gamma = \frac{p(n-1)}{n-p} > 1$

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \le \gamma \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{1/p}.$$

$\frac{\gamma n}{n-1} = \frac{np}{n-p}$

$\frac{n-1}{n} - \frac{p-1}{p} = \frac{np-p-np+n}{np}$

Next, going from $C_0^\infty(\mathbb{R}^n)$ to $\overset{\circ}{W}{}_p^1(G)$

$u \in \overset{\circ}{W}{}_p^1(G)$, $\exists \{u_m\} \subset C_0^\infty(\mathbb{R}^n)$ $\|u_m - u\|_{1,p} \xrightarrow[m\to\infty]{} 0$.

Extend $u_m = 0$ on $\mathbb{R}^n - \overline{G}$. we get the $\|u\|_{L_{p^*}(G)} \le C \|Du\|_{L_p(G)}$.

$\|u\|_{L_q(G)} \le C \|u\|_{L_{p^*}(G)}$ if $1 \le q \le p^*$. We get the Poincaré.

An application of Theorem 1

let G open, bounded and ∂G is C^1. $1 \le p < n$

Then if $u \in W^{1,p}(G)$, $u \in L_{p^*}(G)$ and

$$\|u\|_{L_{p^*}(G)} \le C \|u\|_{1,p}.$$

Constant C depends on p, n, G.

Case $p > n$ Morrey's Inequality (See Evans' book)

If $u \in W^{1,p}(G)$ then $u \in C^{0,\gamma}(G)$

Hölder continuous.

Given G open, bounded.

Application The existence Theorem for the Dirichlet Problem.

Theorem 4 (Dirichlet Principle)

Let G be a nonempty bounded open set in $\mathbb{R}^n$, $n = 1, 2, \ldots$

Given $f \in L_2(G)$, $g \in W_2^1(G)$

Consider the generalized Dirichlet problem

$$\min_{u - g \in \mathring{W}_2^1(G)} \frac{1}{2} \int_G \sum_{j=1}^n (\partial_j u)^2 \, dx - \int_G f u \, dx \qquad (1*)$$

along with the generalized boundary-value problem

$$\int_G \sum_j \partial_j u \, \partial_j v \, dx = \int_G f v \, dx \quad \text{for all } v \in \mathring{W}_2^1(G), \quad u - g \in \mathring{W}_2^1(G). \qquad (2*)$$

<u>Notation</u> "$u-g\in \mathring{W}_2^1(G)$" corresponds to the boundary condition

$u-g=0$ on ∂G.

<u>Conclusion</u> (1) The problem (1*) has a unique solution $u\in W_2^1(G)$

(2) This is also the unique solution $u\in W_2^1(G)$ for (2*)

<u>Proof</u> <u>Recall</u> Main Theorem on Quadratic variational Problem

H: Hilbert, $a: H\times H\to\mathbb{R}$ symmetric, bilinear, continuous, coercive and $b\in H^*$

Then the variational problem $\min\limits_{v\in H} \frac{1}{2}a(v,v)-b(v)$

has a unique solution

~~[crossed out]~~ for all $u,v\in W_2^1(G)$

$$a(u,v):=\int_G \sum^2 \partial_j u\,\partial_j v\,dx \quad , \quad b_1(v)=\int_G f v\,dx .$$

Let $w=u-g\in\mathring{W}_2^1(G)$ then (1*) can be rewritten as

$$\min_w \frac{1}{2}a(w+g,w+g)-b_1(w+g) \quad , \quad w\in\mathring{W}_2^1(G)$$

$$\min_w \frac{1}{2}a(w,w)-\underbrace{(-a(w,g)+b_1(w))}_{b(w)}$$

$$\min_{w\in\mathring{W}_2^1(G)} \frac{1}{2}a(w,w)-b(w) \qquad H=\mathring{W}_2^1(G)$$

<u>Verify</u> a is bilinear, symmetric   easy ✓

<u>Verify a is bounded</u>

$$|a(v,w)| \leq \int_G \sum |\partial_j u \, \partial_j w| \, dx$$

Cauchy-Schwarz

$$\leq \sum \left[\int_G |\partial_j u|^2 dx\right]^{1/2} \left[\int_G |\partial_j w|^2 dx\right]^{1/2}$$

$$\leq N \, \|u\|_{1,2} \, \|w\|_{1,2} .$$

<u>Verify a is coercive</u>

$$a(v,v) = \sum \int_G (\partial_j v)^2 dx \geq \frac{c}{c+1} \|v\|_{1,2} \quad \text{for all } v \in H .$$

<u>Verify $b \in H^*$</u> b is linear ✓

b is bounded ✓