AMath 731 Fall 2019 Questions - To Be Updated

(1) State the following theorems:
   (a) Riesz-Fischer theorem
   (b) Banach fixed point theorem, Brower fixed point theorem, and Schauder fixed point theorem
   (c) Picard-Lindelof theorem, Peano theorem
   (d) Arzela-Ascoli theorem, Frechet-Kolomogorov theorem
   (e) State the Neumann series and its properties
   (f) The Hahn-Banach theorems, the generalized mean value theorem
   (g) Orthhogonal projection theorem, Riesz representation theorem
   (h) Lax-Milgram theorem for Hilbert spaces
   (i) Spectral theorem for compact positive self-adjoint linear operators
   (j) to be continued ...

(2) Prove local existence and uniqueness for the initial value problem

$$\begin{cases} u(0) & = 1 \\ \dfrac{du}{dt} & = u^2(t) + t, \quad t \in (0, T). \end{cases}$$

Give a lower bound for T and the length of the time interval for which the solution is guaranteed to exist.
Hint: Use Banach fixed point theorem or the Picard-Lindelof theorem.

(3) Show that the boundary value problem

$$u'' - \varepsilon u^2 = f(x), \quad x \in (0,1)$$
$$u(0) = \alpha \quad \text{and} \quad u'(1) = \beta$$

has a unique, continuous solution if $\varepsilon > 0$ is small enough, where $f(x)$ is a smooth function on $[0, 1]$.

(4) Let X be a Banach space and $F : X \to X$ be a smooth map. Suppose that $x_* \in X$ such that $F(x_*) = 0$ and the Frechet derivative $DF(x_*)$ is invertible. Given any starting point x_0, consider the Newton iteration scheme

$$x_{k+1} = G(x_k) \quad \text{where} \quad G(x) = x - DF(x)^{-1}F(x).$$

   (a) Not in the FINAL. Show that $G(x_*) = x_*$, $DG(x_*) = 0$ and there exists a closed ball B about x_* such that $\|DG(x)\| \leq \dfrac{1}{2}$ for all $x \in B$. (Hint: you do not need to compute $DG(x)$, only $DG(x_*)$)
   (b) Show that $G(x) \in B$ for all $x \in B$.
   (c) Show that $G : B \to B$ is a contraction.
   (d) Show that $\lim_{k\to\infty} x_k = x_*$ for any $x_0 \in B$.

(5) Let K be a continuous function $[0,1]\times[0,1]\to\mathbb{R}$. Show that for λ small enough, for any $g\in C[0,1]$, there exists a unique solution $f\in C[0,1]$ to

$$f(x)=g(x)+\lambda\int_0^1 K(x,y)f(y)dy.$$

(6) Let X and Y be two Banach spaces and M be a dense subset of X (i.e., for every $x\in X$, there exists a sequence $\{x_n\}\subset M$ such that $x_n\to x$). Prove that every bounded linear operator $T:M\to Y$ has a unique bounded linear extension $\hat{T}\ :\ X\to Y$ (i.e., $\hat{T}\in B(X,Y)$ and $\hat{T}|_M=T$) . Show that $\|T\|=\|\hat{T}\|$.
Hint: For each $x\in X$, by assumption, there exists a sequence $\{x_n\}\subset M$ such that $x_n\to x$. Define $\hat{T}(x)=\lim_{n\to\infty}T(x_n)$.

(7) Let $\Omega\subset\mathbb{R}$ be open and bounded. Suppose $K\in C(\overline{\Omega}\times\overline{\Omega})$ and $T:L_2(\Omega)\to L_2(\Omega)$ is defined by

$$Tf(x):=\int_\Omega K(x,y)f(y)dy.$$

(a) Show that T is well-defined.
(b) Prove that T is a compact operator.
(c) Prove that T is a bounded linear operator. Compute $\|T\|$.

(8) Let X be a Banach space and consider $GL(X,X)$ be the set of all isomorphisms from X to X. Prove that $GL(X,X)$ is an open set of $B(X)$. (Hint: Recall that $(1-x)^{-1}=\sum_{n=0}^{\infty}x^n$, for $x\in\mathbb{C},|x|<1$.)

(9) Let X be a normed linear space, Y be a linear subspace of X, and $z\in X\setminus Y$ such that

$$d=\operatorname{dist}(z,Y)=\inf_{y\in Y}\|y-z\|>0.$$

Prove that there exists $f\in X^*$ such that $\|f\|\le 1$, $f(z)=d$, and $f(y)=0$ for all $y\in Y$. (Hint: Work in $X_0=Y+\mathbb{K}z$ and extend to X using the Hahn-Banach Theorem.)

(10) (a) Show that $Y_1=\{x=(x_1,x_2,\ldots)\in\ell_2\mid x_{2n}=0,\forall n\in\mathbb{N}\}$ is a closed subspacce of ℓ_2 and find $Y_1^\perp$.
(b) What is $Y_2^\perp$ if $Y_2=Span(e_1,\ldots,e_n)\subset\ell_2$ where $e_{j,k}=\delta_{jk}$?
(c) Take $x=(1,2,3,4,0,0,\ldots)$. What are the orthogonal projection $P_{Y_1}(x)$ and $P_{Y_2}(x)$ and $dist(x,Y_1)$ and $dist(x,Y_2)$?

(11) Let H be a Hilbert space and $A:H\to H$ be a linear operator. If for all $x,y\in H$, the following holds

$$\langle x,Ay\rangle=\langle Ax,y\rangle,$$

prove that A is bounded.

(12) Consider an operator $A : L_2[0,1] \to L_2[0,1]$ defined by

$$Af(x) = \int_0^x f(t)dt.$$

(a) Show that $A^*f(x) = \int_x^1 f(t)dt$ and $A^*Af(x) = \int_0^1 (1 - \max(x,t))f(t)dt$.
(b) Show that A^*A is self-adjoint, positive, and compact on $L_2[0,1]$.
(c) Show that if $\lambda \neq 0$ is an eigenvalue of A^*A with eigenfunction f, then $\lambda f'' = -f$ almost everywhere on $[0,1]$, $f'(0) = 0$, and $f(1) = 0$.
(d) Show that $\|A^*A\| = 4/\pi^2$ and $\|A\| = 2/\pi$.

(13) Define the linear operator $T : L_2[0,1] \to L_2[0,1]$ by

$$Tf(x) = \int_0^x \int_y^1 f(z)dz\, dy.$$

(a) Prove that T is self-adjoint.
(b) Prove that T is compact.
(c) Find an orthogonal basis for $L_2[0,1]$ based on the eigenvalues of this operator. (Hint: differentiate twice and consider carefully the boundary conditions that must be satisfied.)

(14) Let H be a separable Hilbert spaces, $\{v_i\}_{i=1}^\infty$ a countable orthonormal base, and $L : H \to H$ the linear operator that $Lv_i = \sum_{j=1}^{\infty} 2^{-(i+j)} v_j$. Show that L is compact.

(15) Suppose that H is a Hilbert space and $V \subset H$ is a nonempty, closed convex set. Let $x \in H$ such that $x \notin V$.
(a) Prove that there is a unique $y \in V$ such that $\|x - y\|$ is minimal.
(b) If V is a linear subspace, prove that $x - y$ is orthogonal to V.

(16) (a) Let H be a real Hilbert space and Y be a closed subspace of H. Consider a bilinear functional $a : H \times H \to \mathbb{R}$ that has the following properties:
(i) There is a constant $C > 0$ such that

$$|a(u,v)| \leq C\|u\|_H \|v\|_H, \quad \text{for all } \; u, v \in H.$$

(ii) There is a constant $\gamma > 0$ such that

$$a(v,v) \geq \alpha \|v\|_H^2, \quad \text{for all } v \in Y.$$

Given $\varphi \in Y^*$ and $y_0 \in H$. Prove that there is a unique element $u \in Y + y_0 \subset H$ such that

$$a(v,u) = \varphi(v) \quad \forall v \in Y.$$

Moreover,

$$\|u\|_H \leq \frac{1}{\alpha}\|\varphi\|_{Y^*} + \left(\frac{C}{\alpha} + 1\right) \|y_0\|_H.$$

Hint: Mimic the proof of the Lax-Milgram Theorem.

(b) Not in the Final. Consider the boundary value problem:

$$-u_{xx} + (1+y)u = f, \quad \text{for } (x,y) \in (0,1)^2,$$
$$u(0,y) = 0, \quad u(1,y) = \cos(y), \quad \text{for } y \in (0,1).$$

(i) Find the associated variational problem. In which space should f lie?
(ii) Show that there exists a unique solution to this problem.

(17) Consider the following linear operator T on the separable Hilbert space $H = (\ \ell_2,\ \|\ \cdot\ \|_2\ )$:

$$T : H \to H$$
$$T(x_1, x_2, x_3, \ldots) = \left(\frac{1}{2}(x_1+x_2), \frac{1}{4}(x_2+x_3), \ldots, \frac{1}{2^n}(x_n+x_{n+1}), \ldots\right)$$

Denote

$$e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots) \quad \text{where 1 is at the } n^{th}\text{-position.}$$

Then it is easy to see that $\{e_n\}$ is an orthonormal basis for H.

(a) Prove that T is bounded and find an upper bound for $\|T\|$.
(b) Determine T^*.
(c) Is T self-adjoint?
(d) Verify that $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$.
(e) Let $T_n(v) = \sum_{k=1}^{n} \langle v, e_k \rangle Te_k,\ \forall v \in H$. Explain why $T_n : H \to H$ is compact.
(f) Prove that $\|T_n - T\| \to 0$ and T is a compact operator.
(g) Prove that for every $k \geq 1$, there exists $v_k \in H$, $v_k \neq 0$ such that $Tv_k = \frac{1}{2^k} v_k$.
(h) Alternate question for the previous question: Find v_1 and v_2 nonzeros in H such that $Tv_k = \frac{1}{2^k} v_k$ for $k = 1, 2$. Are v_1 and v_2 orthogonal to each other? Why does it not violate the Spectral Theorem that we studied in class?

(18) Find the set of eigenvalues of the linear operator $T : (\ell_2, \|\cdot\|_2) \to (\ell_2, \|\cdot\|_2)$ defined through

$$T(x_1, x_2, x_3, \ldots,) := (x_2, x_3, x_4, \ldots).$$

(19) Consider the set $X = C([0,a])$ of continuous and real-valued functions on the interval $[0,a]$. We define the following norms on X,

$$\|f\|_\infty := \max_{x \in [0,a]} |f(x)|, \quad \|f\|_a := \max_{x \in [0,a]} |f(x)|\, e^{-ax}.$$

Determine the operator norms $\|T\|_\infty$ for $T : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|_\infty)$ and $\|T\|_a$ for $T : (X, \|\cdot\|_a) \to (X, \|\cdot\|_a)$, where the operator T is defined as follows,

$$(Tf)(x) := \int_0^x t f(t)\, dt.$$

(20) Consider the Sobolev norm $\|f\|_{1,2} := \left[\int_a^b \left(f^2(x)+(f'(x))^2\right) dx\right]^{1/2}$ for any real valued function $f \in C^1[a,b] = \{g : [a,b] \to \mathbb{R} \mid g \text{ and } g' \text{ are continuous functions on } [a,b]\}$.

(a) Show that $C^1[a,b]$ is not complete with respect to the Sobolev norm $\|\cdot\|_{1,2}$. (Hint: Consider $a=-1, b=1$, and the sequence

$$f_n(x) = \begin{cases} 0, & \text{for } -1 \le x \le 0, \\ x^{1+\frac{1}{n}}, & \text{for } 0 \le x \le 1. \end{cases}$$

and show that $\{f_n\}$ is a Cauchy sequence w.r.t $\|\cdot\|_{1,2}$, converges pointwise to some function $f(x)$ but $f \notin C^1[-1,1]$.)

(b) Show that $\|f\|_\infty \le \left[(b-a)^{-1/2} + (b-a)^{1/2}\right] \|f\|_{1,2}$ for all $f \in C^1[a,b]$.

(c) Given the for every $g \in W_2^1[a,b]$, there exists a sequence $\{g_n\}_{n=1}^\infty \subset C[a,b]$ such that $\|g_n - g\|_{1,2} \to 0$ as $n \to \infty$. Prove that $W_2^1[a,b] \subset C[a,b]$.

(d) Find a function in $C[a,b]$ but not in $W_2^1[a,b]$.

(21) If $f \in L^p(\Omega)$ show that

$$\|f\|_p = \sup \left| \int_\Omega fg\, dx \right| = \sup_\Omega |fg|\, dx,$$

where the supremum is taken over all $g \in L^q(\Omega)$ such that $\|g\|_q \le 1$, where $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(22) Let H be a Hilbert space and $T \in B(H)$, $T \neq 0$. Let $N = \ker T$ and suppose M is a closed linear subspace such that $H = N \bigoplus M$ (i.e., $H = N + M$ and $N \cap M = \{0\}$). Suppose $Im(T)$ is closed. Note that we may take $M = N^\perp$, but this is not required.

(a) Show that T is one-to-one on M.

(b) Show that there is some $\gamma > 0$ such that

$$\gamma \|x\| \le \|Tx\| \quad \forall x \in M.$$

(c) Let γ_M^* be the maximal γ that can be taken above for M. Show that $\gamma_M^* \le \gamma_{N^\perp}^*$.

(23) Define $T : \ell_2 \to \ell_2$ by $Tx = y$ where $y_j = \alpha_j x_j$ and $\alpha_j \to 0$ as $j \to \infty$. Show that T is compact.

(24) Let H be a complex Hilbert space and $T : H \to H$ a bounded linear operator. Note that both T^*T and TT^* are positive self-adjoint operators (An operator L is positive if $\langle Tx, x \rangle \ge 0$ for all $x \in H$). Prove that $I + T^*T$ and $I + TT^*$ must be boundedly invertible.