Research Statement

Gabriel Islambouli

Overview

The topology of smooth 4-dimensional manifolds is among the deepest, yet poorly understood areas of low dimensional topology. My research is focused on trying to understand smooth 4-manifolds using singularity theory and simplicial complexes associated to surfaces. To pass to these theories, I primarily use trisections and, more generally, multisections, which are decompositions of smooth, compact, orientable 4-manifolds into simple pieces. A nice feature of these decompositions is that they encode the smooth topology of the manifold as sets of curves on a surface. This allows us to recast questions about 4-manifolds into questions about these curves, which tend to be more combinatorial in nature. In this vein, Section 3 of this statement describes joint work with Michael Klug which provides a combinatorial answer to the basic question: "When is an orientable 4-manifold the boundary of an orientable 5-manifold."

The theory of 4-manifolds presents unique difficulties due to the vast difference between the smooth and topological categories. In contrast to other dimensions, a fixed homeomorphism class of 4-manifolds may have infinitely many diffeomorphism types within it. Section 2 describes joint work with Patrick Naylor which explains how this difference is reflected in a multisection, giving a construction for passing between multisections of any two topologically, but not smoothly equivalent 4-manifolds. In a related direction, multisections also hold the potential for developing new combinatorial invariants of smooth 4-manifolds. Towards this end, Section 4 describes joint work which provides a set of moves relating any two multisections of a fixed 4-manifold. This section also describes independent work constructing the only known trisections which require these moves.

1 Background and Constructions of multisections

A common technique in topology is to build a space out of simple pieces, and analyze how these pieces fit together to make a more complex space. A useful building block in this regard is an n-dimensional genus g handlebody, which is the manifold obtained by taking a neighbourhood of a bouquet of g circles in $\mathbb{R}^n$. In 1898, Poul Heegaard [7] introduced a decomposition of 3-manifolds into two handlebodies which we now call a Heegaard splitting. Motivated by this decomposition, Gay and Kirby [4] introduced the notion of a trisection, which is a decomposition of a 4-manifold into three 4-dimensional handlebodies. They also showed that an arbitrary smooth, orientable, compact 4-manifold admits a trisection. In [11], we generalize their decomposition to an arbitrary number of handlebodies, and we present this definition below. A schematic of a trisection and a 4-section can be seen in Figure 1, and can be used as a reference while absorbing the formal definition.

Definition 1.1. Let X be a smooth, orientable, closed, connected 4-manifold. A genus g *n-section*, or *multisection* of X is a decomposition $X = X_1 \cup X_2 \cup \cdots \cup X_n$ such that:

1. X_i is a 4-dimensional handlebody;
2. $X_1 \cap X_2 \cap \ldots \cap X_n = \Sigma_g$, a closed orientable surface of genus g;
3. $X_i \cap X_j = H_{i,j}$ is a 3-dimensional handlebody if $|i-j| = 1$, and $X_i \cap X_j = \Sigma_g$ if $|i-j| > 1$;
4. $\partial X_i \cong \#^{k_i} S^1 \times S^2$ has a Heegaard splitting given by $H_{(i-1),i} \cup_\Sigma H_{i,(i+1)}$.

We call the X_i **sectors** and a union of one or more consecutive sectors a **subsection**

One of the nice features that multisections share with Heegaard splittings is that they can be described as sets of curves on a surface. A **cut system** for a genus g surface is a collection of g disjoint simple closed curves which cut the surface into a $2g$ punctured sphere. An elementary application of a theorem of Laudenbach and Poenaru [14] shows that all of the structure of an n-section can be encoded by an ordered list of n cut systems on the trisection surface. Some examples of such diagrams can be found at the bottom of Figure 1. We call the surface, together with the cut systems, a **multisection diagram**.

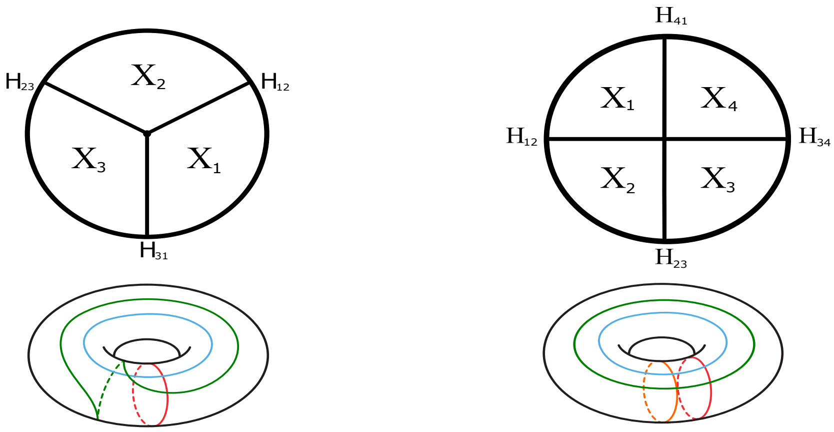

Figure 1: Top: Schematics for a trisection and a 4-section. Each X_i is diffeomorphic to a 4-dimensional handlebody and each H_{ij} is diffeomorphic to a 3-dimensional handlebody. The H_{ij} meet in a closed surface indicated by a dot in the center of the disk. Bottom: Diagrams for the trisection of $\mathbb{C}P^2$ (left) and the 4-section of $S^2 \times S^2$ (right).

Multisections arise in a wide array of natural settings, which may be surprising given the abundance of structure present. Trisections were initially constructed by Gay and Kirby in [4] by taking a particular map of a 4-manifold to a disk, dividing the disk into three pieces, and taking the inverse image of each piece. Later, Baykur and Saeki [1] showed how to take an arbitrary stable map to the disk and modify it so that it can be divided into three nice pieces as before. To get the flavor of how one can modify such functions, Figure 3 shows some valid moves on the critical image of a map of a 4-manifold to the disk. More indirectly, Gay [5] showed how to associate a multisection to a loop of generalized Morse functions on a surface. Inspired by this work, joint work with Michael Klug [10] shows how to construct a multisection from a loop in the pants complex; a viewpoint which will be elaborated on in Section 3.

2 Cut and Paste operations on multisections

Since multisections present manifolds as curves on a surface, it is natural to attempt to realize the cut and paste operations ubiquitous in 4-manifold topology as operations on these curves. For the case of trisections, there has been slow but steady progress on this problem. In practice however, implementing cut and paste operations on a trisection can be quite unwieldy. In [11], we introduced the notion of a multisection with the purpose of simplifying this procedure.

When cutting along a subsection of a multisection, the two resulting pieces inherit the structure of a Heegaard splitting on their boundary. If we choose a gluing map that respects such a Heegaard splitting, then the resulting manifold inherits a multisection structure as well. If one has a map of a 3-manifold in mind, it is always possible to make the gluing compatible with some Heegaard splitting. Moreover, in many important cases, it is actually practical to find concrete representatives of such maps.

Once a map respects a Heegaard splitting, a nice consequence is that the map is entirely determined by its restriction to the surface. This allows one to locate and carry out cut and paste operations entirely within the diagrams. A schematic of the effect of regluing on the diagram, as well as a particular example of this effect is shown in Figure 2. The diagram in the bottom left of this figure represents a 4-section of $S^2 \times S^2$ and a Dehn twist about the red curve takes this to a 4-section of $S^2 \tilde{\times} S^2$ shown on the right. The main result of [11] is the following theorem, which states that this regluing procedure completely captures the subtle difference between homeomorphism and diffeomorphism in dimension 4.

Theorem 2.1. *Suppose that X and X' are smooth, closed, oriented simply connected 4-manifolds that are homeomorphic, but not diffeomorphic. Then, there exists a surface Σ, and cut systems C_1, C_2, C_3, C_4, and C_4', such that:*

1. *$(\Sigma; C_1, C_2, C_3, C_4)$ is a 4-section diagram for X;*
2. *$(\Sigma; C_1, C_2, C_3, C_4')$ is a 4-section diagram for X';*
3. *There exists a map $\tau : \Sigma \to \Sigma$ such that, $\tau(C_1) = C_1$, $\tau(C_3) = C_3$, and $\tau(C_4) = C_4'$ where τ is the restriction of a cork twist to Σ.*

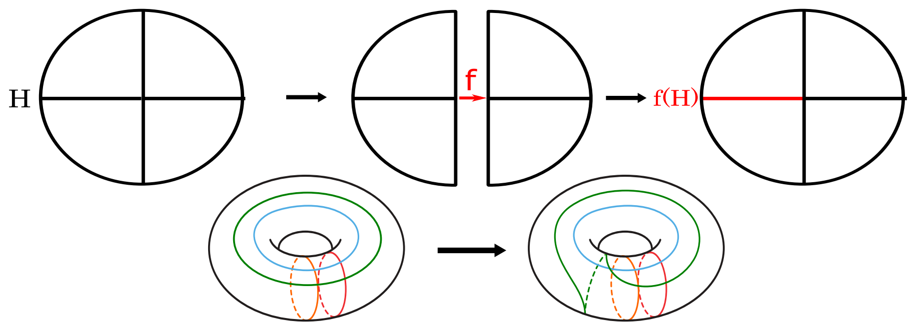

Figure 2: Top: A schematic illustrating the process of cutting out and regluing a subsection by a map respecting the boundary Heegaard splitting. The handlebodies on the boundary do not change, but the handlebodies in the subsection being re-glued may change. Bottom: The gluing map transforming $S^2 \times S^2$ to the nontrivial S^2 bundle over S^2 amounts to modifying the green curve by a Dehn twist about the common orange and red curve.

3 Multisections and complexes associated to surfaces

Among the deepest and most subtle relations in 3-manifold theory is the connection between the topology of a 3-manifold and the Hempel distances of its Heegaard splittings. The Hempel distance is defined as a distance in the curve complex of the Heegaard surface, and contains deep geometric information about the underlying 3-manifold. A central part of my research has been exploring how to make use of the complexes associated to a multisection surface to gain knowledge about the underlying 4-manifold.

A particularly important complex associated to a surface is the pants complex. The vertices of this complex correspond to isotopy classes of $3g-3$ mutually non-separating disjoint curves which separate the surface into $2g-2$ 3-hold spheres, these being the namesake "pants." Edges and 2-cells are associated to configurations of curves in such a way that the complex becomes connected and simply connected. In joint work with Michael Klug [10], we show how to take a loop L in the pants complex, L, and construct a closed, smooth 4-manifold, $\mathcal{X}^4_C(L)$. Our main theorem is the following existence result.

Theorem 3.1. *For every closed, smooth, orientable 4-manifold X^4, there exists a closed loop L in the pants complex of a closed surface so that X is diffeomorphic to $\mathcal{X}^4_C(L)$.*

Moreover, given two such loops and a homotopy between them, we can build a unique cobordism between the manifolds corresponding to the loops. Since the pants complex is simply connected [6], we can contract a loop to give a null-bordism of the corresponding manifold. This gives our main application, which is an elementary proof that the cobordism group of closed, smooth, orientable 4-manifolds is isomorphic to the integers. We are also able to go backwards and gain information about the pants complex using this correspondence, gaining insight into the types of simplicial disks a loop can bound.

In order to explore the connection between 4-manifolds with boundary and complexes associated to surfaces, joint work with Castro, Miller, and Tomova [2] introduces a new complex which we call the p-cut complex. As before, a loop in this complex corresponds to a smooth orientable manifold, where in this case the manifold has boundary. Inspired by work of Kirby and Thompson [13], we define a new invariant of a 4-manifold, $r\mathcal{L}(\mathcal{M}^4)$, which is defined to be minimum length of a loop representing M^4 in this complex. One nice feature of this invariant is that it detects B^4 among rational homology balls.

Theorem 3.2. *If M is a rational homology ball with $r\mathcal{L}(\mathcal{M}) = 0$, then $M \cong B^4$.*

Shifting viewpoints slightly, we can also use the pants complex to "compare" 4-manifolds. This is the subject of my work in [8] which extends work of Johnson [12] to four dimensions. To compare two trisections we can cut one sector out of each trisection and glue along the resulting boundary components. This identifies two of the sets of curve in each trisection diagram. Since each trisection was completely described by three sets of curves, the "difference" between these two trisections is contained in the "difference" between the unidentified curves. More precisely, we extend these different sets of curves to pants decompositions and compute the distance between these pants decompositions in the pants

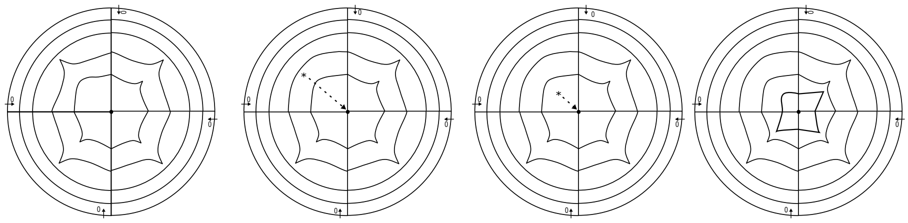

Figure 3: The sequence of moves making up a UPW move. Here, it is used to remove the cusp in the top left sector, leaving this sector with no cusps in the final critical image. A sector with no cusps can be contracted to give a multisection with fewer sectors (in this case, the result will be a trisection).

complex. There were a few choices suppressed in this description, but minimizing over all such choices gives a well defined distance, $\boldsymbol{D(T_1, T_2)}$, between two trisections.

Given two $(g, k)-$trisections, we can stabilize both of them to obtain $(g + 3, k + 1)$-trisections, which we can again compare in the pants complex. One may of course repeat this action, and compare the twice stabilized trisections to each other. Given a trisection T, let $\boldsymbol{T^n}$ be the trisection obtained from T by stabilizing n times. The main result of [8] is the following theorem.

Theorem 3.3. *Let T_1 and T_2 be trisections of M_1 and M_2 with some sector of T_1 diffeomorphic to some sector of T_2. The natural number $lim_{n\to\infty} D(T_1^n, T_2^n)$ exists, and depends only on the underlying manifolds M_1 and M_2. We can then define a distance between manifolds, $\boldsymbol{D(M_1, M_2)}$, to be $lim_{n\to\infty} D(T_1^n, T_2^n)$ where T_1 is any trisection of M_1, and T_2 is any trisection of M_2.*

4 Stable equivalence of Multisections

Two trisections $X = X_1 \cup X_2 \cup X_3$ and $X' = X_1' \cup X_2' \cup X_3'$ are said to be diffeomorphic if there exists a diffeomorphism, $f : X \to X'$, such that $f(X_i) = X_i'$. In [4], Gay and Kirby defined a stabilization operation on trisections, and showed that any two trisections of a fixed 4-manifold become diffeomorphic after some number of stabilizations. This theorem brings up a natural question: can a manifold admit non-diffeomorphic trisections of the same genus? Drawing inspiration from geometric group theoretic techniques used in dimension three, my work in [9] affirmatively answers this question in a strong sense.

Theorem 4.1. *For any $k > 1$, there exist infinitely many manifolds admitting $2^k - 1$ non-diffeomorphic $(3k, k)$-trisections.*

It is straightforward to show that the stabilization operation which Gay and Kirby use to relate trisections is insufficient to relate any two multisections of a 4-manifold. To account for this, in [11] Naylor and I introduce an additional move, which we call a UPW move. This move is defined as a modification of the map a multisection induces to the disk. The modification of the critical image of this map is shown in Figure 3. Repeated applications of this move can turn any multisection into a trisection, whereby we prove the following theorem.

Theorem 4.2. *Let X be a smooth, oriented, closed, and connected 4-manifold. Any two multisections of X are related by a sequence of UPW moves, stabilizations, and isotopy through multisections.*

5 Future Work

5.1 Invariants: old and new

Our understanding of the landscape of smooth 4-dimensional manifolds has, time and time again, been challenged by the introduction of new invariants. The first of the truly perspective altering invariants came from work of Donaldson [3]. In modern days, a more computable, and conjecturally equivalent, invariant comes from the theory of Heegaard-Floer homology, as developed by Ozsváth and Szabó [17]. The Ozsváth-Szabó closed 4-manifold invariant is defined using decompositions which resemble multisections quite closely, so it is natural to ask the following question.

Problem 5.1. Compute the Ozsváth-Szabó closed 4-manifold invariant directly from a multisection diagram. Use this calculation to deduce new cut and paste results and distinguish new classes of smooth 4-manifolds.

In addition to being an interesting way of building new 4-manifolds, an understanding of Problem 5.1 would also give subtle invariants of loops of Morse functions and loops in the pants complex. Going the other way, these alternate characterizations of multisections give unorthodox pathways for deducing new invariants of 4-manifolds. My work in [10] shows how to extract a classical invariant, the signature, from a loop in the pants complex. The signature is also known to appear as an element of the cohomology group of the mapping class group of a surface. Madsen and Weiss [15] showed that the cohomology in the "stable range" is generated by the Mumford-Miller-Morita classes, one of which is a multiple of the aforementioned signature element. Since the genus of the surfaces which multisections are represented on can be made arbitrarily large, we can push each surface into a desired stable range leading to the following problem.

Problem 5.2. Interpret the Mumford-Miller-Morita classes of a multisection. Understand how these behave under stabilizations to define invariants of 4-manifolds.

5.2 Interpreting manifolds in complexes associated to surfaces

Work of Meier and Zupan [16] shows how to represent a knotted surface in a 4-manifold on a trisection surface. This amounts to puncturing the original surface in some number of points and connecting these punctures via arcs. By analogy, a wide variety of the techniques used to construct multisections should work to construct a 4-manifold together with an embedded surface.

Problem 5.3. Given a loop of generic smooth functions on a surface with boundary, construct a unique 4-manifold together with an embedded surface. Show that every 4-manifold/surface pair arises in this fashion.

Smoothly knotted surfaces in 4-manifolds are still poorly understood and it is likely that such a construction will lead to novel invariants. In addition, a representation of knotted surfaces as loops in a complex could lead to combinatorial proofs of classical results.

We next turn to cobordisms. By Theorem 3.1, 4-manifolds can be represented by loops in the pants complex. Some more work shows that a map of an annulus corresponds to a 5-dimensional cobordism. It is straightforward to show that not every cobordism arises in this fashion, but it is likely that the most interesting cobordisms can be represented like this.

Problem 5.4. Show that every smooth 5-dimensional h-cobordism can be represented as an annulus in the pants complex. Study the complexity measure of the h-cobordism given by the simplicial area of this annulus.

5.3 Constructions of new multisections

My work described in Section 4 describes the only known way to distinguish trisections of a fixed manifold, and hinges on fundamental group arguments. By contrast, the complexity of smooth 4-dimensional topology is present even among simply connected manifolds. This leads naturally to the following problem.

Problem 5.5. Find two inequivalent trisections of a simply connected 4-manifold of the same genus.

The cut and paste techniques described in Section 2 give a very flexible set of tools to construct new multisections. Perhaps the most interesting of these is the cork twist operation which underlies Theorem 2.1. A result of that theorem is that the entire cork twist operation can be described on the surface as a mapping class. By the Nielsen-Thurston classification, mapping classes of a surface are either reducible, finite order, or pseudo-Anosov. The previously cited work has examples of the first two classes of maps, leading to the following problem.

Problem 5.6. Find an infinite order cork whose restriction to the multisection surface is a pseudo-Anosov map.

Pseudo-Anosov maps usually increase distances in complexes associated to a surface. It is therefore likely that the 4-manifolds constructed in this way are very different from each other, which could be reflected as a complexity in the h-cobordism between them, as in Problem 5.4.

References

[1] R. I. Baykur and O. Saeki. Simplifying indefinite fibrations on 4-manifolds. *ArXiv e-prints*, May 2017.

[2] Nickolas A. Castro, Gabriel Islambouli, Maggie Miller, and Maggy Tomova. The relative $\mathcal{L}$-invariant of a compact 4-manifold. *arXiv e-prints*, page arXiv:1908.05371, August 2019.

[3] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[4] David Gay and Robion Kirby. Trisecting 4–manifolds. *Geom. Topol.*, 20(6):3097–3132, 2016.

[5] David T. Gay. Functions on surfaces and constructions of manifolds in dimensions three, four and five. In *Breadth in contemporary topology*, volume 102 of *Proc. Sympos. Pure Math.*, pages 79–94. Amer. Math. Soc., Providence, RI, 2019.

[6] Allen Hatcher. Pants Decompositions of Surfaces. *arXiv Mathematics e-prints*, page math/9906084, Jun 1999.

[7] P. Heegaard. Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang. Kjöbenhavn. 104 S. 8° (1898)., 1898.

[8] Gabriel Islambouli. Comparing 4-manifolds in the pants complex via trisections. *Algebr. Geom. Topol.*, 18(3):1799–1822, 2018.

[9] Gabriel Islambouli. Nielsen equivalence and trisections of 4-manifolds. *ArXiv e-prints*, April 2018.

[10] Gabriel Islambouli and Michael Klug. Representing smooth 4-manifolds as loops in the pants complex. *arXiv e-prints*, page arXiv:1912.02325, December 2019.

[11] Gabriel Islambouli and Patrick Naylor. Multisections of 4-manifolds. *arXiv e-prints*, page arXiv:2010.03057, October 2020.

[12] Jesse Johnson. Heegaard splittings and the pants complex. *Algebr. Geom. Topol.*, 6:853–874, 2006.

[13] Robion Kirby and Abigail Thompson. A new invariant of 4-manifolds. *Proc. Natl. Acad. Sci. USA*, 115(43):10857–10860, 2018.

[14] François Laudenbach and Valentin Poénaru. A note on 4-dimensional handlebodies. *Bull. Soc. Math. France*, 100:337–344, 1972.

[15] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.

[16] Jeffrey Meier and Alexander Zupan. Genus-two trisections are standard. *Geometry & Topology*, 21(3):1583–1630, 2017.

[17] Peter Ozsváth and Zoltán Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math.*, 202(2):326–400, 2006.