On the bit complexity of some randomized algorithms in real algebraic geometry

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Outline

1. Computing one point in each connected component of a smooth real algebraic set
   - Problem statement
   - Polar varieties
   - The algorithm
   - Weak transversality
   - Quantitative genericity statements
   - Proving the main result
2. Deciding connectivity queries in smooth and bounded real hypersurfaces
   - Problem statement
   - Other work on roadmap computation
   - Estimating the height of the output

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Problem statement

Problem 1

- Suppose that $F = (f_1, \ldots, f_p) \in \mathbb{Z}[X_1, \ldots, X_n]^p$ is a sequence of polynomials.
- Suppose that the ideal $\langle f_1, \ldots, f_p \rangle$ is radical and that $V = V(F) \subset \mathbb{C}^n$ is smooth and equidimensional with dimension $n - p$.
- Compute at least one point in each connected component of $V(F) \cap \mathbb{R}^n$.

Applications

- Used in higher level algorithms.
- Decide if $V(F) \cap \mathbb{R}^n$ has solutions.
- Determine an upper bound on the number of connected components of $V(F) \cap \mathbb{R}^n$.

Starting point

- An algorithm by [Safey El Din, Schost, 2003]
  - Uses random changes of variables that are proven to generically ensure certain desirable geometric properties.
  - Cost given in an algebraic complexity model.

Contributions

- We determine the bit complexity and error probability.
- We provide a quantitative analysis of the genericity properties:
  - Weak transversality.
  - Noether normalization for polar varieties.

Future work

- Reuse the techniques in the analysis of other algorithms.
  - Randomized algorithms for deciding connectivity queries on smooth and bounded real hypersurfaces.

Main result

- Let $F = (f_1, \ldots, f_p) \in \mathbb{Z}[X_1, \ldots, X_n]^p$ with $\deg(f_i) \le d$ and $\mathrm{ht}(f_i) \le b$. Suppose $\langle f_1, \ldots, f_p \rangle$ is radical and $V = V(F) \subset \mathbb{C}^n$ is smooth and equidimensional with dimension $n - p$.
  - The height of a polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$ is the maximum of the logarithms of the absolute values of the coefficients of f.

Theorem

- For $0 < \epsilon < 1$, there exists a randomized algorithm that takes F and ϵ as input and returns a finite set including at least one point on each connected component of $V(F) \cap \mathbb{R}^n$.
- The algorithm succeeds with probability at least $1 - \epsilon$, and otherwise returns a proper subset of the points or FAIL.
- The algorithm uses

$$O^{\sim}(d^{3n+2p+1}(\log 1/\epsilon)(b + \log 1/\epsilon))$$

bit operations. The polynomials in the output have degree at most d^{n+p}, and height

$$O^{\sim}(d^{n+p+1}(b + \log 1/\epsilon)).$$

Main result

- The algorithm uses
$$O^{\sim}(d^{3n+2p+1}(\log 1/\epsilon)(b+\log 1/\epsilon))$$
bit operations. The polynomials in the output have degree at most d^{n+p}, and height
$$O^{\sim}(d^{n+p+1}(b+\log 1/\epsilon)).$$
- Roughly optimal: equal to the output bit-size times the algebraic complexity.
- Close to matching what is implemented in practice, in Maple, available through RAGlib.
- A different algorithm with bit complexity $d^{O(n)}$ [Basu, Pollack, Roy, 2003].
  - This algorithm is general and makes no assumptions on the input polynomials.
  - Given the generality of the algorithm, the constant in the exponent is large (not used in practice).

Outline

- Let $T_{\boldsymbol{x}}V$ denote the Zariski-tangent space to V at $\boldsymbol{x} \in V$. And for $i \in \{1, \ldots, n-p+1\}$, denote by π_i the projection

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{C}^i \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_i). \end{aligned}$$

- The i-th polar variety

$$W(i, V) := \{\boldsymbol{x} \in V \mid \dim \pi_i(T_{\boldsymbol{x}}V) < i\}$$

is the set of critical points of π_i on V.

Determinantal modeling of polar varieties

- Let $\mathrm{jac}(F, i)$ denote the truncated Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial X_{i+1}} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial X_{i+1}} & \cdots & \frac{\partial f_p}{\partial X_n} \end{bmatrix}.$$

  - $W(i, V(F)) = \{\boldsymbol{x} \in \mathbb{C}^n \mid F(\boldsymbol{x}) = 0 \text{ and } \mathrm{rank}(\mathrm{jac}_{\boldsymbol{x}}(F, i)) < p\}$.

- Let $M_{i,1}, \ldots, M_{i,S_i}$ be the p-minors of $\mathrm{jac}(F, i)$.
  - $W(i, V(F)) = V\left(F, M_{i,1}, \ldots, M_{i,S_i}\right)$.

- When $V = V(F) = V(f)$ is a hypersurface, then

$$W(i, V(f)) = V\left(f, \frac{\partial f}{\partial X_{i+1}}, \ldots, \frac{\partial f}{\partial X_n}\right).$$

Example

- Let $f = X_1^2 + X_2^2 + X_3^2 - 1$ and consider
$$V = V(X_1^2 + X_2^2 + X_3^2 - 1) \subset \mathbb{C}^3.$$
- The critical points of the projection
$$\pi_2 : (x_1, x_2, x_3) \mapsto (x_1, x_2)$$
on $V(f)$ are defined by $V\left(f, \frac{\partial f}{\partial X_3}\right)$. Hence, the polar variety is defined by those points where
$$X_1^2 + X_2^2 + X_3^2 - 1 = X_3 = 0.$$

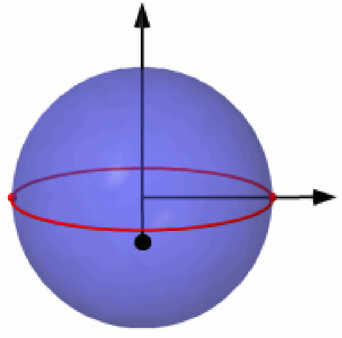


Image from
[Safey El Din, Schost, 2017]

Lagrangian modeling of polar varieties

- Due to the relations between minors of a matrix, the equations

$$\left(F, M_{i,1}, \ldots, M_{i,S_i}\right)$$

  are in general not a complete intersection.
- For both the polynomial system algorithm we use, and an effective Nullstellensatz application, we want equations that define a complete intersection.
- By introducing new indeterminates $(L_1, \ldots, L_p)$, we can model polar varieties as projections of Lagrange systems:

$$V\left(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i), \sum_{i=1}^{p} u_i L_i - 1\right).$$

- The existence of a solution characterizes the set where $\mathrm{jac}(F, i)$ is rank deficient.

Example

- Consider $f = X_1^2 + X_2^2 + X_3^2 - 1$ and $V(X_1^2 + X_2^2 + X_3^2 - 1) \subset \mathbb{C}^3$, then

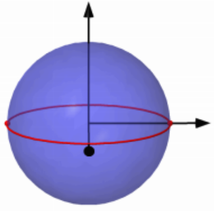
$$\mathrm{jac}(X_1^2 + X_2^2 + X_3^2 - 1, 2) = 2X_3.$$

- The Lagrangian modeling gives

$$V(X_1^2 + X_2^2 + X_3^2 - 1, LX_3, L - 1) = V(X_1^2 + X_2^2 - 1, X_3, L - 1).$$

  The equations on the right hand side are a lexicographically ordered Gröebner basis of the ideal $\langle X_1^2 + X_2^2 + X_3^2 - 1, LX_3, L - 1\rangle$.

- $\pi_{\mathbf{X}}\left(V(X_1^2 + X_2^2 - 1, X_3, L - 1)\right)$ describes $W(2, V(f))$:



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The algorithm

- If we apply a generic change of coordinates $\boldsymbol{A} \in \mathbb{Z}^{n\times n}$ to $F = (f_1, \ldots, f_p)$:

$$F^{\boldsymbol{A}} = \big(f_1(\boldsymbol{AX}), \ldots, f_p(\boldsymbol{AX})\big),$$

  then $W(i, V(F^{\boldsymbol{A}})$ is known to be equidimensional of dimension $(i-1)$ or empty [Bank, Giusti, Heintz, Mbakop, 1997]
  and to be in *Noether position.*

- It then suffices to choose a generic $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{n-p})$ in $\mathbb{Z}^{n-p}$ and solve the systems defined by

$$X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, \left(F^{\boldsymbol{A}}, M_{i,1}^{\boldsymbol{A}}, \ldots, M_{i,S_i}^{\boldsymbol{A}}\right)$$

  for $i = 1, \ldots, n-p+1$.

  - Computes the intersection of $W(i, F^{\boldsymbol{A}})$ with the fiber $\pi_i^{-1}(\sigma_1, \ldots, \sigma_{i-1})$.

- They all admit finitely many solutions.
- The union of their solution sets contains one point on each connected component of $V(F) \cap \mathbb{R}^n$.
  [Safey El Din, Schost, 2003]

The algorithm

- Since the equations

$$\left(F, M_{i,1}, \ldots, M_{i,S_i}\right)$$

are in general not a complete intersection, we instead use the Lagrangian modeling of polar varieties and solve the equations

$$X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, \left(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i), \sum_{i=1}^{p} u_i L_i - 1\right),$$

for $i = 1, \ldots, n-p+1$, and then compute the projections of each solution set on the $\boldsymbol{X}$-space.

Main contributions

- We analyze precisely what conditions on our change of coordinates $\boldsymbol{A} \in \mathbb{Z}^{n\times n}$ guarantee success.
- We revisit key ingredients in the proofs given in
  [Bank, Giusti, Heintz, Mbakop, 1997],
  [Safey El Din, Schost, 2003]
  and we give quantitative versions of these results, bounding the degrees of the hypersurfaces we have to avoid.
- To solve the equations

$$X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, \left(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i), \sum_{i=1}^{p} u_i L_i - 1\right)$$

  we use the algorithm in [Safey El Din, Schost, 2018] for which a complete bit complexity analysis is available.

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Weak transversality

Thom's weak transversality theorem

- Generalizes Sard's lemma: *the set of critical values of a smooth function $\mathbb{R}^n \to \mathbb{R}^m$ have measure zero.*
  - Algebraic versions exist for which the sets of critical values are contained in algebraic sets in the codomain.
- We develop a quantitative version which allows us to bound the degrees of the hypersurfaces we have to avoid.
- The bad parameters show up as the critical values of a smooth function.

Weak transversality

- Let n, s, and m be positive integers, with $m \leq n$, and let

$$\mathbf{\Phi} : \mathbb{C}^n \times \mathbb{C}^s \ \to \mathbb{C}^m$$

be a mapping defined by polynomials in $\mathbb{C}[\boldsymbol{X}, \boldsymbol{\Theta}]$.

- For $\boldsymbol{\vartheta}$ in $\mathbb{C}^s$, let

$$\begin{aligned} \mathbf{\Phi}_{\boldsymbol{\vartheta}} : \mathbb{C}^n &\to \mathbb{C}^m \\ \boldsymbol{x} &\mapsto \mathbf{\Phi}(\boldsymbol{x}, \boldsymbol{\vartheta}). \end{aligned}$$

- A point $\boldsymbol{z}$ is a regular value of $\mathbf{\Phi}$ iff for all $(\boldsymbol{y}, \boldsymbol{\vartheta})$ with $\mathbf{\Phi}(\boldsymbol{y}, \boldsymbol{\vartheta}) = \boldsymbol{z}$, the Jacobian of $\mathbf{\Phi}$ has full rank at $(\boldsymbol{y}, \boldsymbol{\vartheta})$.

Proposition (weak transversality)

- Suppose that $\mathbf{0}$ is a regular value of $\mathbf{\Phi}$. Then there exists a non-zero polynomial $\Gamma \in \mathbb{C}[\boldsymbol{\Theta}]$ of degree at most d^{m+n} such that for $\boldsymbol{\vartheta}$ in $\mathbb{C}^s$, if $\Gamma(\boldsymbol{\vartheta}) \neq 0$, then $\mathbf{0}$ is a regular value of $\mathbf{\Phi}_{\boldsymbol{\vartheta}}$.

Our contribution is the degree estimate.

Example

- Consider $f \in \mathbb{C}[X_1, X_2]$, squarefree, $\deg(f) \leq d$ and $V(f) \subset \mathbb{C}^2$ smooth.
- Let the mapping $\mathbf{\Phi} : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ be defined by

$$\mathbf{\Phi}(X_1, X_2, \Theta) = (X_1 - \Theta, f(X_1, X_2)).$$

- The Jacobian of $\mathbf{\Phi}$ has rank two at any point in $\mathbf{\Phi}^{-1}(\mathbf{0})$:

$$\mathrm{jac}(\mathbf{\Phi}) = \begin{bmatrix} 1 & 0 & -1 \\ \partial f/\partial X_1 & \partial f/\partial X_2 & 0 \end{bmatrix}.$$

  - The assumptions of the proposition apply.

Example (continued)

- Thus $\Gamma \in \mathbb{C}[\Theta]$ exists, with $\deg(\Gamma) \le d^4$, such that when $\Gamma(\vartheta) \neq \boldsymbol{0}$ then $\boldsymbol{0}$ is a regular value of
$$\mathbf{\Phi}_\vartheta(X_1, X_2) = (X_1 - \vartheta, f(X_1, X_2)).$$
- The Jacobian of $\mathbf{\Phi}_{\boldsymbol{\vartheta}}$ has rank two at any point in $\mathbf{\Phi}_{\boldsymbol{\vartheta}}^{-1}(0)$:
$$\mathrm{jac}(\mathbf{\Phi}_{\boldsymbol{\vartheta}}) = \begin{bmatrix} 1 & 0 \\ \partial f/\partial X_1 & \partial f/\partial X_2 \end{bmatrix}.$$
- By the Jacobian Criterion, the ideal $(X_1 - \vartheta, f(X_1, X_2))$ is radical; equivalently, $f(\vartheta, X_2)$ is squarefree.
  - For all ϑ in $\mathbb{C}$ except at most d^4 values.
- Note that using the discriminant of f with respect to X_2 produces the same result.
- This examples illustrates how we apply the result when solving the equations in the main algorithm.

Example (continued)

Compare with the equations solved in the main algorithm

- In the example, $\mathbf{\Phi} : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ is defined by the polynomials

$$X_1 - \Theta, f(X_1, X_2).$$

- Compare with the equations we solve in the main algorithm: (for $i = 1 \ldots, n - p + 1$)

$$X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, \left(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i), \sum_{i=1}^{p} u_i L_i - 1 \right).$$

- Following the same steps as in the example, we bound the degree of a polynomial such that if $\boldsymbol{\sigma}$ is not a zero then these equations have finitely many solutions.

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Genericity properties

- Let $F = (f_1, \dots, f_p) \in \mathbb{C}[X_1, \dots, X_n]^p$.
- For $i = 1, \dots, n - p + 1$,

F satisfies $\boldsymbol{H}_i$ if

1. $W(i, V(F))$ is either empty or $(i-1)$-equidimensional.
2. The Jacobian matrix of the polynomials

$$\big(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i)\big)$$

   has full rank at any $(\boldsymbol{x}, \boldsymbol{l})$ that cancels equations.
3. $W(i, V(F))$ is either empty or in Noether position for π_{i-1}.

Noether position

- An equidimensional algebraic set $X \subset \mathbb{C}^n$ of dimension d is in Noether position for the projection
$$\pi_d : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d)$$
when the extension
$$\mathbb{C}[X_1, \ldots, X_d] \to \mathbb{C}[X_1, \ldots, X_n]/I(X)$$
is integral.
- Consequently, for any $\boldsymbol{x} \in \mathbb{C}^d$, the fiber $X \cap \pi_d^{-1}(\boldsymbol{x})$ has dimension zero (so it is finite and not empty).

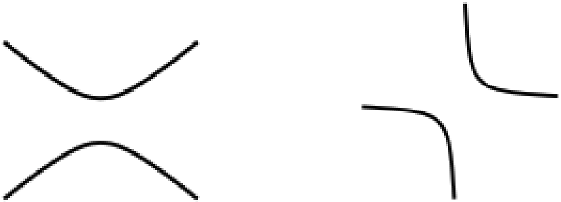

Figure 1: X is in Noether position for π_1 on the left, but not on the right.

- Let $F = (f_1, \ldots, f_p) \in \mathbb{C}[X_1, \ldots, X_n]^p$.
- For $i = 1, \ldots, n - p + 1$,

F satisfies $\boldsymbol{H}_i$ if

1. $W(i, V(F))$ is either empty or $(i-1)$-equidimensional.
2. The Jacobian matrix of the polynomials

$$\big(F, [L_1 \ldots L_p] \cdot \mathrm{jac}(F, i)\big)$$

   has full rank at any $(\boldsymbol{x}, \boldsymbol{l})$ that cancels equations.
3. $W(i, V(F))$ is either empty or in Noether position for π_{i-1}.

Note that $W(i, V(F^{\boldsymbol{A}}))$ may not equal $W(i, V(F))^{\boldsymbol{A}}$, as, for instance, their dimensions may vary.

- For $i = 1, \ldots, n-p+1$, if F satisfies $\boldsymbol{H}_i$, then given $\boldsymbol{\sigma}$ in $\mathbb{Z}^{i-1}$, we further say

F and $\boldsymbol{\sigma}$ satisfy $\boldsymbol{H}_i^{'}$ if

1. 0 is a regular value of the polynomials

$$\big(X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i)\big),$$

in the open set defined by $(L_1 \cdots L_p) \neq (0 \cdots 0)$.

- For $i = 1, \dots, n-p+1$, if F satisfies $\boldsymbol{H}_i$, and F and $\boldsymbol{\sigma}$ satisfy $\boldsymbol{H}'_i$, then given $\boldsymbol{u} \in \mathbb{C}^p$ we further say

$\boldsymbol{u}$ satisfies $\boldsymbol{H}''_i$ if

1. $\boldsymbol{u}$ is such that the projections on the $\boldsymbol{X}$-space of the solutions of

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, \left(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i), \sum_{i=1}^{p} u_i L_i - 1\right) \tag{1}$$

are the solutions of

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, \left(F, M_{i,1}, \dots, M_{i,S_i}\right). \tag{2}$$

Genericity statements

Proposition

- There exists a polynomial

$$\Delta_1 \in \mathbb{C}[(\mathfrak{A}_{j,k})_{1 \leq j,k \leq n}]$$

of degree at most $5n^3(2d)^{5n}$ such that if $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ does not cancel Δ_1, then $\boldsymbol{A}$ is invertible and $F^{\boldsymbol{A}} = F(\boldsymbol{AX})$ satisfies $\boldsymbol{H}_i$:

1. $W(i, V(F))$ is either empty or $(i-1)$-equidimensional.
2. The Jacobian matrix of
$$\big(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i)\big)$$
has full rank at any $(\boldsymbol{x}, \boldsymbol{l})$ that cancels equations.
3. $W(i, V(F))$ is either empty or in Noether position for π_{i-1}.

For all $i \in \{1, \ldots, n-p+1\}$.

Genericity statements

Proposition

- Suppose that $F = (f_1, \dots, f_p)$ satisfies $\boldsymbol{H}_i$ for all $i = 1, \dots, n - p + 1$.
- There exists a polynomial
$$\Delta_2 \in \mathbb{C}[S_1, \dots, S_{i-1}]$$
of degree at most nd^{4n} such that if $\boldsymbol{\sigma} \in \mathbb{C}^{i-1}$ does not cancel Δ_2, then F and $\boldsymbol{\sigma}$ satisfy $\boldsymbol{H}'_i$:
  1. 0 is a regular value of the polynomials
  $$\big(X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i)\big),$$
  in the open set defined by $[L_1 \cdots L_s] \neq [0 \cdots 0]$.

For all $i \in \{1, \dots, n - p + 1\}$.

Proposition

- Suppose that F satisfies $\boldsymbol{H}_i$ and F and $\boldsymbol{\sigma}$ satisfy $\boldsymbol{H}_i^{'}$, for all $i \in \{1, \ldots, n-p+1\}$.
- There exists a polynomial

$$\Delta_3 \in \mathbb{C}[T_1, \ldots, T_p]$$

of degree at most $n(n(d-1))^n$ such that if $\boldsymbol{u} \in \mathbb{C}^p$ does not cancel Δ_3, then $\boldsymbol{u}$ satisfies $\boldsymbol{H}_i''$:

1. $\boldsymbol{u}$ is such that the projections on the $\boldsymbol{X}$-space of the solutions of

$$X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, \left(F, [L_1 \cdots L_p] \cdot \mathrm{jac}(F, i), \sum_{i=1}^{p} u_i L_i - 1\right) \qquad (1)$$

are the solutions of

$$X_1 - \sigma_1, \ldots, X_{i-1} - \sigma_{i-1}, \left(F, M_{i,1}, \ldots, M_{i,S_i}\right). \qquad (2)$$

For all $i \in \{1, \ldots, n-p+1\}$.

Outline

Proving the main result

1. The algorithm first randomly chooses $\boldsymbol{A} \in \mathbb{Z}^{n\times n}$. Using the degree bound for Δ_1, the entries of $\boldsymbol{A}$ are chosen from a sufficiently large set so that by the Schwartz–Zippel lemma
$$\mathbb{P}[\Delta_1(\boldsymbol{A}) = 0] \leq 1 - \epsilon.$$
2. Next, the algorithm chooses $\boldsymbol{\sigma} \in \mathbb{Z}^{n-p}$ at random and we again quantify using the Schwartz–Zippel lemma: we bound
$$\mathbb{P}[\Delta_2(\boldsymbol{\sigma}) = 0 \mid \Delta_1(\boldsymbol{A}) \neq 0] \leq 1 - \epsilon.$$
3. Finally, the algorithm randomly chooses $\boldsymbol{u} \in \mathbb{Z}^p$ and we quantify once more using the Schwartz–Zippel lemma: we bound
$$\mathbb{P}[\Delta_3(\boldsymbol{u}) = 0 \mid \Delta_1(\boldsymbol{A})\Delta_2(\boldsymbol{\sigma}) \neq 0] \leq 1 - \epsilon.$$

Outline

Problem statement

Roadmaps

- A roadmap $\mathscr{R}$ for an algebraic set X is a curve with non-empty and connected intersection with all connected components of X.

Applications

- Deciding connectivity queries.
- Robot motion planning.

Problem 2

- Let f be a squarefree polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ such that $V(f)$ has a finite number of singular points and $V(f) \cap \mathbb{R}^n$ is bounded.
- Compute a roadmap $\mathscr{R}$ of $V(f) \cap \mathbb{R}^n$.

Starting point

- Another algorithm by [Safey El Din, Schost, 2011].
  - Also uses random changes of variables proven to generically ensure weak transversality and Noether position.
  - Recursive algorithm based on calculating polar curves of polar varieties.
  - Cost given in an algebraic complexity model:

$$(nd)^{O(n^{1.5})} \text{ operations in } \mathbb{Q}.$$

Contributions (ongoing)

- Determining the bit complexity and error probability.
- Giving a quantitative analysis of the genericity properties.
  - Weak transversality (reusing techniques from previous analysis).
  - Noether normalization for polar varieties (reusing techniques from previous analysis).
  - Additional genericity properties.

Another genericity property

- Let $F = (f_1, \ldots, f_p) \in \mathbb{Z}[X_1, \ldots, X_n]^p$ with degree $f_i \leq d$. Suppose the F defines a radical ideal and $V(F)$ is equidimensional of dimension $n - p$ with a finite number of singular points and $V(F) \cap \mathbb{R}^n$ bounded.
- For $i = 2, \ldots, (n - p + 3)/2$,

F satisfies $\boldsymbol{G}_i$ if

- $W(1, W(i, V(F)))$ is finite.

  - Proven to hold generically in [Safey El Din, Schost, 2011].

Proposition

- There exists a hypersurface $\Delta \subset \mathbb{C}[(\mathfrak{A}_{j,k})_{1 \leq j,k \leq n}]$ with degree at most

$$n(p+n)^n (2d)^{p+n}$$

with the property that if $\Delta(\boldsymbol{A}) \neq 0$ then $F^{\boldsymbol{A}} = F(\boldsymbol{A}\boldsymbol{X})$ satisfies $\boldsymbol{G}_i$, for all $i \in \{2, \ldots, (n - p + 3)/2\}$.

Outline

Other work on roadmap computation

- $d^{O(n^4)}$, deterministic, semi-algebraic sets, no assumptions [Canny, 1987]
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- $d^{O(n^2)}$, deterministic, semi-algebraic sets, no assumptions [Basu, Pollack, Roy, 1999]

- $(nd)^{O(n^{1.5})}$, randomized, real hypersurfaces, smooth and bounded [Safey El Din, Schost, 2011]
- $d^{O(n^{1.5})}$, deterministic, real algebraic sets, no assumptions [Basu, Roy, Safey El Din, Schost, 2014]

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  [Basu, Roy, Safey El Din, Schost, 2014]

- $(nd)^{O^{\sim}(n)}$, deterministic, real hypersurfaces, no assumptions
  [Basu, Roy, 2014]

Other work on roadmap computation

- $d^{O(n^4)}$, deterministic, semi-algebraic sets, no assumptions
  [Canny, 1987]
- $d^{O(n^2)}$, randomized, semi-algebraic sets, no assumptions
  [Canny, 1987]
- $d^{O(n^2)}$, deterministic, semi-algebraic sets, no assumptions
  [Basu, Pollack, Roy, 1999]

- $(nd)^{O(n^{1.5})}$, randomized, real hypersurfaces, smooth and bounded
  [Safey El Din, Schost, 2011]
- $d^{O(n^{1.5})}$, deterministic, real algebraic sets, no assumptions
  [Basu, Roy, Safey El Din, Schost, 2014]

- $(nd)^{O^{\sim}(n)}$, deterministic, real hypersurfaces, no assumptions
  [Basu, Roy, 2014]
- $(nd)^{O(n \log d)}$, real algebraic sets, smooth and bounded
  [Safey El Din, Schost, 2017]

Other work on roadmap computation

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- $(nd)^{O^{\sim}(n)}$, deterministic, real hypersurfaces, no assumptions [Basu, Roy, 2014]
- $(nd)^{O(n \log d)}$, real algebraic sets, smooth and bounded [Safey El Din, Schost, 2017]
- $(nd)^{O(n \log d)}$, real algebraic sets, smooth (unbounded) [Prebet, Safey El Din, Schost, 2022]

Outline

1. Computing one point in each connected component of a smooth real algebraic set
   - Problem statement
   - Polar varieties
   - The algorithm
   - Weak transversality
   - Quantitative genericity statements
   - Proving the main result
2. Deciding connectivity queries in smooth and bounded real hypersurfaces
   - Problem statement
   - Other work on roadmap computation
   - Estimating the height of the output

Height of the output

- Algebraic complexity and degree of the output: $(nd)^{O(n^{1.5})}$.
- Expect height to be $(nd)^{O(n^{1.5})}$.

Difficulties

- Need to solve polynomial equations with a special shape.
- Two blocks of variables subject to different constraints:
  - $X_1, \ldots, X_i$, high degree and bit-size.
  - $X_{i+1}, \ldots, X_n$, low degree and bit-size.
- Classical arithmetic Bézout gives height $(nd)^{O(n^2)}$.

Solutions

- Multi-projective height techniques that involve the arithmetic Chow ring [Krick, Sombra, D'Andrea, 2012] precisely allow you to handle the two blocks separately.
- We get height $(nd)^{O(n^{1.5})}$.

Thank you.