UNIVERSITY OF WATERLOO
Faculty of Mathematics

Can paradoxical decompositions be explained in pictures?

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MEMORANDUM OF SUBMITTAL

To:	Nico Spronk
From:	Jose Luis Avilez
Address:	200 University Avenue, Waterloo, ON N2L 3G1
Date:	August 20, 2019
Re:	Can paradoxical decompositions be explained in pictures?

As agreed, I have prepared the enclosed report, “Can paradoxical decompositions be explained in pictures?” for my 4B work report and for the Pure Mathematics Department. This report, the fourth of five work reports that the Co-operative Education Program requires that I successfully complete as part of my BMath Co-op degree requirements, has not received academic credit.

The analysis group at the University of Waterloo studies a wide range of topics, amongst which is abstract harmonic analysis. My job required that I give weekly lectures on a subtopic of harmonic analysis called amenability theory.

The Faculty of Mathematics requests that you evaluate this report for command of topic and technical content/analysis. This report is written for someone who is interested in the communication of mathematics; no formal mathematical training is required to understand this report. Following your assessment, the report, together with your evaluation, will be submitted to the Math Undergrad Office for evaluation on campus by qualified work report markers. The combined marks determine whether the report will receive credit.

Thank you for your assistance in preparing this report.

Jose Luis Avilez

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EXECUTIVE SUMMARY

Determining whether a group is amenable or paradoxical is a question of interest in the field of abstract harmonic analysis. Several tests and characterisations discovered in the twentieth century are usually taught to students reading pure mathematics at the graduate level. This report studies one such characterisation and provides an intuitive and graphical framework for it to be understood by students in other field of mathematics as well as non-mathematicians.

In particular, this paper defines amenable and paradoxical groups and studies Følner's theorem from a graphical perspective; intuitively, this states that a group is paradoxical if its graphical representation exhibits many "bottlenecks".

The analysis of Cayley graphs of several groups, such as $\mathbb{Z}_n, \mathbb{Z}, \mathbb{Z}^2, A_5$, and $\mathbb{F}_2$ revealed that the graphical approach to amenability is fairly useful in de-mystifying the statement of Følner's theorem and providing intuition to eyeball test the amenable or paradoxical nature of a group.

Finally, the paper extrapolates from these findings to argue that graphical representations are useful at communicating abstract mathematical findings to people in and outside the field of harmonic analysis.

1 INTRODUCTION

In 1924, mathematicians discovered how to do something alchemists never could: conjure matter out of nothing. That year Stephan Banach and Alfred Tarski, an analyst and a logician, published an algorithm that allowed them to cut a ball into five pieces and re-arrange them in a particular way that would yield two copies of the original ball—all without stretching any of the pieces [4, 5].

As a response to the so-called Banach-Tarski paradox, John von Neumann constructed an object called an "invariant mean" which allowed analysts to study when such paradoxical decompositions occur and when mathematical structures are safeguarded against them [6]. Intuitively, an invariant mean is a uniform probability distribution define over all the subsets of a space—given this we say a space is paradoxical if no such distribution exists.

The field has come a long way since the discovery of Banach and Tarski, but providing intuition about their results remains a challenge. This paper strives to provide a graphical representation of the paradox and the defence against it.

1.1 Mathematical Preliminaries

For the remainder of the paper, let G denote a group; that is a non-empty set G with a binary operation $\cdot$ such that for elements a, b, and c in G we always have four conditions: (i) (closure) $a \cdot b$ is in G, (ii) (associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c$, (iii) (identity) there is an element e in G such that $a \cdot e = a$, and (iv) (inverses) there is an element a^{-1} in G such that $aa^{-1} = e$.

A group G is finitely generated if there is a finite set $S = \{s_1, \ldots, s_n\}$ such that every element in G can be realised as a combination of finitely many elements in S and their inverses.

Two prototypical examples of groups are the integers $\mathbb{Z}$ and the real numbers $\mathbb{R}$, where the operation is addition. The integers are finitely generated, with generating set $S = \{1, -1\}$, as every whole number can be expressed by adding or subtracting 1 sufficiently many times. The real numbers, however, are not finitely generated because their size is an infinity far too large to ever be able to adopt a finite generating set.

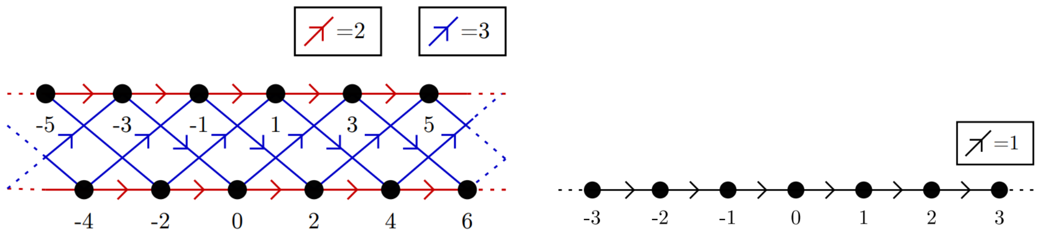
Whenever a group G exhibits a Banach-Tarski paradox, G shall be called paradoxical; in the case G is not paradoxical, it is called amenable.

This paper's tool of choice to study the amenability of a group is called a Cayley graph. Given a group G and a generating set S a coloured graph $\Gamma = (V, E)$ can be constructed as follows:

1. Let the elements g of the group G be the vertices of the graph; in symbols, $V = V(\Gamma) = G$;

2. To each generator element s in S, we assign a colour c_s and for any group element g, we let the pair (g, gs) be an edge in the graph of colour c_s; in symbols, $E = E(\Gamma) = \{(g, gs; c_s) \, : \, g \in G, \, s \in S\}$

For example, the group of integers $\mathbb{Z}$ with the generating sets $S = \{2, 3\}$ and $R = \{1\}$ has the Cayley graphs displayed on Figure 1:



(a) Graph for $S = \{2, 3\}$

(b) Graph for $R = \{1\}$

Figure 1: Cayley graphs for two different generating sets for $\mathbb{Z}$. Adapted from [1].

The philosophy behind the study of amenability of groups is to find as many easy tests—so-called characterisations—to determine whether a group is amenable or not. Some of the more exciting characterisations requires a significant amount of mathematical background, an exercise which is not profitable for the purpose of this manuscript (the curious reader may consult Runde's *Lectures in Amenability Theory* [7]).

2.1 Examples of amenable and paradoxical groups

The easiest source of examples for amenable groups are those which are finite and those which are abelian (that is $a \cdot b = b \cdot a$ for all group elements). In that case, the groups of integers and real numbers are amenable, and so are their products—namely, $\mathbb{Z}^n$ and $\mathbb{R}^n$, the n-dimensional integers and real numbers, are amenable too. Furthermore, the groups of clock arithmetic, denoted $\mathbb{Z}_n$, are also amenable because they are finite—in this group, if we take two elements a and b, the group operation takes their sum, $a + b$, and returns what the remainder after division by n is.

Conversely, the first example of a paradoxical group is $\mathbb{F}_2 = \langle a, b\rangle$, the free group on two generators. More specifically, $\mathbb{F}_2$ stands for all the words that can be written using the letters a, b, a^{-1}, b^{-1}, where group operation is simply word concatenation. Why is this group paradoxical? Imagine $\mathbb{F}_2$ as a dictionary with four volumes: each volume corresponding to the words starting with a, b, a^{-1}, b^{-1}. Suppose you want to sell the whole dictionary[1] to two friends, Alice and Bob, but you only have one copy of each volume. Do you hoodwink your friends? No need. Take the volumes that begin with a^{-1} and b^{-1} and erase the first letter of every word; now, give Alice the a

[1]Those endowed with marketing skills may want to call it a Hyper-Webster, or a Hyper-OED if the Queen's is preferred.

and erased-a^{-1} volumes and give Bob the b and erased-b^{-1} volumes. It will turn out that they each have a complete copy of $\mathbb{F}_2$.

Can this explanation be put in a picture?

2.2 Cayley graphs of amenable and paradoxical groups

The answer to the question above is codified in the following theorem [8, 9]:

(Følner's Theorem - Graph Theoretical Version) *Let G be an infinite discrete finitely generated group. Then G is amenable if and only if the Cheeger constant, $h(G)$, of its Cayley graph is zero.*

The only missing step to demystify this theorem is defining what the Cheeger constant of a Cayley graph is. The easiest way to understand it is by declaring it to be the likelihood of a bottleneck in a finite section of the graph. In symbols, this is:

$$h(G) = \inf \left\{ \frac{|\partial A|}{|A|} \,:\, A \subseteq V(G),\, 0 < |A| < \frac{1}{2}|V(G)| \right\}$$

Where, the boundary of the finite subgraph A is the set of vertices of A connected to an edge outside of A; in set notation:

$$\partial A = \{(x, y) \in E(G) \,:\, x \in A, y \in V(G) \setminus A\}$$

Another formulation, more palatable to those in mathematical finance, is the following: at each vertex $g \in G$, a person is holding a loonie; at any given time, they are told to toss the loonie to their neighbour who lives at a fixed distance away; then, a group is paradoxical precisely when each person's net worth increases. In other words, a group is paradoxical if it can support a Ponzi scheme!

Within the context of this exciting mathematical problem, the Pure Mathematics Department was interested in having readily available examples of amenable and paradoxical Cayley graphs for future teaching of amenability theory. To that end, Wolfram Alpha, Maple, and LaTeX code was written to compute the Cayley graphs of several groups. In particular, several graph visualisation libraries in each programming language were explored and applied, in an *ad hoc* fashion, to each group, with the choice depending on the suitability of the language to the group.

2.3 Results

This section lists and analyses Cayley graphs of one paradoxical group—$\mathbb{F}_2$—as well as the Cayley graphs of several amenable groups—$\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}_n$, amongst others. The Cayley graph of the free group on two generators is displayed on Figure 2.

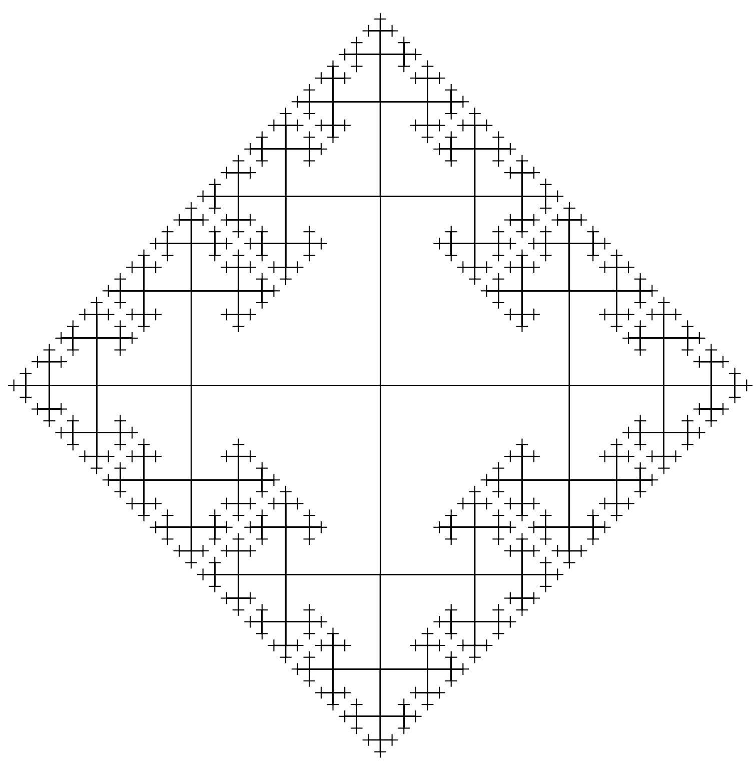


Figure 2: Cayley graph for $\mathbb{F}_2$, adapted from [2].

Although computing the exact value of the Cheeger constant of this Cayley graph is a hard computational exercise, it can be proven to be positive, thus indicating that $\mathbb{F}_2$ is paradoxical. The proof of this fact requires observing the approximate inequality $|K| \leq 2|\partial K|$ for any subgraph K of $\mathbb{F}_2$. Furthermore, the intuition provided to understand the central theorem of this paper is exhibited on this graph, as bottlenecks become more likely the further from the centre one travels. Another way to view this is by noting that the perimeter of the graph becomes highly detailed with respect to the centre of the graph, which exhibits increasingly large boundaries and forces positivity of the Cheeger constant of the graph.

For the group of integers $\mathbb{Z}$, it was observed in Figure 1(a) and 1(b) that no visual bottlenecks appear. Indeed, the Cayley graphs of $\mathbb{Z}$ exhibit an equidistant nature which allow for the choice of arbitrarily large subgraphs of finite size whose boundary size is just two—in fact, a sequence of connected line segments of finite size will do. This picture becomes even more apparent in the two-dimensional lattice: $\mathbb{Z} \times \mathbb{Z}$. Indeed, this group (Figure 3) is generated by the elements $S = \{(1,0),(0,1)\}$, that is, by repeatedly combining north and east steps. This forms the usual two-dimensional grid; over time, the perimeter of the grid becomes small with respect to the area contained inside the grid. Indeed, the former grows at a linear rate n, whereas the latter grows at a quadratic rate n^2, leading to the conclusion that the Cheeger constant vanishes as n grows.

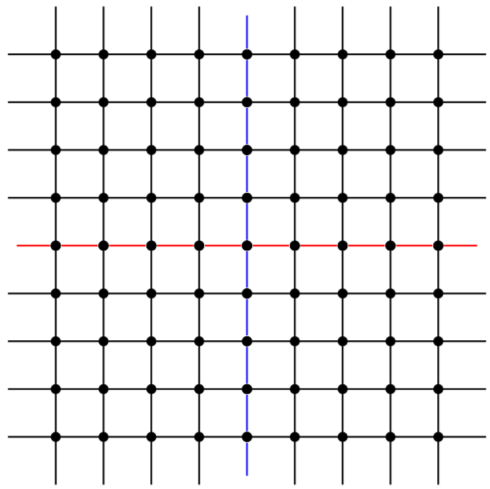
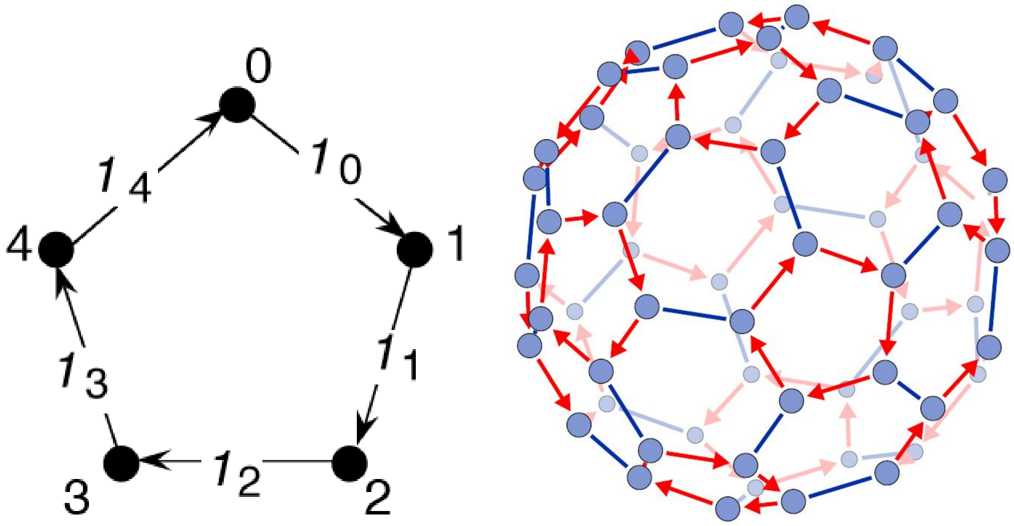


Figure 3: Cayley graph for $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$, the two-dimensional lattice group

Finally, Figure 4 shows two Cayley graphs for finite groups. The first one is $\mathbb{Z}_5$ with generator $\{1\}$ and the other one is A_5, the alternating group of order five, otherwise known as the icosahedral group. It is known that all finite groups are amenable as they allow a discrete uniform distribution to be supported on them, making the invariant mean computation weighted counting [7]. As opposed to the Cayley graph for $\mathbb{F}_2$, the nodes remain equidistant from their adjacent nodes, hence preventing the formation of bottlenecks at when zooming in to the graphs.



(a) Cayley graph for $\mathbb{Z}_5$

(b) Cayley graph for A_5

Figure 4: Cayley graphs for finite groups. Adapted from [3].

3 CONCLUSIONS

Banach, Tarski, and von Neumann would probably not be surprised to know that the field they started had profound consequences in abstract harmonic analysis. In many cases in modern mathematics, explaining such deep concepts is complicated and requires several years of formal training before a mathematics student can study them. The same is true for amenability, but, as this report shows, all is not lost: many of the concepts that the theory of amenable groups explores can be captured in simple graphical representations.

Furthermore, several mathematical software packages can be used in unison to further elucidate findings in abstract mathematics. These graphs can then be used to provide students of harmonic analysis further intuition about the theory of amenability. Cleverly choosing groups and Cayley graphs to represent amenability could serve as an effective tool to communicate different formulations of Følner's theorem, as exhibited in this report.

4 RECOMMENDATIONS

In mathematics, a student can get a performance advantage by having an intuitive understanding of a topic. This advantage can later lead to significant improvements in class performance and research productivity. In that context, this report makes the case for three recommendations: (i) to use graphical representations to further explain abstract mathematical concepts in amenability theory, (ii) to replicate the code used to produce the Cayley graphs in this paper to save time when preparing future courses in harmonic analysis, and (iii) to replicate the approach used here when communicating abstract mathematical findings to the community at large.

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