STAT 929

Thoughts on the functional GARCH model

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0.1 Introduction

When Engle and Bollerslev introduced the ARCH and GARCH models in the 1980s, they proved to be a significant stepping stone in modelling financial volatility [4, 1]. These proved to be successful as they are mathematically tractable and they manage to model well-known "stylised facts" that financiers use. For an extensive treatise on GARCH models, we refer the interested reader to the book by Francq and Zakoïan [5].

In practice, GARCH models are useful when the time series has a daily resolution. With advances in technology and computation, financial modellers now have access to ultra high-frequency data, a setting in which the usefulness of GARCH models may be diluted. To address this paradigm, functional time series models have been proposed. We say that a discrete-time stochastic process $(Y_t)_{t\in\mathbb{Z}}$ is a functional time series if it takes values in a function space. We may thus interpret a realisation $Y_t(u)$ as the volatility of an asset on day t at time u.

Since the seminal work by Bosq on functional linear time series models [2], there has been interest in extending these models to the non-linear setting. As in the scalar setting, ARCH and GARCH models prove to be the natural first step in studying non-linear models.

In this paper we study the current state of functional GARCH models and propose mild generalisations and directions in which these models can be further generalised. The first generalisation of the GARCH model was made by Hörmann et al. [6], where they introduced a functional version of the ARCH(1) model, or fARCH(1). This model was later generalised by Cerovecki et al. [3] into a fGARCH$(1, 1)$ model in the $L^2[0, 1]$ space.

Here, we provide some comments on the model proposed by Cerovecki et al. and provide a mild extension to $C[0, 1]$ using the methods used by Hörmann et al.

0.2 The functional GARCH model

0.2.1 Preliminaries

To begin, we establish some notation. Let $\mathcal{H}$ denote the real Hilbert space $L^2[0, 1]$, where our time series will be observed. This Hilbert space is equipped with the inner product $\langle f, g\rangle = \int_{[0,1]} fg$, which induces the norm $\|f\|_{\mathcal{H}} = \left(\int_{[0,1]} f^2\right)^{1/2}$. Two metric subspaces of interest are $\mathcal{H}^+$ and $\mathcal{H}^+_*$, the elements in $\mathcal{H}$ which are non-negative and positive almost everywhere, respectively. Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of bounded operators on $\mathcal{H}$, equipped with the operator norm $\|T\|_{\mathcal{B}(\mathcal{H})} = \sup_{\|x\|\le 1} \|Tx\|_{\mathcal{H}}$. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is positive if $T(x) \in \mathcal{H}^+$ for all $x \in \mathcal{H}^+$ Let $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ denote the subspace of kernel operators on $\mathcal{H}$; that is, we say $\alpha \in \mathcal{K}(\mathcal{H})$ if $\alpha(x)(u) = \int_{[0,1]} K_\alpha(u, v)x(v)dv$ (for almost every $u \in [0, 1]$) for some $K_\alpha \in L^2\left([0, 1]^2\right)$. We will sometimes abuse notation and write α for K_α when the context is clear.

More generally, we will let X be a separable Banach space with norm $\|\cdot\|_X$. For definiteness, we will consider $C[0, 1]$ be the Banach space of continuous functions whose domain is the compact unit interval, endowed with the uniform norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Note that $C[0, 1]$ with its usual norm is a Banach algebra; that is, $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$ for all $f, g \in C[0, 1]$.

Following the convention established in [2], we establish some terminology for Banach space valued random variables. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(X, \mathrm{Bor}\,(X))$ be a tuple containing a Banach space X with its Borel σ-algebra, $\mathrm{Bor}\,(X)$. We say that $y : \Omega \to X$ is an X-valued random variable if it is

Ω-X-measurable. We say that y has weak expectation if there exists a vector $\mathbb{E}(y) \in X$ such that:

$$\mathbb{E}(f(y)) = f(\mathbb{E}(y)) \qquad \forall f \in X^*$$

Where X^* is the dual space of X. We say that y is integrable if $\mathbb{E}(\|y\|) < \infty$. If $y : \Omega \to X$ satisfies $\mathbb{E}\left(\|y\|^2\right) < \infty$ and $\mathbb{E}(y) = 0$, we call the function $C_y : X^* \to X$ the covariance operator associated to y and specify it as:

$$C_y(f) = \mathbb{E}(f(y)y) \qquad f \in X^*$$

We remark the if $X = \mathcal{H}$ then an operator $C : \mathcal{H} \to \mathcal{H}$ is a covariance operator if and only if it is positive, compact, and has absolutely summable eigenvalues.

If $y = (y_t)_{t\in\mathbb{Z}}$ is an X-valued process, we say it is weakly stationary if $\mathbb{E}(\|y_t\|_X) < \infty$ for all $t \in \mathbb{Z}$, $\mathbb{E}(y_t) = \mu$ does not depend on t and the covariances of y are lag-dependent:

$$\mathbb{E}(f(y_{n+h-\mu})g(y_{m+h} - \mu)) = \mathbb{E}(f(y_n - \mu)g(y_m - \mu))$$

for all $n, m, h \in \mathbb{Z}$ and $f, g \in X^*$.

0.2.2 Cerovecki's model

We introduce the fGARCH(p, q) model.

Definition 0.1 Let $(\epsilon_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathcal{H}$-valued random variables. We say that $(y_t)_{t\in\mathbb{Z}} \sim$ fGARCH(p, q) if the process $(y_t)_{t\in\mathbb{Z}}$ is a stationary solution of the equations

$$y_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \delta + \sum_{i=1}^{q} \alpha_i(y_{t-i}^2) + \sum_{j=1}^{p} \beta_j(\sigma_{t-j}^2)$$

where $\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p \in \mathcal{K}(\mathcal{H})$ are positive kernel operators and $\delta \in \mathcal{H}_*^+$.

While the above model is useful, especially in high-frequency financial applications, it has a few weaknesses. Firstly, an ideal model for intraday volatility, like the fGARCH, should be able to provide pointwise estimates of volatility. However, the stochastic process $(y_t)_{t\in\mathbb{Z}}$ in Definition 0.1 takes values in $L^2[0, 1]$, which is a space of equivalence classes, not functions. Thus, for any set of measure zero (which may correspond to every time sample an financial modeller might take!) we may specify the values of σ as we wish. To remedy this, it is natural to consider defining the process above in a reproducing kernel Hilbert space, where point-evaluation functionals are well-defined. For instance, if one is willing to accept some smoothness in volatility processes, the model in Definition 0.1 may be specified in the space

$$H^2[0, 1] = \left\{f : [0, 1] \to \mathbb{R} \,:\, f \text{ is absolutely continuous }, f(0) = f(1) = 0, f' \in L^2[0, 1]\right\}$$

Where the inner product is given by $\langle f, g\rangle = \int_{[0,1]} f'g'dx$, and pointwise evaluation is well-defined and bounded, as, by the Cauchy-Schwartz inequality:

$$|f(x)| \leq \|f'\|_{L^2[0,1]} \left(\int_{[0,1]} \chi_{[0,x]}(t)dt\right)^{1/2} = \|f\| \sqrt{x}$$

Sadly, a model like this does not allow for rough processes to live in it. Recall that if a function is absolutely continuous, then it is differentiable almost everywhere. In this case, the sample paths of

Brownian motions or Ornstein-Uhlenbeck processes, which are popular in modelling financial volatility, would not be allowed to live in $H^2[0,1]$.

Another possibility is to consider the space A_a of functions $f : \mathbb{R} \to \mathbb{R}$ which are integrable and whose Fourier transform F is square integrable and has support in $[-a, a]$ for $a > 0$. By Plancherel's theorem, the functions f are necessarily square integrable; the space is also a reproducing kernel Hilbert space as, using Fourier's inversion theorem and Parseval's theorem, the following estimate can obtained for every $x \in \mathbb{R}$:

$$|f(x)| \leq \sqrt{\frac{a}{\pi}}\, \|f\|_{L^2[0,1]}$$

Again, this space has a weakness: Hardy's uncertainty principle tells us that such an f cannot be compactly supported on $\mathbb{R}$. Furthermore, Fourier's inversion theorem tells us that such an f must be analytic, as $f(x) = \int_{[-a,a]} F(w)e^{iwx}dx$, and F, being supported in a set of finite measure, must be integrable; thus, uniform convergence of the Taylor series of e^{iwx} on $[-a, a]$ gives a power series representation for f, implying analyticity. Once again, this space fails to be the right setting for volatility processes, as all its elements are smooth. Furthermore, in financial applications, where we might standardise a day into the interval $[0, 1]$, such a space proves to be unhelpful.

Other candidate reproducing kernel Hilbert spaces we have considered suffer from the insurmountable difficulty that they are function spaces supported in complex domains, which opens up another can of worms when it comes to interpreting what "complex time" may be.

All is not lost. If we are willing to depart from Hilbert spaces, we can specify the model in Definition 0.1 in $C[0, 1]$ and even obtain results about its stationarity. We defer these results to the next section.

A second weakness of the fGARCH model is that it does not truly generalise a multivariate GARCH model. Indeed, recall that in an $\mathbb{R}^n$-valued GARCH model, the stochastic representation of $(y_t)_{t\in\mathbb{Z}}$ is given by:

$$y_t = \Sigma_t^{1/2}\epsilon_t$$

Where Σ_t is a positive definite matrix admitting a suitable stochastic representation. In particular, the conditional covariance of y_t is given by Σ_t. Being a linear operator on a finite-dimensional vector space, Σ_t is trivially compact with summable eigenvalues. If the specification of Σ_t is symmetric, then with our framework, Σ_t is truly a covariance operator.

The same is not true for σ_t in the fGARCH model. Observe that we may write the fGARCH model as $M_{\sigma_t}(y_t) = \sigma_t y_t$, where $M_{\sigma_t} \in \mathcal{B}(\mathcal{H})$ is a multiplication operator. But since σ_t is positive almost everywhere, M_{σ_t} is invertible and thus can neither be a compact operator nor a covariance operator.

It may be argued that the actual weakness of the fGARCH model is not truly the non-compactness of σ_t, but rather that it is a pointwise or diagonal model. Indeed, letting σ_t be a multiplication operator simply scales an innovation process on every point of $[0, 1]$. To our knowledge, more general non-diagonal models have not yet been considered.

With this in mind, it is highly desirable to specify an operator-level functional GARCH model. We list some ideas in Section 0.4.

0.3 A $C[0,1]$-valued fGARCH process

It turns out the same methods used by Hörmann et al. to prove thatin [6] under mild conditions a $\mathcal{H}$-valued fARCH(1) process admits a stationary process can be used to extend Cerovecki's model to a $C[0,1]$ valued model.

To do so, we introduce a result from the theory of iterated function systems.

For the setting, let (S,ρ) be a complete, separable metric space and Θ be another metric space. Let $M : \Theta \times S \to S$ be a measurable function. Associate to random elements $\theta \in \Theta$ the random function M_θ. We may define the random iterated function system $X_n = M_{\theta_n}(X_{n-1})$ where $(\theta_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence.

Further define $\Phi_n, \Psi_n : S \to S$ by

$$\begin{aligned} \Phi_n &= M_{\theta_n} \circ M_{\theta_{n-1}} \circ \ldots \circ M_{\theta_1} \qquad\qquad n \in \mathbb{Z}^+ \\ \Psi_n &= M_{\theta_{-1}} \circ M_{\theta_{-2}} \circ \ldots \circ M_{\theta_{-n}} \end{aligned}$$

Theorem 0.2 *(Wu and Shao, 2004) Assume that:*

(A) There are $y_0 \in S$ and $\alpha > 0$ such that $\mathbb{E}\left([\rho(y_0, M_{\theta_0}(y_0))]^\alpha\right) < \infty$, and

(B) There are $x_0 \in S, \alpha > 0, 0 < r_1 = r_1(\alpha) < 1$, and $c = c(\alpha) < \infty$ such that

$$\mathbb{E}\left([\rho(\Phi_n(x), \Phi_n(x_0))]^\alpha\right) \leq c r_1^n [\rho(x, x_0)]^\alpha$$

for all $x \in S$ and $n \in \mathbb{Z}^+$.

Then for all $x \in S$, we have that $\Psi_n(x)$ converges almost surely to some Ψ_∞ which is independent of x. Furthermore, $\Psi_\infty = g(\theta_0, \theta_{-1}, \ldots)$ and $\mathbb{E}\left([\rho(\Psi_n(x), \Psi_\infty)]^\alpha\right) \leq c_1 r^n$ where $c_1 = c_1(x, x_0, y_0, \alpha) < \infty$ and $0 < r = r(\alpha) < 1$. Moreover, the process $X_n = g(\theta_n, \theta_{n-1}, \ldots)$ is a stationary solution of the iterated function system $X_n = M_{\theta_n}(X_{n-1})$.

We now give conditions for a $C[0,1]$-valued fGARCH$(1,1)$ model to be stationary.

Theorem 0.3 *Let $(\epsilon_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of random variables in $C[0,1]$ with $\mathbb{E}(\epsilon_t) = 0$ and $\mathbb{E}\left(\epsilon_t^2\right) = 1$. Let $(y_t)_{t\in\mathbb{Z}}$ be the $C[0,1]$-valued process given by:*

$$\begin{aligned} y_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= \delta + \alpha(y_{t-1}^2) + \beta(\sigma_{t-1}^2) \end{aligned}$$

Where $\alpha(f)(t) = \int_{[0,1]} K_\alpha(t,s) f(s) ds$ and $\beta(f)(t) = \int_{[0,1]} K_\beta(t,s) f(s) ds$ are assumed to be bounded and positive operators and $\delta(t) > 0$ for all $0 \leq t \leq 1$. Define $H(\epsilon_1^2) = \sup_{0\leq t\leq 1} \int_{[0,1]} \left(\beta(t,s)\epsilon_1^2 + \alpha(t,s)\right) ds$. If there is some $\alpha > 0$ such that $\mathbb{E}\left([H(\epsilon_1^2)]^\alpha\right) < 1$ then $(y_t)_{t\in\mathbb{Z}}$ admits a unique stationary representation in $C[0,1]$ and $\sigma_t^2 = g(\epsilon_{t-1}, \epsilon_{t-2}, \ldots)$ for a Borel measurable function $f : C[0,1]^{\mathbb{N}} \to C[0,1]$.

Proof. Define $M_{\theta_n} : C[0,1] \to C[0,1]$ by

$$M_{\theta_n}(x)(t) = \delta(t) + \int_{[0,1]} \alpha(t,s)\epsilon_{n-1}^2(s) x(s) ds + \int_{[0,1]} \beta(t,s) x(s) ds$$

To prove this result, we must only prove that conditions (A) and (B) hold in Theorem 0.2 for M_{θ_n}.

Condition (A) holds trivially. Set $y_0(t) = 0$ for $0 \leq t \leq 1$, so that $y_0 \in C[0,1]$. Then,

$$\|y_0 - M_{\theta_0}(y_0)\|_\infty = \sup_{0 \leq t \leq 1} |\delta(t)| < \infty$$

Since δ is a deterministic function, it follows that $\mathbb{E}\left(\|y_0 - M_{\theta_0}(y_0)\|_\infty\right) < \infty$, so that condition (A) is met.

To see that condition (B) holds, observe that for all $x, x_0 \in C[0,1]$ we have:

$$\begin{aligned}
\|S_n(x) - S_n(x_0)\|_\infty &= \|M_{\theta_n}(S_{n-1}(x) - M_{\theta_n}(S_{n-1}(x_0))\|_\infty \\
&= \left\| \int_{[0,1]} (\alpha(t,s)\epsilon_{n-1}^2 + \beta(t,s))(S_{n-1}(x)(s) - S_{n-1}(x_0)(s) ds \right\|_\infty \\
&\leq \|S_{n-1}(x) - S_{n-1}(x_0)\|_\infty \left\| \int_{[0,1]} (\alpha(t,s)\epsilon_{n-1}^2 + \beta(t,s)) ds \right\|_\infty \\
&= \|S_{n-1}(x) - S_{n-1}(x_0)\|_\infty H(\epsilon_{n-1}^2) \\
&\leq \|S_{n-2}(x) - S_{n-2}(x_0)\|_\infty H(\epsilon_{n-1}^2) H(\epsilon_{n-2}^2) \\
&\vdots \\
&\leq \|x - x_0\|_\infty \prod_{i=0}^{n-1} H(\epsilon_{n-i}^2)
\end{aligned}$$

Raising to the power of $\alpha > 0$ and taking expectations, we get:

$$\begin{aligned}
\mathbb{E}\left(\|S_n(x) - S_n(x_0)\|_\infty^\alpha\right) &\leq \|x - x_0\|_\infty^\alpha \, \mathbb{E}\left(\prod_{i=0}^{n-1} H(\epsilon_{n-i}^2)^\alpha\right) \\
&= \|x - x_0\|_\infty^\alpha \left[\prod_{i=0}^{n-1} \mathbb{E}\left(H(\epsilon_{n-i}^2)^\alpha\right)\right] \\
&= \|x - x_0\|_\infty^\alpha \, \mathbb{E}\left([H(\epsilon_1^2)]^\alpha\right)
\end{aligned}$$

Where the last two lines follow since $(\epsilon_t)_{t \in \mathbb{Z}}$ is i.i.d. ∎

We end this section by remarking that the proofs given in [3, 8] using the top Lyapunov exponent of this system adapt, *mutatis mutandis*, to the $C[0,1]$ setting. For completeness, we present their proof below.

Observe that we may write the fGARCH(p,q) model in the state-space form $z_t = b_t + \Pi_t z_{t-1}$ where:

$$z_t = \begin{pmatrix} y_t^2 \\ \vdots \\ y_{t-q+1}^2 \\ \sigma_t^2 \\ \vdots \\ \sigma_{t-p+1}^2 \end{pmatrix} \in C[0,1]^{p+q} \qquad b_t = \begin{pmatrix} \epsilon_t^2 \delta \\ 0 \\ \vdots \\ 0 \\ \delta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in C[0,1]^{p+q} \tag{1}$$

$$\Pi_t = \begin{pmatrix} \Upsilon_t\alpha_1 & \dots & \Upsilon_t\alpha_{q-1} & \Upsilon_t\alpha_q & \Upsilon_t\beta_1 & \dots & \Upsilon_t\beta_{p-1} & \Upsilon_t\beta_p \\ I_{C[0,1]} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & I_{C[0,1]} & 0 & 0 & \dots & 0 & 0 \\ \alpha_1 & \dots & \alpha_{q-1} & \alpha_q & \beta_1 & \dots & \beta_{p-1} & \beta_p \\ 0 & \dots & 0 & 0 & I_{C[0,1]} & 0 & 0 & \\ 0 & \dots & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & I_{C[0,1]} & 0 \end{pmatrix} \tag{2}$$

Where Υ_t is pointwise multiplication by ϵ_t^2 and $I_{C[0,1]}$ is the identity operator in $C[0,1]$. It is immediate that $\|\Upsilon_t\| = \left\|\epsilon_t^2\right\|_\infty$. Furthermore, $(\Pi_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence. The following is a useful lemma regarding the matrix Π_t:

Lemma 0.4 *Let $\mathbb{E}\left(\log\left(\max(1, \left\|\epsilon_0^2\right\|\right)\right) < \infty$. For the Markov state-space matrix in Equation 2, the following equality holds almost surely with values in $[-\infty, \infty)$:*

$$\gamma = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left(\log \|\Pi_t\Pi_{t-1}\dots\Pi_1\|\right) = \lim_{t\to\infty} \frac{1}{t}\log \|\Pi_t\Pi_{t-1}\dots\Pi_1\|$$

Proof. The sequence (Π_t) is stationary in the Banach algebra $\mathcal{B}\left(C[0,1]\right)$. Since $\mathbb{E}\left(\log\left(\max(1, \epsilon_t^2)\right)\right) < \infty$, the conditions in Theorem 6 in [7] given the desired result. ■

Definition 0.5 The parameter γ in Lemma 0.4 is called the *top Lyapunov exponent* of Π_t.

We observe that if a strictly stationary solution exists for the state-space model $(z_t)_{t\in\mathbb{Z}}$, then by projecting into the $q+1$ coordinate of z_t, we recover strictly stationary solutions for σ_t^2 and, thus, for y_t.

Theorem 0.6 *If $\gamma < 0$, then there exists a unique non-anticipative strictly stationary solution in $C[0,1]$ to the* fGARCH(p,q) *equations.*

Proof. By formally expanding the recursion $z_t = b_t + \Pi_t z_{t-1}$, we have that

$$z_t = b_t + \sum_{k=1}^{\infty} \Pi_t\Pi_{t-1}\dots\Pi_{t-k+1}(z_{t-k}) \tag{3}$$

if the limit exists almost surely. This limit in fact does exist, as:

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\|\Pi_t\Pi_{t-1}\dots\Pi_{t-k+1}(b_{t-k})\| &\leq \limsup_{t\to\infty}\frac{1}{t}\log\|\Pi_t\Pi_{t-1}\dots\Pi_{t-k+1}\|\,\|b_{t-k}\| \\ &= \limsup_{t\to\infty}\frac{1}{t}\log\|\Pi_t\Pi_{t-1}\dots\Pi_{t-k+1}\| + \limsup_{t\to\infty}\frac{1}{t}\log\|b_{t-k}\| \\ &= \gamma + \limsup_{t\to\infty}\frac{1}{t}\log\|b_{t-k}\|\end{aligned}$$

Having assumed that $\mathbb{E}\left(\log\left(\max(1,\left\|\epsilon_0^2\right\|)\right)\right) < \infty$, and since $\|b_{t-k}\| \geq \|\delta\| > 0$, it follows that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|b_{t-k}\| = 0 \quad (a.s.)$$

Applying that Cauchy root test, it follows that Equation 3 is a well-defined almost surely convergent expression. Almost sure convergence gives us that the $q+1$ component of z_t is a non-anticipative solution to the fGARCH(p,q) equations.

To show uniqueness suppose, to the contrary, that z_t^* is another solution. Repeated iteration of the Markov state-space recurrence gives:

$$z_t^* = b_t + \sum_{k=1}^{N}\Pi_t\dots\Pi_{t-k+1}(b_{t-k}) + \Pi_t\dots\Pi_{t-N}(z_{t-N-1}^*)$$

Then,

$$\begin{aligned}\|z_t^* - z_t\| &= \left\|b_t + \sum_{k=1}^{N}\Pi_t\dots\Pi_{t-k+1}(b_{t-k}) - z_t + \Pi_t\dots\Pi_{t-N}(z_{t-N-1}^*)\right\| \\ &\leq \left\|b_t + \sum_{k=1}^{N}\Pi_t\dots\Pi_{t-k+1}(b_{t-k}) - z_t\right\| + \left\|\Pi_t\dots\Pi_{t-N}(z_{t-N-1}^*)\right\| \\ &\leq \left\|b_t + \sum_{k=1}^{N}\Pi_t\dots\Pi_{t-k+1}(b_{t-k}) - z_t\right\| + \|\Pi_t\dots\Pi_{t-N}\|\left\|(z_{t-N-1}^*)\right\| \\ &= \left\|z^N - z_t\right\| + \|\Pi_t\dots\Pi_{t-N}\|\left\|(z_{t-N-1}^*)\right\|\end{aligned}$$

Since $\gamma < 0$, it follows that $\left\|z_t^N - z_t\right\| \to 0$. Since $\|\Pi_t\dots\Pi_{t-N}\| \to 0$ as $N\to\infty$ and since the law of $\left\|z_{t-N-1}^*\right\|$ does not depend on N, it follows that $\|z_t^* - z_t\| \to 0$ in probability, so that $P(z_t^* = z_t) = 1$. ■

We end this section by remarking that it is not known whether $\gamma < 0$ is a necessary condition, like it is in the scalar case.

0.4 Operator-level fGARCH

We propose operator-level fGARCH models where the volatility σ_t acts as a compact operator.

Definition 0.7 Let $(\epsilon_t)_{t\in\mathbb{Z}}$ be a given i.i.d. sequence in $L^2[0,1]$. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis for

$\mathcal{H}$. We say that a sequence $(y_t)_{t\in\mathbb{Z}}$ is a $\ell^\infty - \text{CCC} - \text{fGARCH}(p,q)$ model if

$$\begin{aligned} y_t &= H^{1/2}\epsilon_t \\ H_t &= D_t R D_t \\ D_t &= \text{diag}\left(\sqrt{h_t}\right) \\ h_t &= \delta + \sum_{i=1}^{q} A_i(y_{t-i}^2) + \sum_{j=1}^{p} B_j(h_{t-j}) \end{aligned}$$

Where D_t is a bounded diagonal operator with respect to $(e_k)_{k=1}^\infty$, $h_t \in \ell^\infty$ for all $t \in \mathbb{Z}$, R is a covariance operator, $A_1,\ldots,A_q \in \mathcal{B}^+(L^2,\ell^\infty)$, and $B_1,\ldots,B_p \in \mathcal{B}^+(\ell^\infty)$.

Since the compact operators on $\mathcal{H}$ are a two sided ideal of $\mathcal{B}(\mathcal{H})$ and R is compact in the model above, it follows that H_t is a covariance operator. In fact, there exists another basis $(f_k)_{k=1}^\infty$ for $\mathcal{H}$ such that H_t can be diagonalised in terms of $(e_i \otimes f_j)_{i,j=1}^\infty$.

In the definition above, we let $h_t \in \ell^\infty$ as this allows for the most possible flexibility in specifying the diagonal of H_t. This has the disadvantage that when forming the Markov state space matrix associated with this process, we may not be able to give it analytic meaning, and hence we would be unable to extract stationarity via a top Lyapunov exponent condition. However, we believe this can be salvaged if h_t is restricted to ℓ^2, as this matrix now acquires meaning. This motivates the following definition:

Definition 0.8 Let $(\epsilon_t)_{t\in\mathbb{Z}}$ be a given i.i.d. sequence in $L^2[0,1]$. Let $(e_k)_{k=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. We say that a sequence $(y_t)_{t\in\mathbb{Z}}$ is a $\ell^2 - \text{CCC} - \text{fGARCH}(p,q)$ model if

$$\begin{aligned} y_t &= H^{1/2}\epsilon_t \\ H_t &= D_t R D_t \\ D_t &= \text{diag}\left(\sqrt{h_t}\right) \\ h_t &= \delta + \sum_{i=1}^{q} A_i(y_{t-i}^2) + \sum_{j=1}^{p} B_j(h_{t-j}) \end{aligned}$$

Where D_t is a compact diagonal operator with respect to $(e_k)_{k=1}^\infty$ whose diagonal is $h_t \in \ell^2$, R is a covariance operator, $A_1,\ldots,A_q \in \mathcal{B}^+(L^2,\ell^2)$, and $B_1,\ldots,B_p \in \mathcal{B}^+(\ell^2)$.

We show that the $\ell^2 - \text{CCC} - \text{fGARCH}(p,q)$ admits a state-space formulation. Suppose that, given the basis $(e_k)_{k=1}^\infty$ for $L^2[0,1]$, we specify a unitary isomorphism $U : L^2[0,1] \to \ell^2$. Letting $\tilde{\epsilon}_t = R^{1/2}\epsilon$, we get:

$$y_t = H_t^{1/2}\epsilon_t = (D_t R D_t)^{1/2}\epsilon_t = D_t R^{1/2}\epsilon_t = D_t\tilde{\epsilon}_t$$

If we define the operator $\Upsilon_t : \ell^2 \to L^2$ by

$$\Upsilon_t\left((x_n)_{n=1}^\infty\right) = U^{-1}\left(\left(\langle \epsilon_t^2, e_j\rangle x_n\right)_{n=1}^\infty\right) \qquad x = (x_n)_{n=1}^\infty \in \ell^2$$

Then Υ_t is a bounded operator. Provided that the equality $y_t^2 = (D_t\tilde{\epsilon}_t)^2 = D_t^2\tilde{\epsilon}_t^2$ holds (e.g. when D_t behaves as a multiplication operator in $L^2[0,1]$) then the following is true:

$$\epsilon_t^2 = \Upsilon_t(\delta) + \sum_{i=1}^{q} \Upsilon_t(A_i y_{t-i}^2) + \sum_{j=1}^{p} \Upsilon_t B_j h_{t-j}$$

Which may then be written as $z_t = b_t + \Pi_t z_{t-1}$, where:

$$b_t = b(\epsilon_t) = \begin{pmatrix} \Upsilon_t \delta \\ 0 \\ \vdots \\ 0 \\ \delta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in (\ell^2)^{p+q} \quad z_t = \begin{pmatrix} y_t^2 \\ \vdots \\ y_{t-q+1}^2 \\ h_t \\ \vdots \\ h_{t-p+1} \end{pmatrix} \in (\ell^2)^{p+q} \tag{4}$$

$$\Pi_t = \begin{pmatrix} \Upsilon_t A_1 & & \dots & & \Upsilon_t A_q & \Upsilon_t B_1 & & \dots & & \Upsilon_t B_p \\ I_{\ell^2} & 0 & \dots & & 0 & 0 & & \dots & & 0 \\ 0 & I_{\ell^2} & \dots & & 0 & 0 & & \dots & & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & & \dots & I_{\ell^2} & 0 & 0 & & \dots & 0 & 0 \\ A_1 & & \dots & & A_q & B_1 & & \dots & & B_p \\ 0 & & \dots & 0 & I_{\ell^2} & 0 & & \dots & & 0 \\ 0 & & \dots & 0 & 0 & I_{\ell^2} & & \dots & & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & & \dots & 0 & 0 & 0 & & \dots & I_{\ell^2} & 0 \end{pmatrix} \tag{5}$$

Where $\Pi_t : (\ell^2)^{p+q} \to (\ell^2)^{p+q}$ and I_{ℓ^2} is the identity in ℓ^2.

Providing a stationary solution to this state-space equation yields a stationary solution for y_t, analogous to the $\mathbb{R}^m$ CCC-GARCH(p,q) model.

Theorem 0.9 *A sufficient condition for there to exist a strictly stationary non-anticipative solution to the* $\ell^2 - \text{CCC} - \text{fGARCH}(p, q)$ *model is that the top Lyapunov exponent associated to* Π_t *satisfy* $\gamma < 0$.

Proof. The proof is similar to the proof for Theorem 11.6 in Francq and Zakoïan [5], replacing $\mathbb{R}^m$ by ℓ^2. ■

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