The Origin and the Resolution of Nonuniqueness in Linear Rational Expectations

John G. Thistle

Department of Electrical and Computer Engineering,
University of Waterloo,
Waterloo, Ontario,
Canada N2L 3G1
jthistle@uwaterloo.ca *

Abstract

The nonuniqueness of rational expectations solutions is traced to free parameters that determine the public's immediate reactions to shocks: a unique solution is defined if and only if those parameter values are fixed. The traditional solution fixes them indirectly, by means of a stability criterion that is effective only in the presence of particular instabilities. But within a broad class of models, the requirement of least-square forecast errors determines a unique solution, irrespective of stability.

In a New Keynesian model, the conventional approach is shown to suppress precisely the dynamics that arise from rational expectations. The new approach uncovers those dynamics, showing rational expectations to be "inertial," and revealing model misspecification.

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1 Introduction

There is growing recognition that modern dynamic macroeconomic models, and specifically DSGE[1] models, incorporate significant flaws. A specific target of criticism is their use of rational expectations. But this paper shows that there is much more to rational expectations than has hitherto been explored.

The traditional approach is based on ad hoc means of contending with the nonuniqueness of rational-expectations solutions. This article performs a general analysis of rational expectations, and traces that nonuniqueness to its economic source, in the dynamics of expectation formation. It identifies the underlying free parameters, and presents a general solution of rational-expectations models.

It has in effect been widely supposed that the heart of the nonuniqueness problem might lie in an infinite regression (if it is possible to speak of "regression" into the future): if the values of endogenous variables depend on contemporary forecasts of their future values, then those forecasts of future values depend on future forecasts, which in turn depend on still later forecasts; and so forth. This apparent regression has pointed to a need for a terminal boundary condition, as a means of determining a unique solution. But it is shown here that the origin of indeterminacy lies not in the infinitely remote future, but in the immediate present: in the stochastic, linear, discrete-time, constant-coefficients case, the associated free parameters are specific coefficients that determine the immediate responses of expectations to shocks – specifically, they determine the initial values of the impulse responses of forecasts to shocks. The requirement of rationality – or model-consistency – generally leaves those parameters unconstrained; yet when their values are appropriately specified, a unique solution is determined.

For instance, if it is required that forecasting errors not only be zero-mean, but that they also be minimized in the least-squares sense, then, for a broad class of models, the free parameters are determined, and solutions are unique.

In contrast, the traditional approach to solving for rational expectations requires models to have just enough unstable eigenvalues to determine a unique 'stable' solution (Blanchard and Kahn, 1980). It implicitly assumes that the immediate, aggregate response of forecasts to shocks is calibrated so as to cancel the associated unstable dynamics, exactly. Such traditional rational expectations have proven insufficiently

[1]Dynamic, stochastic, general-equilibrium.

inertial (Blanchard, 2018): it is shown here that, in producing a unique solution to a standard New Keynesian model – the prototype of DSGE models – the conventional approach suppresses precisely those dynamics that arise from rational expectations.

But the methods of this paper do not presuppose any stability or instability properties, nor do they generally suppress any model dynamics; and the resulting 'inertial' dynamics of rational expectations bring to light additional flaws in the New Keynesian model: the new I-S equation and the expectational Phillips curve are incompatible with the dynamics of rational expectations.

The limitations of the conventional approach are thrown into relief when it is asserted that good policy should avoid indeterminacy (in the conventional sense of a lack of unstable eigenvalues) (Barthélemy and Marx, 2019; Cho and McCallum, 2015; Lubik and Schorfheide, 2004). This amounts to calling for policymakers to promote dynamical instability. The new framework frees policymakers from that artificial burden: determinacy is no longer dependent on policy. It thereby allows them to attend to the more realistic and altogether more constructive task of doing precisely the opposite – ensuring economic stability. But policy prescriptions may turn out not be radically altered: when the aforementioned model misspecifications are corrected, policies that ensure economic stability may resemble those that formerly promoted instability. In the suitably 'corrected' New Keynesian model, economic stability calls for interest-rate policy to satisfy the Taylor principle – to meet a given percentage increase in inflation with a greater percentage increase in the policy rate.

1.1 Taylor's example

The nature of the free parameters that underlie nonuniqueness is illustrated by an early example of Taylor (1977). Consider the following equation,

$$\hat{p}_{2,t-1} = \hat{p}_{1,t-1} + \delta_1 p_t + u_t$$

where δ_1 is a nonzero constant, u_t a sequence of independent, identically distributed, zero-mean random variables with finite variance, p_t an endogenous variable (proportional to the logarithm of the price of output)[2], and, for $i = 1, 2$, $\hat{p}_{i,t}$ is a forecast of p_{t+i}, formulated at time t (that is, formulated in terms of u_τ, for $\tau \leq t$).

[2]This version of Taylor's equation has been linearized through a change of coordinates: p_t denotes $p_t - \delta_0/\delta_1$ in Taylor's coordinates.

Seeking solutions of the form

$$p_t = \sum_{i=0}^{\infty} \pi_i u_{t-i} \ ,$$

Taylor imposes rational expectations by setting

$$\hat{p}_{1,t-1} = \sum_{i=1}^{\infty} \pi_i u_{t-i} \quad \& \quad \hat{p}_{2,t-1} = \sum_{i=2}^{\infty} \pi_i u_{t+1-i} \ .$$

Substituting these expressions into the model equation, he finds:

$$\pi_0 = -\delta_1^{-1} \ , \quad \text{and} \quad \pi_{i+1} = (1+\delta_1)\pi_i \ , \ \forall i \geq 1 \ .$$

The coefficient π_0 is determined,[3] but π_1 is free, and its value determines all other coefficients. Taylor then turns to the imposition of a finite-variance condition (dynamical stability) and a minimum-variance condition, as means of limiting the possible values of π_1.

But what is the economic significance of the quantity π_1? According to the parameterization of $\hat{p}_{1,t-1}$,

$$\hat{p}_{1,t-1} = \pi_1 u_{t-1} + \pi_2 u_{t-2} + \dots \ ,$$

so the coefficient π_1 determines the immediate effect of u_{t-1} on this forecast; it models the integration of new information into expectations. This is an undeniably important parameter of the "expectations mechanism."

Yet, π_1 is completely free under the assumption of rational expectations. As strong a condition as rational expectations is, it does not determine, either in whole or in part, the integration of new information into the forecast. The results of this paper generalize this finding; they show that the free variables that account for the nonuniqueness of rational expectations are precisely the coefficients that govern the immediate effects of shocks on forecasts. Consequently, nonuniqueness is resolved if and only if those immediate effects are modeled unambiguously – by means of criteria other than model-consistency.

[3]The analysis of appendix A shows that π_0 is determined because the model equation contains no unlagged forecasts.

The prevailing approach is to constrain such effects indirectly, principally by imposing a terminal condition, which requires dynamical stability. The use of the stability criterion is arbitrary, because it bears no particular relationship to the cause of nonuniqueness. It is also unrealistic, because it depends on infinite-precision cancellation of unstable dynamics. Taylor shows that, if δ_1 is positive, then his model is stable if and only if π_1 is exactly zero. He derives the following recurrence:

$$p_t - (1+\delta_1)p_{t-1} = -\delta_1^{-1}u_t + (\pi_1 + (1+\delta_1)\delta_1^{-1})u_{t-1} \ .$$

The moving averages in p_t and u_t are of the same form if $\pi_1 = 0$, leading to a so-called 'pole-zero cancellation' that suppresses the unstable eigenvalue at $1+\delta_1$. The cancellation is displayed more explicitly by bringing in a 'left-shift' operator z, so that zx_t stands for x_{t+1}, and $z^{-1}x_t$ for x_{t-1}, so z^{-1} is a 'right-shift' or 'lag' operator. Then, assuming that normal algebraic operations apply, the above equation can be rewritten as,

$$p_t = -\delta_1^{-1}\frac{z - (\pi_1\delta_1 + (1+\delta_1))}{z-(1+\delta_1)}u_t \ .$$

(In the sequel, the operator notation is formalized with the use of the z-transform.) Hence, if $\pi_1 = 0$,

$$p_t = -\delta_1^{-1}\frac{z-(1+\delta_1)}{z-(1+\delta_1)}u_t = -\delta_1^{-1}u_t \ .$$

The zero of the rational function in z coincides with (and cancels) the unstable pole – the root of the denominator polynomial – if and only if $\pi_1 = 0$. Stability requires that this cancellation – the result of aggregate actions of the public – be performed *exactly* – with infinite precision. The slightest collective error on the part of economic actors will mean that the model is unstable; on the other hand, if this infinite-precision cancellation is actually carried out, then the resulting model must have reduced-order dynamics. Moreover, this approach is only effective if the exactly the number of unstable model eigenvalues happens to correspond to the number of degrees of freedom in the space of rational-expectations solutions.

Another way of specifying the same unique value for π_1 in Taylor's example is to require in addition that the variance of the forecast errors be minimized. This paper shows that, in a broad class of models, that requirement determines a unique solution, regardless of stability properties, and generally without entailing pole-zero cancellation.

1.2 Background

The cause of the nonuniqueness of rational expectations lies in very the reason for modeling expectations: forecasts have a bearing on the behavior of economic variables; and in particular, they may affect the very quantities being forecast. The study of this self-referential phenomenon dates at least to Tinbergen (1933), who examined the effect of forecast horizons on the movement of commodity prices. The work of Grunberg and Modigliani (1954), on the public prediction of social events, makes the circularity of the problem explicit, treating model-based expectations as fixed points. This property was later summarized by Shiller (1978) as that of an "expectations mechanism which 'reproduces itself' in [the] model." The fixed-point characterization immediately raises the possibilities of nonexistence and nonuniqueness.

Muth (1961) applied similar ideas to market dynamics, hypothesising that the forecasts of market participants did not deviate systematically from model predictions. He called such forecasts *rational expectations*.[4] Lucas (1972, 1976) applied the rational expectations hypothesis to macroeconomic models, to show how changes in anticipated policy may give rise to changes in the behavior of economic agents. But in the context of models with significant dynamics, and corresponding forecasts of the future outcomes of those dynamics, it was found that rational expectations were not defined uniquely (Taylor, 1977; Shiller, 1978);[5] the potential for nonuniqueness inherent in Grunberg and Modigliani's fixed-point characterization was realized.

In the absence of an explanation of the source of this nonuniqueness, the predominant response has been to impose the aforementioned stability criterion.[6] Indeed,

[4]Keuzenkamp (1991) compares Muth's contribution to Tinbergen's.

[5]Black (1974) focused on nonuniqueness, but under the relatively strong requirement that the initial conditions consistent with continual economic equilibrium, and a bounded rate of inflation, be unique.

[6]See, for example, Sargent and Wallace (1973); Shiller (1978); Minford et al. (1979); Blanchard and Kahn (1980); Binder and Pesaran (1997); King and Watson (1998); Klein (2000); Sims (2002); Lubik and Schorfheide (2004).

the term "solution" has almost come to mean "stable solution" (Funovits, 2017). The effect of the cancellation of unstable dynamics has been to obliterate key dynamical features of the model, with result that proponents and critics alike have confined their studies of rational expectations to pathologically special cases. Moreover, these ad hoc methods may still be insufficient to resolve nonuniqueness: for such instances, a variety of other ideas, such as minimum-variance solutions, minimal state-variable realizations, or considerations of 'learnability' have been proposed (Taylor, 1977; Başar, 1989; McCallum, 1999; Evans and Honkapohja, 2001). However, like dynamical stability, such criteria do not get to the heart of the matter.

1.3 Overview

The main point of this article is to explain nonuniqueness by identifying the associated free parameters, which turn out to be parameters of the expectations mechanism itself. For the sake of generality, the analysis proceeds exclusively from simple, minimal assumptions, that are satisfied by traditional approaches to rational expectations. Beyond a standard, technical assumption, it is supposed only that forecasts depend linearly on initial conditions and on the model's driving variables, their dependence on the driving variables being representable by means of a linear difference equation with constant coefficients. This mild condition echoes assumption 3 of Muth (1961); it is satisfied by conventional rational-expectations solutions; and it is necessary for the preservation of the linear, constant-coefficient structure of the model equations. It is formalized in the form of equations called *forecasting mechanisms*.

The problem of deriving a general rational-expectations solution consists in solving for a forecasting mechanism subject to the constraint of model-consistency – that is, subject to the constraint that the forecasting mechanism "reproduce itself in the model." That problem in turn reduces to the solution of a generally singular system of deterministic matrix difference equations that describe the interrelationships of model parameters. It is simple and convenient to solve it in the frequency domain, by means of the z-transform (and without loss of generality). Uniqueness demands only the appropriate specification of a parameter of the forecasting mechanism that governs the immediate response of forecasts to shocks; it is that parameter that distinguishes one fixed-point forecasting mechanism from another. In other words, it is that parameter that distinguishes different rational-expectations solutions.

The determination of the necessary form of any model-consistent forecasting mechanism (in sections 3.1 and 3.2) leads to a necessary and sufficient condition for existence, and to a characterization of the general solution in terms of the aforementioned parameter (section 3.3). This result explains the nonuniqueness of rational expectations, with reference to the dynamics of expectation formation. Based on minimal assumptions, all of this analysis is general: it encompasses any reasonable approach to rational expectations under linear, constant-coefficient models.

This general picture can be tidied up through a simple structural assumption, called well-posedness. Well-posedness ensures existence of a model-consistent forecasting mechanism, and permits the realization of that forecasting mechanism in the relatively robust form of a predictor that incorporates feedback (section 3.4).

For purposes of comparison, it is shown in section 4 that the present framework can reproduce well known observations on the "determinacy" (in the conventional sense) of a small New Keynesian model. That exercise explains the criticism of Blanchard (2018) that rational expectations are insufficiently inertial: it turns out that the pole-zero cancellation required for stability suppresses precisely those dynamics that arise from expectations (section 5).

For a broad class of models, nonuniqueness can be eliminated simply by taking the assumption of rational expectations – or unbiased forecasts – one step further, and assuming that forecast errors are not only zero-mean, but are minimized, in the least-squares sense (section 6). The strengthened assumption is arguably milder than that of unstable pole-zero cancellations, and does not require any particular stability or instability properties of the model. Nor does it generally result in the cancellation of dynamics. Moreover, it represents the important limiting case of forecasts that are as precise as possible.

Because the methods of the paper are independent of dynamical stability considerations, they allow for the study of stabilization via policy. This point is illustrated in section 7, where it emerges that the standard formulation of the New Keynesian model is incompatible with the dynamics of rational expectations.

A brief review is given in section 8 of the vast related literature, and the paper concludes with some general suggestions for research. Appendices show how key results of the paper extend to more general models, and give details of some proofs omitted from the main body of the paper. The supplementary material outlines relevant mathematical background, for consultation as necessary.

2 Problem formulation

The exposition is based on a simple, abstract model of Cho and McCallum (2015), though the principles generalize (see appendix A).

For all $t \in \mathbb{Z}$,

$$x_t = Ax_{t-1} + \hat{A}\hat{x}_{1,t} + Bu_t \tag{1}$$

$$u_t = Ru_{t-1} + w_t \ . \tag{2}$$

The matrices $A, \hat{A} \in \mathbb{R}^{n\times n}$ ($\hat{A} \neq 0$), $B \in \mathbb{R}^{n\times m}$ are constants.

The variables include the independent variable $t \in \mathbb{Z}$, representing discrete time instants, a vector of endogenous variables $x_t \in \mathbb{R}^{n\times 1}$, the vector $\hat{x}_{1,t} \in \mathbb{R}^{n\times 1}$ representing a one-time-step "forecast" of the value of x_t, and a vector of exogenous inputs $u_t \in \mathbb{R}^{m\times 1}$, driven by a sequence $w_t \in \mathbb{R}^{m\times 1}$ of real-valued, independent, zero-mean random variables, whose covariance matrix contains only finite entries, defined on a common probability space. The model will be solved for $t \geq 0$, so the forecast $\hat{x}_{1,t}$ may depend on the initial conditions x_{-1}, $\hat{x}_{1,-1}$ and u_{-1}, which are formally treated as constants, and on the sequence of random variables $u_0, u_1, \ldots, u_t$ – or equivalently, the sequence $w_0, w_1, \ldots, w_t$.

It will be assumed that the polynomial matrix $[z^2\hat{A} - zI + z]$ is *regular* – meaning that its determinant does not vanish for all $z \in \mathbb{C}$, in which case the model too will be called regular. This is a common assumption, which serves to rule out a source of nonuniqueness that is unrelated to expectations.[7]

Blanchard example: For a simple illustration, consider a version of the univariate model of Blanchard (1985); namely, let all variables be scalars, and suppose for simplicity that A is zero. Blanchard points out that such a model has many possible interpretations, such as the movement of share prices under arbitrage against a constant interest rate. Some applications may require more general specification of the driving variables, but to allow for calculations to be performed essentially by inspection, it will here be assumed that $R = 0$. That yields exactly the univariate example

[7]The model is intended by Cho and McCallum to capture a local, linear approximation around the steady state of a DSGE model; see, for instance, the New Keynesian model of section 4. Under the conventional rational-expectations paradigm, the Cho-McCallum model has been used to capture equations with arbitrary finite numbers of expectation terms, having arbitrary expectational leads and lags (Broze et al., 1995; Binder and Pesaran, 1997; McCallum, 2007); see Appendix A for a direct extension of the results of the paper to such general models.

employed by Lubik and Schorfheide (2004). It will serve as a running example in the next section. □

Of course, the random variables $\hat{x}_{1,t}$ and x_t are undefined, in the absence of additional equations. For the sake of generality, it will be assumed only that the forecasts $\hat{x}_{1,t}$ depend linearly on the initial conditions and the driving variables w_t – and moreover, that their dependence on the driving variables can be represented by a linear, constant-coefficient, stochastic difference equation. This is in essence a version of assumption 3 of Muth (1961), and it is satisfied by other approaches to rational expectations.[8] The theory of linear, constant-coefficient difference equations implies that expectations must then obey an equation of the following form:

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau + \overline{x}_{t+1} \ , \tag{3}$$

where $\tilde{F}_t \in \mathbb{R}^{n\times m}$ vanishes for negative t, and $\overline{x}_{t+1}$ depends linearly on the initial conditions, but does not depend on the w_t. Such an equation will be called a *forecasting mechanism*. The convolution kernel determines the matrix of impulse responses of the forecasting mechanism, and the limits of the convolution sum ensure that forecasts are based on the appropriate information set.

The variables w_t are in essence merely a convenient means of defining the stochastic structure of the u_t, so it is also of interest to consider forecasting mechanisms driven by the economic, exogenous variables u_t:

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} F_{t-\tau} \tilde{u}_\tau + \overline{x}_{t+1} \ , \ \forall t \geq 0, \tag{4}$$

For every $t \geq 0$, $F_t \in \mathbb{R}^{n\times m}$. The sequence $\tilde{u}_t := \sum_{\tau=0}^{t} R^{t-\tau} w_\tau = u_t - R^{t+1} u_{-1}$ denotes the component of the sequence of endogenous variables u_t that depends only on the w_t and not on the initial conditions.

All of the analysis carried out in the next section follows from the above mild assumption on the form of forecasts, together with the aforementioned regularity property. It therefore represents a general analysis of rational expectations, within the context of linear, constant-coefficient, stochastic difference equations.

[8] See, for example, (Blanchard and Kahn, 1980; Binder and Pesaran, 1997; King and Watson, 1998; Klein, 2000; Sims, 2002; Lubik and Schorfheide, 2004).

Of specific interest are forecasting mechanisms that are unbiased. Let y be any square-integrable random variable defined on the common probability space of the w_t. For $t \geq -1$, $E_t(y)$ denotes the expected value of y, conditioned on the driving variables $w_0, w_1, \ldots, w_t$ – and subject to the full model, including the forecasting mechanism, and its initial conditions. Given consistent initial conditions, a forecasting mechanism (3) (respectively, (4)) is *model-consistent* if the full model (1–3) (respectively, (1,2,4)) satisfies

$$E_t(x_{t+1} - \hat{x}_{1,t}) = 0 \text{ , or equivalently, } \hat{x}_{1,t} = E_t(x_{t+1}) \text{ ,} \forall t \geq -1 \text{ .}$$

Such a forecasting mechanism embodies the "strong" rational-expectations hypothesis, whereby economic agents behave, in aggregate, as if they have access to all relevant information about the economy, and, on that basis, form expectations that do not incorporate any systematic errors.

It should be emphasized that the appropriate conditional expectations are subject to the forecasting mechanism, even if that forecasting mechanism is initially unknown to the modeler or analyst. This is a crucial point, which the author considers to be logically implied by, and entirely in the spirit of, the strong rational-expectations hypothesis. In economic terms, it represents the assumption that economic actors (in their aggregate) behave as if they know not only how the economy responds to shocks and to actors' expectations, but also how actors' own expectations are formed, and revised in response to shocks.

The main technical results of the paper include a general existence-and-uniqueness result for model-consistent forecasting mechanisms (Theorem 3.5, page 22), and identification of a simple structural condition that ensures existence, as well as realizability in the form of combined feedforward/feedback implementations (Theorem 3.6, page 24). On the basis of the form of the general solution, an asymptotic analysis explains the effect of expectations terms on dynamical stability (Corollary 5.1, page 30)). Finally, it is shown that uniqueness can be ensured, for a broad class of models, by requiring not only that forecasting errors be zero-mean, but that they be minimized, in the least-squares sense (Corollary 6.1, page 32).

3 The general model-consistent solution

To derive a general representation of model-consistent forecasting mechanisms, consider separately the case where the initial conditions are zero-valued, and that where the driving variables are zero-valued; their respective solutions can then be superimposed, by linearity, to provide a general solution.

It is in the first of these cases that the reason for nonuniqueness becomes apparent.

3.1 Zero-state response

Suppose then that the initial conditions x_{-1}, $\hat{x}_{1,-1}$, and u_{-1} are zero. In the terminology of linear, time-invariant systems, the resultant model trajectories comprise the *zero-state response*.

By assumption, the zero-state response of the forecasting mechanism itself must have the form of a convolution, and it is convenient in the first instance to consider a convolution with the driving variables w_t:

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau \ , \ \forall t \geq 0 \ . \tag{5}$$

Because the model (1,2) is then effectively a system of linear, constant-coefficient equations, the sequence of endogenous-variable vectors must have the form of a similar convolution:

$$x_t = \sum_{\tau=0}^{t} \tilde{G}_{t-\tau} w_\tau \ , \ \forall t \geq 0 \ . \tag{6}$$

Naturally, $\tilde{G}_t$ will depend on $\tilde{F}_t$, and vice versa. The model and the model-consistency condition will furnish two equations describing their relationship.

Substitute the convolution sums (5,6) into the equation (1), and use (2) to eliminate u_t:

$$\sum_{\tau=0}^{t} \tilde{G}_{t-\tau} w_\tau = A \sum_{\tau=0}^{t-1} \tilde{G}_{t-1-\tau} w_\tau + \hat{A} \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau + B \sum_{\tau=0}^{t} R^{t-\tau} w_\tau \ \forall t \in \mathbb{Z} \ . \tag{7}$$

Applying the conditional expectation operator E_0,

$$\tilde{G}_t w_0 = A\tilde{G}_{t-1} w_0 + \hat{A}\tilde{F}_t w_0 + BR^t w_0 \ , \ \forall t \geq 0 \ .$$

But here, $w_0 \in \mathbb{R}^{m\times 1}$ is arbitrary, so it must be that

$$\tilde{G}_t = A\tilde{G}_{t-1} + \hat{A}\tilde{F}_t + BR^t \ , \ \forall t \geq 0 \ . \tag{8}$$

For a second equation relating the two convolution kernels, bring in the model-consistency condition for $t \geq 0$:

$$\begin{aligned} \hat{x}_{1,t} = E_t(x_{t+1}) \iff & \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau = E_t(\sum_{\tau=0}^{t+1} \tilde{G}_{t+1-\tau} w_\tau) \ , \\ \iff & \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau = \sum_{\tau=0}^{t} \tilde{G}_{t+1-\tau} w_\tau \ . \end{aligned}$$

Once again taking expected values conditioned on w_0, and then factoring out w_0 on the grounds that its value is arbitrary,

$$\tilde{F}_t = \tilde{G}_{t+1} \ , \ \forall t \geq 0 \ . \tag{9}$$

The calculation of the zero-state response amounts to solving the system (8,9), and thus effectively reduces to the solution of the equation obtained by substituting $\tilde{G}_{t+1}$ for $\tilde{F}_t$ in (8):

$$\tilde{G}_t = A\tilde{G}_{t-1} + \hat{A}\tilde{G}_{t+1} + BR^t, \ t \geq 0 \ . \tag{10}$$

For this, it is assumed that the matrix polynomial $[z^2\hat{A} - zI + A]$ is *regular* – that its determinant is not identically zero. This is a standard assumption, for which there is ample justification. See, for example, (King and Watson, 1998; McCallum, 1998). In particular, it is necessary for the uniqueness of solutions, irrespective of expectations.

One means of solving the equation is by so-called *linearization* – a transformation into an equivalent first-order equation that preserves the regularity property:

$$\begin{bmatrix} I & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} \tilde{G}_t \\ \tilde{G}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & I \end{bmatrix} \begin{bmatrix} \tilde{G}_{t-1} \\ \tilde{G}_t \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} R^t \ , \ t \geq 0 \ . \tag{11}$$

Call the first two matrix coefficients $\underline{E}$ and $\underline{A}$. Then the polynomial matrix $z\underline{E} - \underline{A}$ is regular if and only if $z^2\hat{A} - zI + A$ is; moreover, the homogeneous part of (11) is a *strong linearization* of that of (10), so the two matrix polynomials share exactly the same eigenvalues, both finite and infinite (Lancaster, 2008).

For the regular case, the general solution of (11) is given by Brüll (2009). It shows that a unique solution for $t \geq 0$ requires only the specification of an initial condition,

$$\begin{bmatrix} \tilde{G}_{-1} \\ \tilde{G}_0 \end{bmatrix} .$$

But in the present context, $\tilde{G}_{-1}$ vanishes, so solution of (10) requires only an initial boundary condition $\tilde{G}_0$. In the absence of a specification of $\tilde{G}_0$, solutions will not be unique; its value is not determined by the rational-expectations hypothesis. But, given the form of the model (1,2), such nonuniqueness can only arise from that of model-consistent forecasting mechanisms: the additional boundary condition should therefore be expressed in terms of $\tilde{F}_t$. Setting $t = 0$ in (8), one finds

$$\tilde{G}_0 = \hat{A}\tilde{F}_0 + B \ . \tag{12}$$

Indeed, the immediate response of the model to a shock is mediated by two channels: a 'direct' one, modeled by the coefficient B, and the expectations channel, represented by $\hat{A}F_0$, so $\tilde{G}_0$ must have the particular form given above. The true free parameter is $\hat{A}\tilde{F}_0$, as will be shown in the existence and uniqueness results that follow.

But first, the equations must be solved. The results of Brüll (2009) also show that, whenever a solution of (11) exists, it is of exponential order. It follows that the same is true of solutions of (10). All solutions therefore possess unilateral z-transforms, which affords the possibility of avoiding the matrix-theoretic methods of Brüll, and solving (8,9) by algebraic, "frequency domain" methods. Indeed, as a practical matter, it will be convenient merely to solve for the z-transforms $\tilde{F}[z]$ and $\tilde{G}[z]$, which can then be *realized*, by standard methods, in the form of state-space representations of the time-domain convolutions (5,6).

Applying the unilateral z-transform, and its left- and right-shift properties, to (8,9), while keeping in mind that $\tilde{G}_t$ must vanish for negative t, one finds

$$\tilde{G}[z] = Az^{-1}\tilde{G}[z] + \hat{A}\tilde{F}[z] + B[I - Rz^{-1}]^{-1} \ , \tag{13}$$

$$\tilde{F}[z] = z[\tilde{G}[z] - \tilde{G}_0] \ . \tag{14}$$

The appearance of $\tilde{G}_0$ in the equations confirms the need for the aforementioned additional boundary condition; nonuniqueness results when, in effect, only the condition $\tilde{G}_{-1} = 0$ is applied. Substituting $\hat{A}\tilde{F}_0 + B$ for $\tilde{G}_0$, solve for $\tilde{F}[z]$ and $\tilde{G}[z]$:

$$\tilde{F}[z] = [z^2\hat{A} - zI + A]^{-1}\left[[zI - A](\hat{A}\tilde{F}_0 + B)[zI - R] - z^2B\right][I - Rz^{-1}]^{-1} \ ,$$
$$\tilde{G}[z] = [z^2\hat{A} - zI + A]^{-1}\left[z\hat{A}(\hat{A}\tilde{F}_0 + B)[zI - R] - zB\right][I - Rz^{-1}]^{-1} \ .$$

This establishes that, if suitable solutions of (8,9) exist, they must have transforms of the above forms, and therefore must be unique. But the form of $\tilde{F}[z]$ also determines existence:

Proposition 3.1 *Suppose that $[z^2\hat{A} - zI + A]$ is regular. Then for any given value of the product $\hat{A}\tilde{F}_0 \in \mathbb{R}^{n\times m}$, there exists a solution of the system (8,9), such that $\tilde{F}_t$ and $\tilde{G}_t$ vanish for negative t, and $\hat{A}\tilde{F}_0$ has the specified value, if and only if the rational matrix*

$$\tilde{F}[z] = [z^2\hat{A} - zI + A]^{-1}\left[[zI - A](\hat{A}\tilde{F}_0 + B)[zI - R] - z^2B\right][I - Rz^{-1}]^{-1}$$

is proper.

In that case, the inverse z-transforms

$$\tilde{F}_t := \mathcal{Z}^{-1}\left\{[z^2\hat{A} - zI + A]^{-1}\left[[zI - A](\hat{A}\tilde{F}_0 + B)[zI - R] - z^2B\right][I - Rz^{-1}]^{-1}\right\}$$
$$\tilde{G}_t := \mathcal{Z}^{-1}\left\{[z^2\hat{A} - zI + A]^{-1}\left[z\hat{A}(\hat{A}\tilde{F}_0 + B)[zI - R] - zB\right][I - Rz^{-1}]^{-1}\right\}$$

comprise the unique such solution.

Proof: Suppose that an appropriate solution of the system exists. Then $\tilde{F}_t$ and $\tilde{G}_t$ vanish for negative t. Their respective z-transforms are then proper rational matrices; and, by the preceding discussion, $\tilde{F}[z]$ and $\tilde{G}[z]$ have the form given in the statement of the proposition, where $\tilde{F}_0$ is the initial value of $\tilde{F}_t$. This establishes uniqueness (by z-transform inversion), and the necessary condition for existence.

Conversely, for any specified value of $\hat{A}\tilde{F}_0$, if the given matrix $\tilde{F}[z]$ is proper, then so is

$$\begin{aligned}\tilde{G}[z] &= [z^2\hat{A} - zI + A]^{-1}[z^2\hat{A}(\hat{A}\tilde{F}_0 + B) - zB[I - Rz^{-1}]^{-1}] \\ &= (\hat{A}\tilde{F}_0 + B) + [z^2\hat{A} - zI + A]^{-1}[[zI - A](\hat{A}\tilde{F}_0 + B) - zB[I - Rz^{-1}]^{-1}] \\ &= (\hat{A}\tilde{F}_0 + B) + z^{-1}\tilde{F}[z] \ .\end{aligned}$$

Both $\tilde{F}[z]$ and $\tilde{G}[z]$ are therefore unilateral z-transforms. Because $z^{-1}\tilde{F}[z]$ is then strictly proper, the initial value of the inverse transform of $\tilde{G}[z]$ must equal $\hat{A}\tilde{F}_0 + B$. It follows that (13) and (14) are satisfied. Transforming back to the time domain then shows that (8) and (9), and consequently (10), are satisfied.

Applying (10) at $t = 0$ gives $\tilde{G}_0 = \hat{A}\tilde{G}_1 + B$, so $\hat{A}\tilde{G}_1$ equals the specified value of $\hat{A}\tilde{F}_0$. But by (9), $\tilde{F}_t = \tilde{G}_{t+1}$ for all $t \geq 0$. So the product $\hat{A}\tilde{F}_t$ indeed has the specified value at $t = 0$.

This proves the sufficient condition for existence. ∎

The above result leads to a necessary condition on the form of model-consistent forecasting mechanisms (3):

Corollary 3.2 *Let the model (1,2) be regular. For any given value of the product $\hat{A}\tilde{F}_0 \in \mathbb{R}^{n\times m}$, any model-consistent forecast mechanism (3) must satisfy*

$$\tilde{F}_t = \mathcal{Z}^{-1}\left\{[z^2\hat{A} - zI + A]^{-1}\left[[zI - A](\hat{A}\tilde{F}_0 + B)[zI - R] - z^2 B\right][I - Rz^{-1}]^{-1}\right\} \ .$$

with $\tilde{F}[z]$ proper.

If x_{-1}, $\hat{x}_{1,-1}$, and u_{-1} are zero, and $\hat{x}_{1,t} = \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau$, then the model (1,2) satisfies

$$x_t = \sum_{\tau=0}^{t} \tilde{G}_{t-\tau} w_\tau \ ,$$

where

$$\tilde{G}_t := \mathcal{Z}^{-1}\left\{[z^2\hat{A} - zI + A]^{-1}\left[z\hat{A}(\hat{A}\tilde{F}_0 + B)[zI - R] - zB\right][I - Rz^{-1}]^{-1}\right\} \ .$$

The forecast error realized at time $t \geq 0$ is $x_t - \hat{x}_{1,t-1} = (\hat{A}\tilde{F}_0 + B)w_t$.

Proof: The necessary conditions on the form of model-consistent forecast mechanisms, and on the solutions of the resulting models, are direct consequences of Proposition 3.1, by the foregoing discussion.

As regards the satisfaction of the model equations, again by Proposition 3.1, equation (8) is satisfied. By linearity, and the fact that $\tilde{G}_t$ vanishes for negative t, so then is (7). It follows that the $\hat{x}_{1,t}$ and the x_t in the statement of the corollary solve the model equations (1,2).

For the forecast dated at $t = -1$, then,

$$x_0 - \hat{x}_{1,-1} = x_0 = \tilde{G}_0 w_0 = (\hat{A}\tilde{F}_0 + B) w_0 \ ;$$

and by equation (9), for all $t \geq 0$,

$$x_{t+1} - \hat{x}_{1,t} = \sum_{\tau=0}^{t+1} \tilde{G}_{t+1-\tau} w_\tau - \sum_{\tau=0}^{t} \tilde{F}_{t-\tau} w_\tau = \tilde{G}_0 w_{t+1} = (\hat{A}\tilde{F}_0 + B) w_{t+1} \ . \tag{15}$$

∎

These forecast errors are zero-mean, confirming model-consistency.

Blanchard example: Application to the univariate example of Blanchard (1985) and Lubik and Schorfheide (2004) (with A, $R = 0$) gives the following, using lower-case letters to emphasize the scalar nature of the quantities:

$$\tilde{f}[z] = \frac{z(\hat{a}\tilde{f}_0 + b) - zb}{z\hat{a} - 1} \ \ \& \ \ \tilde{g}[z] = \frac{z\hat{a}(\hat{a}\tilde{f}_0 + b) - b}{z\hat{a} - 1} \ .$$

By means of simple algebra, these transforms can be rewritten in terms of that of the exponential $\hat{a}^{-t}$, and inverted by inspection:

$$\tilde{f}_t = \tilde{f}_0 \ \hat{a}^{-t} \ \ \& \ \ \tilde{g}_t = \hat{a}\tilde{f}_0 \ \hat{a}^{-t} + b \ \delta_t \ ;$$

where δ_t is the Kronecker delta function, which is unity if $t = 0$, but vanishes otherwise. Convolving the respective impulse responses with w_t,

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} \tilde{f}_0 \ \hat{a}^{-(t-\tau)} w_\tau \ \ \& \ \ x_t = \sum_{\tau=0}^{t} \hat{a} \ \tilde{f}_0 \ \hat{a}^{-(t-\tau)} \ w_\tau + b \ w_t \ .$$

Whatever the value of $\tilde{f}_0$, these sequences satisfy (1,2), and give rise to the zero-mean forecast errors established in the theorem. $\square$

The solution of the model can be expressed in terms of the exogenous inputs u_t rather than the driving variables w_t, by simply right-multiplying $\tilde{F}[z]$ and $\tilde{G}[z]$ by $[I - Rz^{-1}]$:

$$F[z] = [z^2\hat{A} - zI + A]^{-1}[[zI - A](\hat{A}\tilde{F}_0 + B)[zI - R] - z^2 B] \; ,$$
$$G[z] = [z^2\hat{A} - zI + A]^{-1}z[\hat{A}(\hat{A}\tilde{F}_0 + B)[zI - R] - B] \; ,$$

The time-domain counterpart of right-multiplying by $[I - Rz^{-1}]^{-1}$ to recover $\tilde{F}[z]$ and $\tilde{G}[z]$ is convolution with R^t. It follows that $F_0 = \tilde{F}_0$. It also follows that convolution of F_t (respectively, G_t) with u_t is equivalent to convolution of $\tilde{F}_t$ (resp., $\tilde{G}_t$) with w_t (by the associativity of convolution). Because $[I - Rz^{-1}]$ and its rational-matrix inverse are both proper, and because the initial value of each of their inverse z-transforms is the identity matrix, multiplication by either of them does nothing to alter properness.

The following counterpart of Corollary 3.2 is immediate:

Corollary 3.3 *Suppose that the model (1,2) is regular. Then, for any specified value of $\hat{A}F_0 \in \mathbb{R}^{n\times m}$, any model-consistent forecasting mechanism (4) has*

$$F_t = \mathcal{Z}^{-1}\{F[z]\} = \mathcal{Z}^{-1}\{[z^2\hat{A} - zI + A]^{-1}[[zI - A](\hat{A}F_0 + B)[zI - R] - z^2 B]\} \; ,$$

where $F[z]$ is proper.

If x_{-1}, $\hat{x}_{1,t}$, and u_{-1} are all zero, and $\hat{x}_{1,t} = \sum_{\tau=0}^{t} F_{t-\tau}u_\tau$, then the resulting full model (1-4) satisfies

$$x_t = \sum_{\tau=0}^{t} G_{t-\tau}u_\tau \; ,$$

where

$$G_t = \mathcal{Z}^{-1}\{G[z]\} = \mathcal{Z}^{-1}\{[z^2\hat{A} - zI + A]^{-1}z[\hat{A}(\hat{A}F_0 + B)[zI - R] - B]\} \; .$$

The forecast error realized at time $t \geq 0$ is $x_t - \hat{x}_{1,t-1} = (\hat{A}F_0 + B)w_t$. $\blacksquare$

(In the running example, $u_t = w_t$, so there is no point in revisiting it at this point.)

The especially alert reader will have noticed that the formulas for $\tilde{G}[z]$ and $G[z]$ implicitly assume the exact cancellation of the term $[I - Az^{-1}]^{-1}$, arising from the model equation (1) via equation (14), by the term $[I - Az^{-1}]$ arising in $\tilde{F}[z]$. (To see the cancellation, substitute the solution for $\tilde{F}[z]$ into equation (13).) However, the implementation of $\tilde{F}[z]$ given in section 3.4 ensures that the latter term also arises from the model equation itself, via feedback, in which case the cancellation is merely a matter of algebra, and does not demand infinite precision in the aggregate actions of the public.

A related concern is that the solution may be very sensitive to the value of $\hat{A}$: mathematically, variations in that value could give rise to singular perturbations of the nominal matrix polynomial $[z^2\hat{A} - zI + A]$, as is shown in section 5. A thorough treatment of the latter robustness issue is beyond the scope of this paper; but an element of a resolution might be to treat $\hat{A}$ solely as a parameter of the forecasting mechanism, whose output would then be the product $\hat{A}\hat{x}_{1,t}$ – the 'economic effect' of the forecast $\hat{x}_{1,t}$, as opposed to the forecast itself.

3.2 Zero-input response

The effect of the driving terms having been analyzed, it now suffices (by linearity) to consider the case where the initial conditions may have nonzero values, but all of the driving variables w_t vanish. In system-theoretic terms, the corresponding solution of the model is called its "zero-input response."

By the definition of forecasting mechanisms, in this case, $\hat{x}_{1,t} = \overline{x}_{t+1}$, where $\overline{x}_{t+1}$ depends linearly on the initial conditions, but not at all on the w_t. For any $t \geq 0$, model-consistency therefore requires that

$$x_{t+1} = E_t(\hat{x}_{1,t}) = \hat{x}_{1,t} \ .$$

In other words, $\hat{x}_{1,t}$ must be an exact forecast of x_{t+1}.

The sequence x_t must therefore be the solution of the following "perfect-foresight" model, which captures the case where the driving variables w_t are zero-valued for all $t \geq 0$:

$$x_t = Ax_{t-1} + \hat{A}\hat{x}_{1,t} + BR^{t+1}u_{-1} \ , \tag{16}$$

$$\hat{x}_{1,t-1} = x_t \ . \tag{17}$$

Substituting for $\hat{x}_{1,t}$ in the first equation,

$$x_t = Ax_{t-1} + \hat{A}x_{t+1} + BR^{t+1}u_{-1} \ , \ \forall t \geq 0 \ . \tag{18}$$

Once again, the results of Brüll (2009) imply that any solution has a unilateral z-transform. Taking transforms and solving for $X[z]$,

$$\overline{X}[z] := X[z] = [z^2\hat{A} - zI + A]^{-1}[z^2\hat{A}\hat{x}_{1,-1} - zAx_{-1} - zBR[I - Rz^{-1}]^{-1}u_{-1}] \ ; \tag{19}$$

Here, the second equation of the perfect-foresight model has been used to replace x_0 with the initial condition $\hat{x}_{1,-1}$.

Proposition 3.4 *Suppose that the model (1,2) is regular. Let*

$$\begin{aligned}\overline{x}_t &:= \mathcal{Z}^{-1}\{\overline{X}[z]\} \\ &= \mathcal{Z}^{-1}\left\{[z^2\hat{A} - zI + A]^{-1}[z^2\hat{A}\hat{x}_{1,-1} - zAx_{-1} - zBR[I - Rz^{-1}]^{-1}u_{-1}]\right\} \ .\end{aligned}$$

Then, a perfect-foresight solution exists if and only if $\overline{X}[z] - \hat{x}_{1,-1}$ is strictly proper. In that case, the unique such solution has $x_t = \overline{x}_t$ and $\hat{x}_{1,t-1} = \overline{x}_{t+1}$.

Proof: By the above discussion, any solution x_t of (18) must have a unilateral z-transform of the form of $\overline{X}[z]$. As a unilateral z-transform, $\overline{X}[z]$ must be proper. Moreover, because x_0 must equal $\hat{x}_{1,-1}$, the matrix $\overline{X}[z] - \hat{x}_{1,-1}$ must be strictly proper. This establishes the necessary condition for existence.

Now, suppose that $\overline{X}[z] - \hat{x}_{1,-1}$ is strictly proper. Then $\overline{X}[z]$ is proper, and $\hat{x}_{1,-1}$ is the value of the inverse transform $\overline{x}_t$ at $t = 0$. Rearrange the expression for $\overline{X}[z]$:

$$\overline{X}[z] = Az^{-1}[\overline{X}[z] + zx_{-1}] + \hat{A}z[\overline{X}[z] - \hat{x}_{1,-1}] + BR[I - Rz^{-1}]^{-1}u_{-1} \ .$$

Transforming to the time domain, that inverse transform $\overline{x}_t$ is seen to satisfy (18). The uniqueness of this solution follows from that of the transform $\overline{X}[z]$. If, in addition, $\hat{x}_{1,t-1} = \overline{x}_t$, for all $t \geq 0$, then (16,17) is satisfied. The full model (1,2,3) or (1,2,4) then satisfies $x_t = \overline{x}_t$, if $w_t = 0$, $\forall t \geq 0$. ∎

Blanchard example: In the case of the running example,

$$\overline{x}[z] = \frac{z\hat{a}\hat{x}_{1,-1}}{z\hat{a} - 1} = \frac{\hat{x}_{-1,1}}{1 - \hat{a}^{-1}z^{-1}} \ .$$

But this is just the unilateral z-transform of $\overline{x}_t = \hat{x}_{-1,1} \ \hat{a}^{-t}$. □

As a consequence of Proposition 3.4, the initial conditions will be said to be *consistent* if $\overline{X}[z] - \hat{x}_{1,-1}$ is strictly proper; and in that case, a model-consistent forecasting mechanism must have $\overline{x}_t = \mathcal{Z}^{-1}\{\overline{X}[z]\}$.[9]

By the last part of the proposition, $\overline{x}_t$ can be said, in system-theoretic terminology, to represent the *zero-input response* of the full model (1-4), under model-consistent forecasts. It shows how the system responds to nonzero initial conditions, when any driving terms vanish for $t \geq 0$.

However, the nature of the derivation should be borne in mind in potential applications that might otherwise exceed the limitations of the results. If the model has been evolving through negative time instants, and the model parameters that hold for $t \geq 0$ differ from those in effect for $t = -1$, it is plausible to suppose that realized values of x_{-1} and u_{-1} are valid initial conditions for the zero-input response for $t \geq 0$. But it is less clear what are the implications of realizations of the forecast $\hat{x}_{1,-1}$. It is not obvious that "perfect foresight" should apply under an unforeseen change in model parameters.

Like the formulas derived in the previous section, the above solution for $\overline{X}[z]$ implicitly assumes that a term $[I - Az^{-1}]^{-1}$ arising from the model is exactly cancelled by a term $[I - Az^{-1}]$ derived from the forecast mechanism. But this concern can be addressed by realizing the forecast mechanism using feedback from the model equation, as in section 3.4.

[9]Like the state-space realization of proper rational matrices $F[z]$ and $G[z]$, the inversion of the proper matrix $\overline{X}[z]$ can be carried out with the use of standard software tools; it amounts to computing the impulse responses of a system with a given proper transfer matrix.

3.3 Total response

The total response is the sum of the zero-state and zero-input responses. Indeed, fix any model-consistent forecasting mechanism; then x_t and $\hat{x}_{1,t}$ must be linear functions of the random variables w_τ, $0 \le \tau \le t$, and of the initial conditions x_{-1}, $\hat{x}_{1,-1}$, and u_{-1}. It follows by linearity that they must then be obtained by summing the separate respective responses to the driving variables and to the initial conditions – namely, the zero-state and zero-input responses.

Theorem 3.5 *Suppose that the model (1,2) is regular. For any $\hat{A}F_0 \in \mathbb{R}^{n\times m}$, define*

$$\begin{aligned}
F_t &= \mathcal{Z}^{-1}\{F[z]\} = \mathcal{Z}^{-1}\{[z^2\hat{A} - zI + A]^{-1}[[zI - A](\hat{A}F_0 + B)[zI - R] - z^2 B]\} \ , \\
G_t &= \mathcal{Z}^{-1}\{[z^2\hat{A} - zI + A]^{-1} z[\hat{A}(\hat{A}F_0 + B)[zI - R] - B]\} \ , \quad \textit{and} \\
\overline{x}_t &= \mathcal{Z}^{-1}\{\overline{X}[z]\} \\
&= \mathcal{Z}^{-1}\left\{[z^2\hat{A} - zI + A]^{-1}[z^2\hat{A}\hat{x}_{1,-1} - zAx_{-1} - zBR[I - Rz^{-1}]^{-1}u_{-1}]\right\} \ .
\end{aligned}$$

Suppose that the initial conditions are consistent ($\overline{X}[z] - \hat{x}_{1,-1}$ is strictly proper). Then there exists a model-consistent forecasting mechanism (4) for (1,2) if and only if $F[z]$ is proper.

In that case, in terms of the above inverse transforms, the unique model-consistent forecasting mechanism (4) is

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} F_{t-\tau}\tilde{u}_\tau + \overline{x}_{t+1} \ , \ \forall t \ge 0 \ ; \tag{20}$$

and the resulting full model (1–4) satisfies

$$x_t = \sum_{\tau=0}^{t} G_{t-\tau}\tilde{u}_\tau + \overline{x}_t \ , \ \forall t \ge 0. \tag{21}$$

The forecast error realized at time $t \ge 0$ is

$$(\hat{A}F_0 + B)w_t = (\hat{A}F_0 + B)(u_t - Ru_{t-1}) \ .$$

Proof: If the initial conditions are consistent, then the necessary forms of model-consistent forecasting mechanisms, and of the unique model solution resulting from such a forecasting mechanism, follow from Corollary 3.3 and Proposition 3.4, by the linearity of the model and of conditional expectations. The zero-input solution has no effect on the forecasting errors, so the overall forecasting error realized at time t is as derived in section 3.1. It follows that the specified forecasting mechanism (20) is indeed model-consistent. ∎

Blanchard example: Summing the results of previous calculations, the general solution of the running example is seen to be

$$\begin{aligned} \hat{x}_{1,t} &= \sum_{\tau=0}^{t} f_0 \ \hat{a}^{-(t-\tau)} \ w_\tau + \hat{x}_{1,-1} \ \hat{a}^{-(t+1)} \ , \\ x_t &= \sum_{\tau=0}^{t} f_0 \ \hat{a}^{-(t-1-\tau)} \ w_\tau + b \ w_t + \hat{x}_{1,-1} \ \hat{a}^{-t} \ . \end{aligned}$$

Note that for any value of f_0, this yields a unique solution, irrespective of the value of $\hat{a}$; but the conventional approach produces a unique solution only when $\hat{a} < 1$, in which case it effectively requires both $\hat{x}_{1,-1}$ and f_0 to vanish, so that $\hat{x}_{1,t} \equiv 0$, and $x_t = b \ w_t$. . □

Suppose that a suitable value of $\hat{A}F_0$, representing the economic effects of of the immediate reactions of forecasts to shocks, can be specified. Then the results of this section resolve the nonuniqueness of rational expectations: as long as a model-consistent forecasting mechanism exists, $\hat{A}F_0$ determines G_0, and "rationality" then determines F_t and G_t for all $t > 0$.

In the next section, a simple assumption is introduced that ensures existence for any value of $\hat{A}F_0$.

3.4 Well-posedness, existence, and feedback

A regular model (1,2) will be called *well-posed* if the inverse of the "characteristic matrix" $[-z\hat{A} + I - Az^{-1}]$ is proper – or equivalently, if $[z^2\hat{A} - zI + A]^{-1}$ is strictly proper. For example, this is so whenever $\hat{A}$ is nonsingular, but not when $\hat{A}$ is nilpotent.

Well-posedness admits a simple sufficient condition for consistency of the initial conditions. The initial conditions will be called *weakly consistent* if

$$\hat{x}_{1,-1} - Ax_{-1} - BRu_{-1} \in \text{ Im } \hat{A} \ .$$

Indeed, this condition is plainly necessary if a solution of (1,2) is to exist when $w_t = 0$ and the prediction $\hat{x}_{1,-1}$ is exact. Together, well-posedness and weak consistency imply the existence of a model-consistent forecasting mechanism for any possible value of $\hat{A}F_0$. In economic terms, well-posedness obviates any assumption that economic actors use their presumed aggregate knowledge of the model to "choose" parameter values $\hat{A}F_0$ for which solutions exist.

Well-posedness also allows for the realization of model-consistent forecasting mechanisms in the form of feedforward/feedback interconnections with the rest of the model, as represented by (1,2). (The condition implies the "well-posedness" of that interconnection, in the specific sense in which that term is applied to feedback systems.) The use of feedback resolves key robustness issues with respect to the model parameter A, but sensitivity to the parameter $\hat{A}$ remains an issue (discussed at the end of section 3.1).

For brevity, proofs for this section are relegated to appendix B. The results themselves are summarized in the following:

Theorem 3.6 *If the model (1,2) is regular and well-posed, and the initial conditions are weakly consistent, then for every possible value of the product $\hat{A}F_0 \in \mathbb{R}^{n\times m}$, there exists a unique model-consistent forecasting mechanism. That forecasting mechanism can be realized by the following feedforward/feedback law:*

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} \varPhi_{t-\tau}[Ax_\tau + BRu_\tau] - \sum_{\tau=0}^{t} \Psi_{t-\tau}\hat{A}F_0 w_\tau - \Psi_t(\hat{x}_{1,-1} - Ax_{-1} - BRu_{-1}).$$

Here, $\varPhi_t := \mathcal{Z}^{-1}\{[I - z\hat{A}]^{-1}\}$ and $\Psi_t := \mathcal{Z}^{-1}\{[I - z\hat{A}]^{-1}z\hat{A}\hat{A}^g\}$, where $\hat{A}^g$ is a generalized inverse of $\hat{A}$ (s.t. $\hat{A}\hat{A}^g\hat{A} = \hat{A}$). ∎

4 Conventional determinacy of a New Keynesian model

In this section it is shown – purely for comparison – how the general solution of the previous section lends itself to the reproduction of conventional results.

Consider for example the loglinearized New Keynesian model of Lubik and Schorfheide (2004):

$$y_t = \hat{y}_{1,t} - \tau(r_t - \hat{\pi}_{1,t}) + g_t \tag{22}$$

$$\pi_t = \beta\hat{\pi}_{1,t} + \kappa(y_t - z_t) \tag{23}$$

$$r_t = \rho_r r_{t-1} + (1 - \rho_r)(\psi_1\pi_t + \psi_2[y_t - z_t]) + \epsilon_{r,t} \tag{24}$$

$$g_t = \rho_s g_{t-1} + \epsilon_{g,t} \tag{25}$$

$$z_t = \rho_z z_{t-1} + \epsilon_{z,t} \tag{26}$$

The scalar variables y_t, π_t, and r_t respectively represent output, inflation, and the nominal interest rate, expressed as percentage deviations from a trend path or a steady state; g_t and z_t represent the effects of exogenous shifts on the first two equations. In accordance with the usual notation, $\hat{y}_{1,t}$ and $\hat{\pi}_{1,t}$ represent forecasts of y_{t+1} and π_{t+1}, dated at time t. Write $w_t = [\epsilon_{g,t}\ \epsilon_{z,t}\ \epsilon_{r,t}]'$, and consider distinct values of the resulting vector-valued sequence to be independent, identically distributed, and zero-mean. Suppose also that their covariance matrices contain only finite elements.

The scalar coefficients are as follows: τ represents intertemporal substitution elasticity, β is the households' discount factor, κ is the slope of the expectational Phillips curve; the third equation describes the monetary authority's behavior, ψ_1 and ψ_2 being 'Taylor-rule' coefficients.

Letting $x_t = [y_t\ \pi_t\ r_t]'$, $u_t = [g_t\ z_t\ \epsilon_{r,t}]'$, and $w_t = [\epsilon_{g,t}\ \epsilon_{z,t}\ \epsilon_{r,t}]'$, it is an easy matter to put the equations into the form (1,2). Setting the coefficients equal to the mean values given in Table 1 of (Lubik and Schorfheide, 2004), with $\beta = 0.99$, according to (Lubik and Schorfheide, 2003), one finds[10]

[10]Figures that are not exact are displayed with seven or eight significant digits, as a reminder that the calculations in principle require infinite precision.

$$x_t = \begin{bmatrix} 0 & 0 & -0.2083333 \\ 0 & 0 & -0.1041667 \\ 0 & 0 & 0.4166667 \end{bmatrix} x_{t-1} + \begin{bmatrix} 0.8333333 & 0.1897917 & 0 \\ 0.4166667 & 1.0848958 & 0 \\ 0.3333333 & 0.6204167 & 0 \end{bmatrix} \hat{x}_{1,t}$$

$$+ \begin{bmatrix} 0.8333333 & 0.1666667 & -0.4166667 \\ 0.4166667 & -0.4166667 & -0.2083333 \\ 0.3333333 & 0.3333333 & 0.8333333 \end{bmatrix} u_t \ , \qquad (27)$$

$$u_t = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_{t-1} + w_t \ . \qquad (28)$$

(the respective matrix coefficients being the values of A, $\hat{A}$, B, and R). This model satisfies well-posedness, because all nine entries of $[z^2\hat{A} - zI + A]^{-1}$ are strictly proper, so a model-consistent forecasting mechanism exists for every possible value of $\hat{A}F_0$.

In order for the conventional approach to yield a unique solution, it has been observed that the model must satisfy the "Taylor principle": ψ_1, the coefficient of inflation in the interest-rate policy Taylor rule, must have a value greater than unity, so that, for instance, a rise in inflation is met with a greater percentage increase in the interest rate. In this case, policy is said to be "active"; if ψ_1 is less than unity, policy is "passive." According to the chosen values of the coefficients, ψ_1, the coefficient of inflation in the interest-rate policy 'Taylor rule,' has the value 1.10. Roughly speaking, when interest-rate policy is passive, the New Keynesian model has only a single unstable eigenvalue, which is insufficient to determine a unique solution under the conventional rational-expectations paradigm; but when policy is active, it has two unstable eigenvalues, which do result in uniqueness. Specifically, when $\psi_1 = 1.10$, the general solution has unstable eigenvalues at $z = 1.4461829$ and $z = 1.0446352$. The corresponding left eigenvectors of the denominator polynomial are respectively,

$$\begin{bmatrix} -0.5818587 & 0.6738827 & -0.4553268 \end{bmatrix} \& \begin{bmatrix} -0.0473748 & 0.6928388 & 0.7195346 \end{bmatrix} .$$

In order to show – strictly for purposes of comparison – that these findings on conventional determinacy are indeed reproducible within the framework of the general solution, dynamical stability will now be imposed on the model. Following Lubik and Schorfheide (2004), assume that all initial conditions are zero-valued, and therefore focus on the zero-state response. Because the matrix R is stable, it suffices to consider the matrix polynomial $G[z]$. The free parameter $\hat{A}F_0$ appears only in the numerator of the matrix-fraction description of $G[z]$, so the only way its value can be chosen so as to stabilize an otherwise unstable model is by arranging for unstable 'poles' of $G[z]$ to be canceled by 'zeros.' In the multivariable case, this means that the numerator matrix polynomial of $G[z]$ must have the same unstable eigenvalues, with the same respective left eigenvectors, as the denominator matrix polynomial.

Note that $\lambda_i \in \mathbb{C}$ is an eigenvalue of the numerator matrix polynomial of $G[z]$, with left eigenvector c_i, for both $i = 1$ and $i = 2$, if and only if

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \hat{A}(\hat{A}F_0 + B) = \begin{bmatrix} c_1 B(\lambda_1 I - R)^{-1} \\ c_2 B(\lambda_2 I - R)^{-1} \end{bmatrix}$$

Each row of the above equation represents three equations, each in a distinct pair of unknowns (the entries from the first two rows of a distinct column of $\hat{A}F_0 + B$). So a single pole-zero cancellation does not determine a unique solution, but the two simultaneous cancellations do. The first two columns of

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \hat{A}$$

are linearly independent, yielding a unique solution for the first two rows of $\hat{A}F_0 + B$; approximately,

$$\begin{bmatrix} 1.6999275 & 0.4900217 & -0.6182074 \\ 1.85166 & -0.5554980 & -0.4620143 \end{bmatrix}.$$

This in turn yields the first two rows of F_0,

$$\begin{bmatrix} 0.8094723 & 0.4571583 & -0.2066718 \\ 1.0118144 & -0.3035443 & -0.1544551 \end{bmatrix}.$$

These determine a unique value for $\hat{A}F_0$ – approximately,

$$\begin{bmatrix} 0.8665942 & 0.3233551 & -0.2015408 \\ 1.4349934 & -0.1388313 & -0.2536809 \\ 0.8975706 & -0.0359379 & -0.1647171 \end{bmatrix} .$$

So the requirement of dynamical stability of the model determines a unique value of $\hat{A}F_0$, and therefore, by Theorem 3.5, a unique model-consistent forecasting mechanism.

The resulting matrix $G[z]$ is the following:

$$(z-0.334)^{-1} \begin{bmatrix} 1.700z - 0.949 & 0.490z - 0.0497 & -0.618z \\ 1.852z - 0.903 & -0.555z + 0.271 & -0.462z \\ 1.231z & -0.369z & 0.669z \end{bmatrix}$$

It can be *realized* in the form of the following state-space model:[11]

$$\begin{aligned} \zeta_{t+1} &= \quad 0.3343081 \quad \zeta_t + \begin{bmatrix} 0.8815320 & -0.2644596 & 0.4788405 \end{bmatrix} u_t \ , \\ x_t &= \begin{bmatrix} -0.4316088 \\ -0.3225607 \\ 0.4668023 \end{bmatrix} \zeta_t + \begin{bmatrix} 1.6999275 & 0.4900217 & -0.6182074 \\ 1.85166 & -0.5554980 & -0.4620143 \\ 1.230904 & -0.3692712 & 0.6686162 \end{bmatrix} u_t \ . \end{aligned}$$

Whereas a minimal state-space realization of $G[z]$ is normally third-order (see section 6 for an example), at this isolated point in the parameter space of $\hat{A}F_0$, it is first-order. This confirms the two pole-zero cancellations, and reproduces the results of the conventional approach to determinacy.

In order for the two unstable eigenvalues to be suppressed in this manner, the initial values of the impulse responses of the forecasts of output and inflation to the three shocks must equal – exactly – the values given by the respective entries of the first two rows of F_0 shown (approximately) above: unless the aggregate forecasts of the public respond to the respective shocks in this manner, with infinite precision, the model will be unstable.[12]

[11] A realization of $G[z]$ is a state-space representation of the time-domain relationship $x_t = \sum_{\tau=0}^{t} G_{t-\tau} u_\tau$. This one was found by means of the Scilab command `tf2ss`, based on a Matlab command of the same name. The forms of state-space realizations of $F[z]$ and $G[z]$ will be discussed in section 6, in the more typical context of a higher-order model.

[12] In double-precision floating-point arithmetic, it takes only a multiplicative perturbation of F_0

Moreover, the next section shows that the effect of this suppression is precisely to eliminate the eigenvalues (and the associated dynamics) that arise from expectations.

5 Rational expectations and stability

The relationship between expectations and stability is a longstanding concern (Arrow and Nerlove, 1958). This section presents an asymptotic analysis of the effects of expectation terms on eigenvalues. It helps to explain why saddlepoint instabilities are common in rational-expectations models, and why solutions may be very sensitive to the value of $\hat{A}$. Finally, its application to the New Keynesian model shows that the conventional solution suppresses precisely the dynamics that arise from expectations.

If a 'small' scalar multiplicative weight ϵ is attached to the matrix coefficient $\hat{A}$ of the forecast term, then unless that matrix is nilpotent, the denominator matrix $[z^2\epsilon\hat{A} - zI + A]$ of the solution is a singular perturbation of that of the lower-order model with $\epsilon = 0$: the degree of the denominator polynomial is a fixed integer greater than n for all $\epsilon > 0$; but for $\epsilon = 0$ it is equal to n.

The solution is always unstable when that weight is positive but sufficiently small. If z remains bounded as ϵ tends to zero, then the denominator polynomial tends to $-[zI-A]$, so n of the eigenvalues approach those of the matrix A. But if $\hat{A}$ has nonzero eigenvalues, then the denominator polynomial has more than n finite eigenvalues: in order for the degree to drop at $\epsilon = 0$, some of those finite eigenvalues must 'escape' to infinity.

Indeed, suppose that $\hat{A}$ has $m > 0$ nonzero eigenvalues. To capture the behavior of eigenvalues that vary like $1/\epsilon$ as ϵ tends to zero, perform the change of variable $z = \lambda/\epsilon$. The matrix polynomial $[z^2\epsilon\hat{A} - zI + A]$ becomes

$$[\frac{\lambda^2}{\epsilon}\hat{A} - \frac{\lambda}{\epsilon}I + A] = \epsilon^{-1}[\lambda^2\hat{A} - \lambda I + \epsilon A] \ .$$

As ϵ tends to zero, m eigenvalues of the matrix polynomial in λ approach the reciprocals of the nonzero eigenvalues of $\hat{A}$. Therefore, the moduli of m eigenvalues of the matrix polynomial in $z = \lambda/\epsilon$ tend to infinity.

On the other hand, if $\hat{A}$ is nonsingular, and the weight applied to expectations is sufficiently *large*, then the eigenvalues of the matrix polynomial in z will be stable:

on the order of 1 ± 10^{-9} to spoil the pole-zero cancellation.

Corollary 5.1

Small expectation gain: *In equation (1) above, replace the coefficient matrix $\hat{A}$ with $\epsilon\hat{A}$, where ϵ is a real, positive scalar. Suppose that $\hat{A}$ has some nonzero eigenvalue. Then, for sufficiently small $\epsilon > 0$, the denominator matrix polynomial $[z^2\epsilon\hat{A} - zI + A]$ has unstable eigenvalues; consequently, barring pole-zero cancellations, the full model (1-4) is dynamically unstable under model-consistent expectations.*

Large expectation gain: *On the other hand, suppose that $\hat{A}$ is nonsingular. Then the modulus of any eigenvalue of $[z^2\hat{A} - zI + A]$ is at most*

$$\frac{1 + \sqrt{1 + 4\|\hat{A}^{-1}\|^{-1}\|A\|}}{2\|\hat{A}^{-1}\|^{-1}}.$$

where $\|\ldots\|$ denotes any subordinate matrix norm.

Proof: The "small-gain" result follows from Corollary 1 of Akian et al. (2004), incorporated into a comprehensive theory in (Akian et al., 2014). The upper bound for the case of nonsingular $\hat{A}$ is from Lemma 3.1 of Higham and Tisseur (2003). ∎

The above analysis shows that the unstable pole-zero cancellation required to eliminate unstable eigenvalues of the New Keynesian model (of section 4) under "active" policy would have the effect of obliterating the dynamical features particularly associated with expectations terms. Indeed, the eigenvalues of $[z^2\epsilon\hat{A} - zI + A]$ vary continuously with the parameter ϵ: letting ϵ range from one to zero, numerical computation shows that it is precisely the two eigenvalues that are unstable under active policy that tend to infinity as ϵ tends to zero (moving along the positive real axis), while the other eigenvalues tend toward those of A. One effect of expectations is therefore to produce a modest shift in the nonzero eigenvalue of A: the more important dynamical effect is to bring into being two additional modes corresponding to the finite eigenvalues 1.045 and 1.45. However, these are precisely the modes that are suppressed in order to select a unique model-consistent forecasting mechanism under the conventional approach to rational expectations.[13] That traditional approach therefore cancels the very eigenvalues that arise from expectations.

[13] The suppression also cancels "zero dynamics" of the model, which can explain behavior such as "price puzzles" (Thistle and Miller, 2016).

6 The least-squared-error solution

This section proposes an alternative means of specifying a unique solution that applies in all cases (regardless of dynamical stability or instability), and does not generally entail pole-zero cancellation. In addition to the requirement that forecast errors be zero-mean, it calls for all forecast errors to be minimized, in the least-squares sense: a unique value of $\hat{A}F_0$ will achieve such minimization.

Specifically, let e_t denote $(\hat{A}F_0 + B)w_t$, the forecast error realized at time t, under model-consistent expectations, and e_t' its transpose. Then it is assumed that $\hat{A}F_0$ is such that, given the sequence of the w_t, every squared-error term $e_t'e_t$ is minimized.

This criterion has features that may be deemed unrealistic, at least in some contexts. For instance, if the column span of $\hat{A}$ contains that of B, then $\hat{A}F_0$ will be chosen so that $\hat{A}F_0 + B$ is zero, and the immediate effects of shocks on the model will be fully blocked: the model will then exhibit 'perfect foresight,' or 'self-fulfilling expectations,' and the minimum squared-error terms will of course all be zero. In particular, the above will apply whenever $\hat{A}$ is nonsingular – as in the scalar case, for example.

But while this new assumption clearly represents a significant strengthening of the usual rational-expectations hypothesis, it is in a similar spirit; and in comparison with the infinite-precision, unstable pole-zero cancellations that are usually assumed in conjunction with rational expectations, it is arguably relatively mild. It does not require specific stability properties of the model, and it generally avoids the feature that the unique solution has reduced-order dynamics, owing to pole-zero cancellation. It also represents the limit of attainable precision in unbiased forecasting: in justifying the use of rational expectations, Blanchard and Johnson (2013) write that, "designing a policy on the assumption that people will make systematic mistakes in responding to it is unwise"; similarly, it may be important to consider the case where economic actors do not make larger forecast errors than necessary.

To define the unique solution, note that each column b of the matrix B can be uniquely decomposed into $b_{\|} + b_{\perp}$, where $b_{\|}$ is its projection onto the column span of $\hat{A}$, and $b_{\perp}$ is orthogonal to that vector space. Let $B_{\|}$ consist of the projections $b_{\|}$ of the respective columns of B, and let $B_{\perp}$ be made up of the respective orthogonal components $b_\perp$. Then, for any $t \geq 0$,

$$\begin{aligned} e_t' e_t &= w_t'(\hat{A}F_0 + B)'(\hat{A}F_0 + B)w_t \\ &= w_t'(\hat{A}F_0 + B_\parallel)'(\hat{A}F_0 + B_\parallel)w_t + w_t' B_\perp' B_\perp w_t \ . \end{aligned}$$

Because this is a sum of nonnegative quantities, its minimum value over all possible $\hat{A}F_0$ must be at least $w_t' B_\perp' B_\perp w_t$.

Now, for any column b of B, the corresponding column f_0 of F_0 can be chosen (not necessarily uniquely) so that $\hat{A}f_0 = -b_\parallel$. Let F_0 be composed of such columns. Then $\hat{A}F_0 = -B_\parallel$, and; for any $t \geq 0$, $e_t' e_t = w_t' B_\perp' B_\perp w_t$. This unique value $\hat{A}F_0 = -B_\parallel$ therefore achieves the minimum squared error, for any given w_t.

The foregoing discussion establishes the following

Corollary 6.1 *Suppose that the model (1,2) is regular. Let $\hat{A}F_0 = -B_\parallel$. Suppose that the resulting $F[z]$ is proper and the initial conditions are consistent. Then the forecasting mechanism given by Theorem 3.5 is the unique model-consistent forecasting mechanism that gives rise to least-square forecasting errors. By Theorem 3.6, a weaker sufficient condition is that the model be well-posed and the initial conditions weakly consistent.* ∎

To be sure, the approach outlined above requires the public, in its aggregate, to calibrate its immediate reaction to shocks so as to meet the requirement of least-square forecasting errors. But here, the unique solution is not as 'brittle' as under the method of pole-zero cancellation: a small error in the value of $\hat{A}F_0$ generally means only that forecast errors will be slightly larger than necessary; whereas in the case of pole-zero cancellation, the slightest error means that the model will be dynamically unstable.[14]

For the example of section 4, one finds,

$$-B_\parallel = \begin{bmatrix} -0.833 & -0.155 & 0.322 \\ -0.417 & 0.469 & -0.209 \\ -0.333 & 0.239 & -0.075 \end{bmatrix} \ .$$

[14]In fact, in instances where the least-square-error criterion calls for pole-zero cancellation, a practical means of producing a less pathological solution might be simply to apply a 'small,' random perturbation to the value of $\hat{A}F_0$.

Consider for simplicity the case of zero-valued initial conditions. Setting $\hat{A}F_0 = -B_{\|}$, and finding a minimal state-space realization for the resulting matrix $F[z]$, yields a representation of the unique, least-squared-error, model-consistent forecasting mechanism:

$$\xi_{t+1} = \begin{bmatrix} 1.574 & -0.937 & -0.835 \\ -0.0942 & 0.978 & 0.285 \\ 0.271 & 0.109 & 0.273 \end{bmatrix} \xi_t + \begin{bmatrix} -0.288 & -1.153 & 0.370 \\ 1.003 & 0 & -0.308 \\ 0 & 0 & -0.671 \end{bmatrix} u_t \ , \tag{29}$$

$$\hat{x}_{1,t} = \begin{bmatrix} 0.488 & -1.171 & -0.528 \\ -0.592 & 0.334 & 0.437 \\ -0.349 & -0.049 & -0.0059 \end{bmatrix} \xi_t + \begin{bmatrix} -1 & -0.311 & 0.471 \\ 0 & 0.552 & -0.374 \\ -0.125 & 0.130 & 0.233 \end{bmatrix} u_t \ . \tag{30}$$

The second matrix coefficient of equation (30) contains the initial values of the forecasting mechanism's impulse responses, and indeed, it equals F_0.[15] It is this coefficient that models the only direct path from u_t to $\hat{x}_{1,t}$: it therefore captures what is commonly referred to as "jump," or "forward-looking," behavior; all other effects of u_t are mediated by the state, ξ_t, and correspond to "predetermined," or "backward-looking," or "inertial," dynamics. Multiplying the coefficient by $\hat{A}$ yields the above $-B_{\|}$, as expected. The model is well-posed, so this forecasting mechanism could alternatively be represented as a feedback/feedforward predictor, according to Theorem 3.6.

Coupling the forecasting mechanism to the model equation (27) yields a system that can be represented in the form of the following state-space model (obtained as a realization of the corresponding rational matrix $G[z]$):

$$\zeta_{t+1} = \begin{bmatrix} 1.574 & -0.937 & 0.835 \\ -0.094 & 0.978 & -0.287 \\ -0.271 & -0.109 & 0.273 \end{bmatrix} \zeta_t + \begin{bmatrix} -0.290 & -1.161 & 0.373 \\ 1.011 & 0 & -0.310 \\ 0 & 0 & 0.676 \end{bmatrix} u_t \ , \tag{31}$$

[15] The respective matrix coefficients of such state-space models are typically denoted A, B, C, and D (not to be confused with the A and B of the model equation (1)); in terms of these coefficients, the respective values of F_0, F_1, F_2, F_3, $\ldots$, are therefore D, CB, CAB, CA^2B, $\ldots$. By finding elements of the $\tilde{F}_t$ and $\tilde{G}_t$ sequences in the same manner from state-space realizations, it is easy to verify that equations (8,9) are indeed satisfied, and that this least-squared-error solution is a rational-expectations solution.

$$x_t = \begin{bmatrix} 0.275 & -0.911 & 0.128 \\ -0.444 & -0.127 & -0.366 \\ -0.169 & -0.172 & 0.358 \end{bmatrix} \zeta_t + \begin{bmatrix} 0 & 0.0118 & -0.095 \\ 0 & 0.0522 & -0.417 \\ 0 & -0.0948 & 0.759 \end{bmatrix} u_t \ . \qquad (32)$$

The second matrix coefficient of equation (32) contains the initial values of the impulse responses of the endogenous variables, and is accordingly equal to $G_0 = \hat{A}F_0 + B = B_\perp$. It captures the "jump," or "forward-looking," behavior of the endogenous variables, and of course determines the model's forecast errors.

In contrast with the traditional solution, this one is not reduced in dynamical order, and the preservation of the inertial dynamics of expectations reveals difficulties with the formulation of the New Keynesian model. Indeed, the above state-space equations are at odds with the standard interpretation of key model equations. For instance, the New I-S equation (22) is typically considered to assert that the higher the real interest rate $r_t - \hat{\pi}_{1,t}$, the lower output y_t; and by the same token, the greater g_t, the higher the output y_t. But the state-space equations disagree. Note that g_t is the first component of the vector u_t, and y_t is the first component of x_t. By (32), g_t has no immediate effect upon any of the components of x_t; and by (30) its only effect on relevant forecasts is a negative effect on $\hat{y}_{1,t}$.

But it is not only the least-squared-error solution that disagrees with the New Keynesian model: it is shown in the next section that the eigenvalues of the general solution's denominator polynomial $[z^2\hat{A} - zI + A]$ conflict with the I-S equation and the expectational Phillips curve.

7 Stabilization of the New Keynesian model

By decoupling rational expectations from dynamical stability, the methods of this paper give a clearer picture of the dynamics of macroeconomic models. An illustration is provided by the formulation of a stabilizing interest-rate policy for the New Keynesian model.

If the R matrix is stable, then for any model-consistent forecasting mechanism, the stability of the solution is determined (in the absence of pole-zero cancellations) by the eigenvalues of the matrix polynomial $[z^2\hat{A} - zI + A]$. One can therefore play at a toy game of central banking by altering the Taylor-rule parameters in such a way that all of that matrix polynomial's eigenvalues become stable. For instance, with $\psi_1 = 1.10$ and $\psi_2 = -1.50$, the eigenvalues are $0.81 \pm 0.045i$ and 0.76 (while their moduli can be decreased further by decreasing the value of ρ_r). These indicate a stable model, with a response that is only slightly oscillatory. Under the conventional approach to rational expectations, these Taylor-rule parameters would not produce a 'determinate' model, dynamical stability being incompatible with determinacy (in the conventional sense). But in general, there is no such difficulty: for instance, this stabilization procedure could be carried out in conjunction with the use of the above least-squared-error criterion, to yield a unique, stable solution, without pole-zero cancellations.

But note that the value of the above Taylor-rule coefficient ψ_2 applied to output is negative, meaning that the higher the level of output, the lower is the policy interest rate. The reason for this surprising feature lies in the peculiarity of the new I-S equation. The equation is derived from microfoundations, and specifically from the solution of optimal planning problems for members of different sectors of the economy. Households, for example, are assumed to schedule their consumption over all time, according to expected values of nonlinear functions of interest rates and other variables. Under that scenario, the higher the expected real interest rate at time t, the lower consumption will indeed be at the same time – moreover, the lower it will be at earlier time instants as well – the better to profit from a higher rate.

Linear model equations are then obtained by loglinearizing the first-order conditions associated with optimality – a process of informal approximation that typically assumes, for instance, that expected values of nonlinear functions can be approximated as expected values of approximations of those functions. The new I-S equation is derived by identifying output with consumption in the resulting system of equations. But in contrast to the offline consumption plan from which it is derived, this equation is then used to model adjustments that occur 'on the fly' – in real time: as such, it of course does not imply that the higher the present real interest rate, the lower earlier levels of consumption will have been; and in fact, contrary to the usual interpretation, it is difficult to argue that it implies a negative relationship even with the current level of consumption/output: for any alteration in the current real interest rate there will almost surely be a cognate change in the forecast $\hat{y}_{1,t}$ of the next period's output, so it is not obvious how y_t will be affected. Better to rearrange the equation as follows,

$$\hat{y}_{1,t} - y_t = \tau(r_t - \hat{\pi}_{1,t}) - g_t \ , \tag{33}$$

and see that, with $\tau > 0$, it means simply that, the larger the real interest rate, the larger the expected increase in output over the next period.

But more can be said in the context of the overall model: note that the denominator matrix polynomial $[z^2\hat{A} - zI + A]$ is exactly the same in the case of the zero-input response (section 3.2) as for the zero-state response (3.1). In order to study model eigenvalues it therefore suffices (in the absence of pole-zero cancellations) to consider the zero-input response. Indeed, if the sequences $\epsilon_{r,t}$, $\epsilon_{g,t}$, $\epsilon_{z,t}$ are identically zero, then, according to the analysis of section 3.2, the I-S equation and the expectational Phillips curve effectively become,

$$y_{t+1} = y_t + \tau(r_t - \pi_{t+1}) - g_t \ , \tag{34}$$

$$\pi_{t+1} = \beta^{-1}\pi_t - \beta^{-1}\kappa(y_t - z_t) \ . \tag{35}$$

The result is a perfect-foresight version of the New Keynesian model, with the same eigenvalues as the general version. For any given initial conditions, it has a unique solution, according to which the values of π_{t+1} and y_{t+1} are determined by their previous values and by r_t, g_t and z_t. It follows that the larger r_t, the *larger* y_{t+1}: so the greater the interest rate, the *greater* will be output at the next instant. Similarly, with β essentially unity and $\kappa > 0$, the expectational Phillips curve (23) effectively implies that the higher the level of output, the *lower* the expected rate of increase of inflation over the next period. For use with general model-consistent expectations, both equations are therefore flawed.

In determining inflation in the full model, the two inversions essentially cancel each other, because the only transmission channel from the interest rate to inflation passes by way of output. That explains why the sign of ψ_1 in the above stabilizing policy rule conforms to economic intuition, while that of ψ_2 does not. On the other hand, if one crudely 'corrects' the two key equations by simply changing the signs of τ and κ, one finds that in stabilizing policy rules, both ψ_1 and ψ_2 are positive – as one would normally expect. Indeed, for stability, the policy reactions to both inflation and the output gap should be 'active': with $\psi_2 = 1.5$, the model is stable provided

$1.03 \leq \psi_1 \leq 1.49$; and with $\psi_1 = 1.03$, the model is stable provided $1.04 \leq \psi_2 \leq 1.50$. In the toy game of central banking, one might opt for the policy gains $\psi_1 = 1.10$ and $\psi_2 = 1.50$: that would give eigenvalues at $0.812 \pm 0.0453i$ and 0.763; these are about as far from the unit circle as can be arranged with those two policy parameters, and they produce a response that is only slightly oscillatory.[16]

In a recent, constructive critique of DSGE models, Blanchard (2018) asserts that the new I-S equation (22) and the expectational Phillips curve (23) are "deeply flawed." The results of this section lend support to that statement, but not for all of the reasons that Blanchard cites. He describes rational expectations as insufficiently inertial. As was seen in sections 4 and 5, the fundamental reason for that feature is the suppression of the associated dynamics that is part and parcel of the conventional application of rational expectations. However, that suppression is not a general feature of rational expectations, but only of the very particular approach that has held sway for the last several decades. Rational expectations have inertial dynamics that may be concealed in the conventional case, and when those dynamics are not suppressed they are seen to be incompatible with current formulations of the I-S equation and the expectational Philips curve.

8 Related work

In seeking a way forward for DSGE models, Blanchard (2018) asks, "how can we deviate from rational expectations, while keeping the notion that people and firms care about the future?" This paper shows that the problem is not that rational expectations have been tried and found wanting, but rather that they have never been fully tried.[17]

8.1 The forecasting mechanism

The main reason is that the cause of nonuniqueness was never before understood, chiefly because earlier work did not include any explicit parameterization of the forecasting mechanism. Taylor (1977) adapted the early method of Muth (1961) to dynamic macroeconomic models by means of a parameterization of the model (see

[16] The moduli of the eigenvalues could be reduced further by reducing the value of ρ_r that serves to smooth changes in the policy interest rate.

[17] With apologies to G.K. Chesterton.

subsection 1.1) in the form of a Wold decomposition. While Taylor's formulation resembles the present one in some respects, it did not explicitly parameterize the forecasting mechanism itself. Shiller (1978) recognized that the key to the solution of rational-expectations models was to solve for the forecasting mechanism, but neither did he include such an explicit parameterization. The same is true of all other previous approaches; for recent examples, see (Tan and Walker, 2015) and (Al-Sadoon, 2017). Yet the free parameters that underlie the nonuniqueness of solutions turn out to be nothing other than parameters of the forecasting mechanism.

Without knowledge of the true reason for nonuniqueness, and because of a broadly perceived need for a terminal boundary condition, stability quickly became entrenched as the primary means of attempting to select a unique solution. Taylor (1977) and Blanchard (1979) considered alternatives, but, invoking the stability criterion himself only shortly afterward, Blanchard (1981) referred to its application as having already become "a standard if not entirely convincing practice." In a later survey of solution methods, Blanchard (1985) stated that some endogenous variables in rational-expectations models do not have natural initial conditions, and explained that terminal conditions, or transversality conditions, are an alternative means of limiting the space of solutions: a version of Blanchard's univariate example was considered in section 3; and indeed, in the general solution given in section 3.3, the only initial condition applies to the forecast variable. But it is now seen to be the initial values of the *impulse responses* of forecast variables – which are properly viewed as parameters of the forecasting mechanism, rather than boundary conditions on model variables – that distinguish the respective solutions.

8.2 Stability constraints

The imposition of stability constraints has remained "standard, if not entirely convincing," even though far from universally applicable, and in spite of the recognition that it obliterates model dynamics and alters stability properties (Lubik and Schorfheide, 2004). Consequently, conventional methods of computing rational-expectations solutions are bound up with the question of stability. It is generally assumed that an equation like the following holds:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi v_t + \Pi \eta_t \ . \tag{36}$$

The vector y_t is made up of endogenous variables, including forecasts, v_t is an exogenous random driving variable, and η_t is a vector of zero-mean forecasting errors, determined as part of the solution process.[18] A matrix decomposition is applied, to allow separate consideration of the respective eigenvalues, and where unstable eigenvalues are concerned, it is arranged (if possible) for the corresponding terms in v_t and η_t to cancel each other, giving rise to a zero that cancels the unstable eigenvalue. If the corresponding calculations determine the forecasting errors uniquely, then there is a unique rational-expectations solution – among those that obey the stability constraint. As this article proves, this is just one means of modeling the public's immediate responses to shocks. But the approach has become so closely identified with the solution of rational-expectations models that it should be stressed that the link with stability is unnecessary – as is the exact cancellation of unstable eigenvalues, and the consequent suppression of dynamics.

8.3 "Forward-looking solutions"

The terminal conditions used to enforce stability have led to misconceptions about "forward-looking solutions" that are sufficiently widespread that they should be addressed here, lest it be thought that an important element is missing from the paper. Sims (2002) did not fall prey to those misconceptions, but his development nevertheless furnishes a useful illustration. Considering the "unstable part" of a decomposition of (36), he arrives, under certain assumptions, at an equation of the form

$$w_t = Mw_{t+1} - \Omega^{-1}x_{t+1} \; , \tag{37}$$

where, for simplicity, some of Sims' subscripts have been dropped. As a terminal condition, he assumes that $v_t := M^t w_t$ tends to zero (say, almost surely) as t goes to infinity. Multiplying equation (37) by M^t and solving for $v_t = M^t w_t$, one finds, for $t \geq 0$,

$$v_t = \sum_{\tau=1}^{t} M^{\tau-1}\Omega^{-1}x_\tau + w_0 \; .$$

[18]See, for example, (Blanchard and Kahn, 1980; Binder and Pesaran, 1997; Klein, 2000; Sims, 2002; Lubik and Schorfheide, 2004).

Sims' terminal condition therefore means that, with probability one, the sum

$$\sum_{\tau=1}^{\infty} M^{\tau-1}\Omega^{-1}x_\tau + w_0$$

exists, and vanishes; so, almost surely,

$$v_t = \sum_{\tau=1}^{t} M^{\tau-1}\Omega^{-1}x_\tau + w_0 = -\sum_{\tau=t+1}^{\infty} M^{\tau-1}\Omega^{-1}x_\tau \ . \tag{38}$$

The solution thus has two distinct forms, one expressed in terms of 'past' values of x_τ, for $0 < \tau \leq t$, and another in terms of 'future' values, with $\tau > t$. The latter is called a "forward-looking solution," though it is really just a "forward-looking" *representation* of the unique solution (for a given w_0). The existence of such a representation is often misunderstood to imply that v_t somehow depends 'only on future values' of x_τ – that the dynamics of v_t are anticausal – but that thinking is fallacious: one sees in this case that v_t can alternatively be represented solely in terms of 'past values' of x_τ (and w_0). In fact, the forward-looking representation owes its existence to the special relationship (38), between past and future, which only holds by assumption: it is a direct consequence of the terminal condition, which requires x_t to behave in such a way that

$$\sum_{\tau=1}^{\infty} M^{\tau-1}\Omega^{-1}x_\tau + w_0 = 0 \ , \ \text{a.s.}$$

8.4 Minimum error variance

As an alternative means of attempting to ensure uniqueness, Taylor (1977) considered the requirement that the variance of the endogenous variables be minimized. But Başar (1989) appears to have been the first to study explicitly the minimization of the variance of forecast errors. He considered a univariate model with a single forecast term $\hat{x}_{2,t-1}$ – a two-period forecast of x_{t+1}, formulated at time $t-1$ – driven by an independent, zero-mean sequence with finite variance. Assuming that the information set underlying forecasts consisted either of x_{t-1}, or of a noisy measurement of that scalar variable, with all random variables being Gaussian in the latter case, Başar showed that a minimum-variance rational-expectations solution could be computed as the limiting case of solutions to finite-horizon problems – under an additional assumption amounting to the realness of the two eigenvalues. Specifically, the finite-horizon problems called for minimization of a discounted sum of squared forecast errors, over a finite time interval, and subject to a terminal boundary condition. Apart from the restriction on information sets, Başar made no assumptions as to the form of forecasting mechanisms, but his solutions satisfied linear recurrences with constant coefficients. General solutions of Taylor's and Başar's models are given by the results of appendix A, together with formulas for the forecast errors.

9 Conclusion

Owing to the performance of DSGE models, some macroeconomists seem to be on the verge of discarding rational expectations. But this paper has shown that prior research has focused on a very special case of rational expectations, adopted as an ad hoc means of dispensing with nonuniqueness.

The article has explained that nonuniqueness. It has shown that the underlying free parameters are coefficients that capture the immediate reactions of economic actors to shocks; and it has presented a general solution that has a unique instance for every appropriate specification of those free parameters. If the model is *well-posed*, and the initial conditions *weakly consistent*, then there exists a (unique) solution for every possible specification of the parameters.

Indeed, solutions are unique only when the values of the free parameters are determined, necessarily by means of criteria extraneous to the rational-expectations hypothesis. For instance, if, the requirement of model-consistency is augmented with that of least-square forecast errors (or, more generally, minimum error variance), then – for a broad class of models – unique values of the free parameters are determined, and so therefore is a unique model-consistent solution. This result is one means of ensuring uniqueness. It avoids the most serious drawbacks of the prevalent approach, while preserving its advantages, and it is an important limiting case, representing forecasts of the greatest possible precision.

But the fundamental objective of this paper has been to explain the nature of nonuniqueness. In that connection, the main finding is that the assumption of model consistency generally implies nothing about the economic effects of the most immediate responses of forecasts to shocks. To put it another way, there is no fundamental reason for any model of the most short-term reactions of economic agents to entail systematic forecasting errors. Indeed, a rough paraphrase suggests itself, in the behaviorist terminology of Kahneman (2011): rational expectations exemplify analytical, "slow" thinking, and as such do not capture the "fast" thinking that underlies the most immediate responses to economic shocks; but clearly, any deliberate and reasoned process of expectation formation must necessarily take account of the economic effects of more instantaneous, reflexive actions. Like most loose paraphrases, the analogy probably should not be pushed too far, but it should be underlined that "fast thinking" does not necessarily imply on-the-fly improvisation; it includes the automatic application of a preconditioned response, such as represented by the free parameters identified here.

The paper therefore shows that the free parameters could be evaluated in the thoroughly Cartesian spirit that has dominated much of recent theoretical macroeconomics, by means of the idealized assumption of minimum forecast-error variance. However, it also opens the door to a rapprochement with more empirical methods, which might serve to identify the "fast thinking" of economic actors, about which rational expectations is agnostic; the "slow thinking" captured by rational expectations could then model longer-term expectation formation.

In order to draw robust conclusions, it is likely necessary to consider multiple alternatives. The values of the free parameters do not affect all aspects of model behavior – in the absence of pole-zero cancellations, dynamical stability is independent of them; but their effects are nevertheless extensive. It may be important to take into consideration a range of possible parameter values.

But no matter how the free parameters are specified, the present approach removes substantial obstacles to the application and the study of rational expectations. By dispensing with the cancellation of unstable dynamics, it greatly expands the range of models to which rational expectations can be applied – beyond those with particular instabilities – and obviates the suppression of model dynamics. Consequently, it allows the examination of issues such as stabilization policy, and the 'inertial' dynamics of rational expectations. It should also simplify questions that are central to the pertinence of the rational expectations hypothesis – such as model estimation, and the "learning" of models by economic agents – where the unstable pole-zero cancellations of conventional solutions have posed serious impediments (Rondina and Walker, 2016).

More generally, these new results paint a richer picture of the behavior of macroeconomic models under rational expectations, and should allow for more realistic means "to attribute to individuals some view of the behavior of the future values of variables of concern to them" (Lucas, 1976).

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A Appendix: a "general" model

Within the traditional rational-expectations paradigm, the model employed in the main body of the paper has been applied to cases of arbitrarily many distinct expectation terms through an increase of the dimension n (Broze et al., 1995; Binder and Pesaran, 1997; McCallum, 2007). The present method extends to allow models with a variety of expectation terms to be treated directly. A direct analysis lends insight into the central question of the paper – that of nonuniqueness of solutions and the associated free parameters – and of course allows direct solution of a broader class of models.

To outline a generalized approach, take the following model:

$$\sum_{i=0}^{h}\sum_{j=0}^{l} A_{ij}\hat{x}_{i,t-j} = Bu_t \ , \ A_{00} = I; \tag{39}$$

$$u_t = Ru_{t-1} + w_t \ . \tag{40}$$

Here, the $A_{ij} \in \mathbb{R}^{n\times n}$ are constant matrix coefficients, as are $B \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$. The sequence w_t is as in the main body of the paper, and the vectors u_t again represent exogenous inputs driven by the w_t.

The vector $\hat{x}_{0,t} \in \mathbb{R}^n$ is a vector x_t of endogenous variables, and, for $0 < j \leq l$, $\hat{x}_{0,t-j} = x_{t-j}$ is a "lagged" version of x_t. For positive i, $0 < i \leq h$, $\hat{x}_{i,t}$ represents a forecast, formulated at time t, of the value of x_{t+i}. For brevity, this appendix focuses on the novel part of the present approach, the formulation and solution of the zero-state response: hence, all initial conditions will be assumed to be zero-valued, and forecasts will depend only on the random variables u_τ, for $0 \leq \tau \leq t$.

Formally, a *forecasting mechanism* takes the form of a system of equations of the form

$$\hat{x}_{i,t-j} = \sum_{\tau=0}^{t-j} \tilde{F}_{ij,t-\tau} w_\tau, \ \forall i,j \ 0 < i \leq h \ , \ 0 \leq j \leq l \ \ t \geq j \ . \tag{41}$$

where $\tilde{F}_{ij,t} \in \mathbb{R}^{n\times m}$. The upper limits of the convolution sums reflect the fact that the forecast $\hat{x}_{i,t-j}$ must be based solely on information available at time $t-j$. More could be assumed about the form of the convolution terms, but additional structure will make itself evident shortly.

If y is a square-integrable random variable defined on the common probability space of the w_t, then $E_t(y)$ denotes the expected value of y, conditioned on w_τ, for $0 \leq \tau \leq t$. A forecasting mechanism is *model-consistent* if the resulting full model satisfies the following for all $t \geq j-1$, $0 \leq i \leq h,\ 0 \leq j \leq l$,

$$
\begin{aligned}
&E_{t-j}(x_{t+i-j} - \hat{x}_{i,t-j}) = 0 \\
\Longleftrightarrow\ &\hat{x}_{i,t-j} = E_{t-j}(x_{t+i-j}) \ .
\end{aligned}
$$

It is important to note that the expected values are subject not only to the model equations (39,40), but also to the forecasting mechanism. In other respects, the above model resembles the "general model" of Broze et al. (1995).

The w_t influence the x_t causally, not only through the expectations terms but also via the exogenous inputs. The x_t must therefore constitute a convolution of the following form:

$$
x_t = \sum_{\tau=0}^{t} \tilde{G}_{t-\tau} w_\tau \ . \tag{42}
$$

For equations relating the $\tilde{G}_t$ and the $\tilde{F}_{ij,t}$, invoke model-consistency, for $0 < i \leq h$, and $t \geq 0$:

$$
\begin{aligned}
\sum_{\tau=0}^{t} \tilde{F}_{ij,t-\tau} w_\tau = \hat{x}_{i,t-j} = E_{t-j}(x_{t+i-j}) &= E_{t-j}\left(\sum_{\tau=0}^{t+i-j} \tilde{G}_{t+i-j-\tau} w_\tau\right) \\
&= \sum_{\tau=0}^{t-j} \tilde{G}_{t+i-j-\tau} w_\tau \ .
\end{aligned}
$$

Taking expected values of the left- and right-hand sides, conditioned on w_0,

$$
\tilde{F}_{ij,t} w_0 = \mathbb{1}_{t-j} \tilde{G}_{t+i-j} w_0 \ , \ \forall t \geq 0 \ .
$$

Here, $\mathbb{1}_t$ denotes the unit-step function, which vanishes for negative arguments but otherwise is unity. Because w_0 is arbitrary,

$$
\tilde{F}_{ij,t} = \mathbb{1}_{t-j} \tilde{G}_{t+i-j} \ , \ 0 < i \leq h \ , \ t \geq 0 \ .
$$

Given the above form of the $\tilde{F}_{ij,t}$, drop the index j and write instead

$$\tilde{F}_{i,t} = \mathbb{1}_t \tilde{G}_{t+i} \ , \ 0 < i \leq h \ , \ t \geq 0 \ , \tag{43}$$

and

$$\hat{x}_{i,t-j} = \sum_{\tau=0}^{t-j} \tilde{F}_{i,t-j-\tau} w_\tau \ . \tag{44}$$

After substitution of the above convolutions into (39), the methods of the main body of the paper yield

$$\sum_{j=0}^{l} A_{0j} \mathbb{1}_{t-j} \tilde{G}_{t-j} + \sum_{i=1}^{h} \sum_{j=0}^{l} A_{ij} \tilde{F}_{i,t-j} = BR^t \ , \ \forall t \geq 0 \ . \tag{45}$$

Application of equation (43) leads to the following difference equation in G_t:

$$\sum_{i=0}^{h} \sum_{j=0}^{l} A_{ij} \mathbb{1}_{t-j} \tilde{G}_{t+i-j} = BR^t \ , \ \forall t \geq 0 \ . \tag{46}$$

In order to ensure unique solutions of difference equations, irrespective of rational expectations, assume that the corresponding matrix polynomial

$$D[z] := \sum_{i=0}^{h} \sum_{j=0}^{l} z^{i+l-j} A_{ij}$$

is regular. In that case, the model (39,40) will also be called *regular*.

Equation (46) has coefficients that vary with time, but only up to $t = l$, at the latest. The theory of time-invariant regular descriptor systems therefore implies that any solutions are of exponential order, and consequently possess z-transforms. Suppose that (46) and (43), and therefore (45), can be solved simultaneously. Taking z-transforms of both sides of (43) yields, for any $0 < i \leq h$,

$$\tilde{F}_i[z] = \mathcal{Z}\left\{\mathbb{1}_t \tilde{G}_{t+i}\right\} = \mathcal{Z}\left\{\tilde{G}_{t+i}\right\} = z^i \left[\tilde{G}[z] - \sum_{k=0}^{i-1} z^{-k} \tilde{G}_k\right] \ . \tag{47}$$

Then taking transforms in (46), and substituting according to (47),

$$\sum_{i=0}^{h}\sum_{j=0}^{l} A_{ij} z^{i-j}\left[\tilde{G}[z] - \sum_{k=0}^{i-1} z^{-k}\tilde{G}_k\right] = B[I - Rz^{-1}]^{-1} \ . \tag{48}$$

By regularity, $\tilde{G}[z]$ must therefore satisfy

$$\tilde{G}[z] = D[z]^{-1} z^l \left[\sum_{i=1}^{h}\sum_{j=0}^{l}\sum_{k=0}^{i-1} A_{ij} z^{i-j-k}\tilde{G}_k + B[I - Rz^{-1}]^{-1}\right] \ . \tag{49}$$

Determination of a unique solution requires the specification of the initial values $\tilde{G}_i$, $0 \le i < h$. These must be related to the initial values of the $\tilde{F}_{i,t}$, for $0 < i < h$, and to that of $A_{h0}\tilde{F}_{h,0}$, by the equations $\tilde{G}_0 = B - \sum_{i=1}^{h} A_{i0}\tilde{F}_{i,0}$ (by (45) and (43)), and $\tilde{G}_i = \tilde{F}_{i,0}$, $0 < i < h$ (by (43)). Any two solutions $\tilde{G}_t$ that satisfied these initial values would have to have this z-transform, and to vanish for negative t. It follows from z-transform inversion that the two solutions would in fact be equal.

Proposition A.1 *Suppose that the model (39,40) is regular. For any possible values of $\tilde{F}_{1,0}$, $\tilde{F}_{2,0}$, $\ldots$, $\tilde{F}_{h-1,0}$ and $A_{h0}\tilde{F}_{h,0}$, define $\tilde{G}_0 := B - \sum_{i=1}^{h} A_{i0}\tilde{F}_{i,0}$ and $\tilde{G}_i := \tilde{F}_{i,0}$, for all $0 < i < h$. Let*

$$\tilde{N}[z] := \left[\sum_{i=1}^{h}\sum_{j=0}^{l}\sum_{k=0}^{i-1} A_{ij} z^{i+l-j-k}\tilde{G}_k + z^l B[I - Rz^{-1}]^{-1}\right] \ .$$

Then there exists a solution of (43,45), consistent with the above initial values of the $\tilde{F}_{i,0}$, $0 < i < h$, and $A_{h,0}\tilde{F}_{h,0}$, and such that the $\tilde{F}_{i,t}$ and $\tilde{G}_t$ vanish for negative t, if and only if the following rational matrix is proper:

$$\tilde{F}_h[z] = z^h \left[D[z]^{-1}\tilde{N}[z] - \sum_{k=0}^{h-1} z^{-k}\tilde{G}_k\right] \ .$$

In that case, the unique such solution is given by

$$\tilde{F}_{i,t} = \mathcal{Z}^{-1}\left\{z^i \left[D[z]^{-1}\tilde{N}[z] - \sum_{k=0}^{i-1} z^{-k}\tilde{G}_k\right]\right\} \ , \ 0 < i \le h \ ,$$
$$\tilde{G}_t = \mathcal{Z}^{-1}\left\{D[z]^{-1}\tilde{N}[z]\right\} \ .$$

A model-consistent forecasting mechanism (41) exists if and only if the above condition is satisfied. Any such forecasting mechanism must have $\tilde{F}_{ij,t} = \tilde{F}_{i,t-j}$*, for all* $0 < i \leq h$*,* $0 \leq j \leq l$*.*

If all initial conditions are zero-valued, and $\hat{x}_{i,t-j} = \sum_{\tau=0}^{t-j} \tilde{F}_{i,t-j-\tau} w_\tau$*, for all* $t \geq j$*, and for all* i, j*,* $0 < i \leq h$*,* $0 \leq j \leq l$*, then the model (39,40) satisfies* $x_t = \sum_{\tau=0}^{t} \tilde{G}_{t-\tau} w_\tau$ *,* $\forall t \geq 0$*. For any* i, j*,* $0 < i \leq h$*,* $0 \leq j \leq l$*, and any* $t \geq j - i$*, the forecast errors are given by:*

$$x_{t+i-j} - \hat{x}_{i,t-j} = \sum_{\substack{\tau= \\ t-j+1}}^{t+i-j} \tilde{G}_{t+i-j-\tau} w_\tau \ . \tag{50}$$

Proof: If a solution of (43,45) exists, then, by the above discussion, the unilateral z-transforms of the $\tilde{F}_{i,t}$ and $\tilde{G}_t$ must have the forms given in the statement of the proposition, and must be proper. This establishes the necessary condition for existence.

For sufficiency, note that

$$\tilde{G}[z] = \sum_{k=0}^{h-1} z^{-k} \tilde{G}_k + z^{-h} \tilde{F}_h[z] \ ;$$

hence, if $\tilde{F}_h[z]$ is proper, so is $\tilde{G}[z]$. Both are therefore unilateral z-transforms: their inverse transforms vanish for negative t. Moreover, the first h values of the inverse transform of $\tilde{G}[z]$ are the respective $\tilde{G}_i$, $0 \leq i < h$. Hence, all the $\tilde{F}_i[z]$ are proper.

By definition, $\tilde{G}[z]$ satisfies (49), and transforming to the time domain shows that its inverse transform $\tilde{G}_t$ satisfies (46). The $\tilde{F}_i[z]$ by definition satisfy (47), for $0 < i \leq h$, and transforming to the time domain shows that their respective inverse transforms satisfy (43). It follows that (45) is satisfied.

Setting $t = 0$ in (43) implies that the initial values of the inverse transforms $\tilde{F}_i[z]$ for $0 < i < h$ are indeed the assigned values, and setting $t = 0$ in (45) shows that $A_{h0} \tilde{F}_{h,t}$ takes on its assigned value at $t = 0$. This establishes the sufficiency of the condition for the existence of an appropriate solution of (43,45). Uniqueness follows from the above discussion.

The respective unique forms of the zero-state response of $\hat{x}_{i,t-j}$ and x_t under a model-consistent forecasting mechanism then follow, by the previous discussion, and the expression for the errors by straightforward subtraction of convolution sums. ∎

(Again, the forecasting mechanism can easily be expressed in terms of u_t.)

The parameters $\tilde{F}_{1,0}$, $\tilde{F}_{2,0}$ $\ldots \tilde{F}_{h-1,0}$, and $A_{h0}\tilde{F}_{h,0}$ therefore determine $\tilde{G}_0$ via (39-41); given these values and model equations, model-consistency then determines $\tilde{F}_{i,t}$, for $0 < i \leq h$, and $\tilde{G}_t$, for all $t > 0$.

The sufficient "well-posedness" condition for existence of solutions generalizes as follows:

Corollary A.2 *Suppose that the model (39,40) is regular, and that $D[z]^{-1}z^{h+l-1}$ is proper. Then for any possible values of $\tilde{F}_{1,0}$, $\tilde{F}_{2,0}$, $\ldots$, $\tilde{F}_{h-1,0}$ and $A_{h0}\tilde{F}_{h,0}$, there exists a (unique) model-consistent forecasting mechanism.*

Proof: In accordance with Proposition A.1, define $\tilde{G}_0 := B - \sum_{i=1}^{h} A_{i0}\tilde{F}_{i,0}$ and $\tilde{G}_i := \tilde{F}_{i,0}$, for all $0 < i < h$. Note that

$$
\begin{aligned}
\tilde{N}[z] &:= \left[\sum_{i=1}^{h}\sum_{j=0}^{l}\sum_{k=0}^{i-1} A_{ij}z^{i+l-j-k}\tilde{G}_k + z^l B[I - Rz^{-1}]^{-1}\right] \\
&= \left[\sum_{i=0}^{h}\sum_{j=0}^{l}\sum_{k=0}^{h-1} A_{ij}z^{i+l-j-k}\tilde{G}_k \right. \\
&\quad \left. -\sum_{i=0}^{h}\sum_{j=0}^{l}\sum_{k=i}^{h-1} A_{ij}z^{i+l-j-k}\tilde{G}_k + z^l B[I - Rz^{-1}]^{-1}\right] \\
&= D[z]\sum_{k=0}^{h-1} z^{-k}\tilde{G}_k \\
&\quad - z^l\left[\sum_{i=0}^{h}\sum_{j=0}^{l}\sum_{k=i}^{h-1} A_{ij}z^{i-j-k}\tilde{G}_k(1-\delta_j\delta_{k-i}) - BRz^{-1}[I-Rz^{-1}]^{-1}\right]
\end{aligned}
$$

where δ_t is the Kronecker delta function, whose value is unity when $t = 0$, but otherwise vanishes. Consequently,

$$
\begin{aligned}
\tilde{F}_h[z] &= D[z]^{-1}z^{h+l}\times \\
&\quad \left[BRz^{-1}[I - Rz^{-1}]^{-1} - \sum_{i=0}^{h}\sum_{j=0}^{l}\sum_{k=i}^{h-1} A_{ij}z^{i-j-k}\tilde{G}_k(1 - \delta_j\delta_{k-i})\right] .
\end{aligned}
$$

The term in square brackets is strictly proper, so, if $D[z]^{-1}z^{h+l-1}$ is proper, then $\tilde{F}_h[z]$ must be proper. ∎

B Proofs for section 3.4

The following simple lemma lists some of the implications of well-posedness.

Lemma B.1 *Suppose that $\hat{A}$ is nonzero and $[z^2\hat{A} - zI + A]$ is regular. Then the following are equivalent:*

$$\begin{aligned} & [z^2\hat{A} - zI + A]^{-1} \text{ is strictly proper} \\ \iff & [z^2\hat{A} - zI + A]^{-1} zI \text{ is proper} \\ \iff & [z^2\hat{A} - zI + A]^{-1}[zI - A] \text{ is proper} \\ \iff & [z^2\hat{A} - zI + A]^{-1} z^2\hat{A} \text{ is proper} \\ \iff & [z^2\hat{A} - zI + A]^{-1} z\hat{A} \text{ is strictly proper} \\ \iff & [z\hat{A} - I]^{-1} \text{ is proper} \\ \iff & z\hat{A}[z\hat{A} - I]^{-1} \text{ is proper.} \end{aligned}$$

Proof: The first four equivalences and the final one are straightforward. For the fifth, note that $[z\hat{A} - I]^{-1}$ can be realized from $[z\hat{A} - I + Az^{-1}]^{-1}$, and vice versa, by feedback through Az^{-1}: hence the one is proper if and only if the other is. ∎

As claimed, well-posedness ensures the existence of model-consistent forecasting mechanisms:

Proposition B.2 *Suppose that the model (1,2) is regular and well-posed, and that the initial conditions are weakly consistent. Then there exists a (unique) model-consistent forecasting mechanism for (1,2) for any given value of $\hat{A}F_0$.*

Proof: Consider that $\overline{X}[z]$ is

$$\begin{aligned} & [z^2\hat{A} - zI + A]^{-1}[z^2\hat{A}\hat{x}_{1,-1} - zAx_{-1} - zBR[I - Rz^{-1}]^{-1}u_{-1}] \\ = {} & \hat{x}_{1,-1} + [z^2\hat{A} - zI + A]^{-1}[[zI - A]\hat{x}_{1,-1} - zAx_{-1} - zBR[I - Rz^{-1}]^{-1}u_{-1}] \\ = {} & \hat{x}_{1,-1} \\ & + [z^2\hat{A} - zI + A]^{-1}[z(\hat{x}_{1,-1} - Ax_{-1} - BRu_{-1}) \\ & \qquad - A\hat{x}_{1,-1} - BR^2[I - Rz^{-1}]^{-1}u_{-1}] \,. \end{aligned}$$

The weak consistency of the initial conditions implies that the term in parentheses lies within the image of $\hat{A}$; if, in addition, $[z^2\hat{A} - zI + A]^{-1}$ is strictly proper, then by Lemma B.1, $\overline{X}[z] - \hat{x}_{1,-1}$ also is strictly proper.

Now write

$$\begin{aligned} F[z] &= [z^2\hat{A} - zI + A]^{-1}[[zI - A](\hat{A}\tilde{F}_0 + B)[zI - R] - z^2 B] \\ &= [z^2\hat{A} - zI + A]^{-1}\Big[z^2\hat{A}\tilde{F}_0 \\ &\qquad\qquad - \Big([zI - A](\hat{A}F_0 + B)R + zA(\hat{A}F_0 + B)\Big)\Big] . \end{aligned}$$

If $[z^2\hat{A} - zI + A]^{-1}$ is strictly proper, then by Lemma B.1, this is a sum of proper rational matrices, so $F[z]$ is proper. By Theorem 3.5 therefore, there exists a unique model-consistent forecasting mechanism, regardless of the value of $\hat{A}\tilde{F}_0$. ∎

This simple sufficient condition for the existence of model-consistent forecasting mechanisms also ensures the existence of realizations that incorporate feedback.[19] Consider that the derivation of of $\tilde{F}[z]$ and $\tilde{G}[z]$ in section 3.1 could have begun with the following system of equations, equivalent to (8,9):

$$\begin{aligned} \tilde{F}_t &= A\tilde{G}_t + \hat{A}\tilde{F}_{t+1} + BR^{t+1} \ , \ \forall t \geq 0 \ , \\ \tilde{G}_t &= A\tilde{G}_{t-1} + \hat{A}\tilde{F}_t + BR^t \ , \ \forall t \geq 0 \ . \end{aligned}$$

This yields the transformed equation

$$\begin{bmatrix} I & -[I-z\hat{A}]^{-1}A \\ -[I-Az^{-1}]^{-1}\hat{A} & I \end{bmatrix} \begin{bmatrix} \tilde{F}[z] \\ \tilde{G}[z] \end{bmatrix} = \begin{bmatrix} [I-z\hat{A}]^{-1}\left[BR[I-Rz^{-1}]^{-1} - z\hat{A}\tilde{F}_0\right] \\ [I-Az^{-1}]^{-1}B[I-Rz^{-1}]^{-1} \end{bmatrix} \quad (51)$$

The left-hand side represents a feedback interconnection, and the right-hand side a vector of exogenous signals that serve as inputs to the feedback loop. The inverse of the left-hand coefficient exists:

$$\begin{bmatrix} -[z^2\hat{A}-zI+A]^{-1}z & 0 \\ 0 & -[z^2\hat{A}-zI+A]^{-1}z \end{bmatrix} \begin{bmatrix} I-Az^{-1} & A \\ \hat{A} & I-z\hat{A} \end{bmatrix} \begin{bmatrix} I-z\hat{A} & 0 \\ 0 & I-Az^{-1} \end{bmatrix}$$

– and, by Lemma B.1, that inverse is proper if $[z^2\hat{A} - zI + A]^{-1}$ is strictly proper

[19] In the (usual) case where A is nonzero.

(with the converse holding if A is nonsingular); in that case, the product of the inverse with the right-hand side of the equation is also proper. The above equation therefore describes $\tilde{F}_t$ and $\tilde{G}_t$ as being uniquely and causally derived from each other, within a feedback loop, and from signals that are exogenous to that feedback loop. By linearity and time-invariance, the zero-state responses of $\hat{x}_{1,t}$ and x_t inherit this relationship from their convolution kernels.

More explicitly, a feedforward/feedback realization of the zero-state response can be obtained by transforming the first equation of (51) to the time domain, and convolving with the w_t sequence:

$$\hat{x}_{1,t} = \sum_{\tau=0}^{t} \Phi_{t-\tau}[Ax_\tau + BRu_\tau] - \sum_{\tau=0}^{t} \Psi_{t-\tau}\hat{A}F_0 w_\tau \ , \ \forall t \geq 0 \ .$$

Here, $\Phi_t := \mathcal{Z}^{-1}\{[I - z\hat{A}]^{-1}\}$ and $\Psi_t := \mathcal{Z}^{-1}\{[I - z\hat{A}]^{-1}z\}$. By Lemma B.1, the first of these is the inverse transform of a proper rational matrix (whose product with $\hat{A}$ is strictly proper), and the second, the inverse transform of a matrix whose product with $\hat{A}$ is proper. Alternatively, Ψ_t could be defined as the inverse transform of a proper matrix, $\mathcal{Z}^{-1}\{[I - z\hat{A}]^{-1}z\hat{A}\hat{A}^g\}$, where $\hat{A}^g$ is a generalized inverse of $\hat{A}$ (such that $\hat{A}\hat{A}^g\hat{A} = \hat{A}$).

By setting $w_t = 0$ for all $t \geq 0$, and transforming to the z-domain, it is easy to check that the extended law in the statement of Theorem 3.6 leads to the same zero-input response as found in section 3.2 (provided that the initial conditions are weakly consistent). This establishes the theorem.

The structure of the feedforward/feedback predictor is readily seen in the form of a block diagram, as in Figure 1 (where, for simplicity, all initial conditions are zero). By well-posedness, all blocks in the diagram are labeled by rational matrices that are proper. Below the dashed line is a representation of the model equations (1,2) (or of the second block row of equation (51)); above is a representation of an implementation of the forecasting mechanism by means of a combination of feedforward of w_t and feedback of x_t (namely, the first block row of equation (51)). It is easily verified algebraically that w_t and $\hat{x}_{1,t}$ are related by $\tilde{F}[z]$, and w_t and x_t by $\tilde{G}[z]$. Moreover, in the latter calculation, the problematic pole-zero cancellation between the forecasting mechanism and the rest of the model is avoided, owing to the use of feedback.

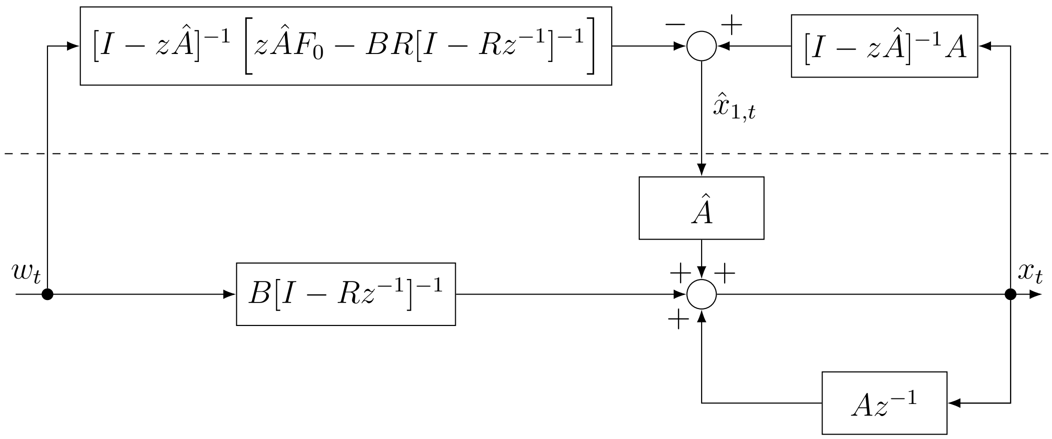

Figure 1: Realization of model-consistent forecasts for well-posed model (with $\overline{x}_t \equiv 0$).

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