Supplementary material: Mathematical preliminaries

This addendum briefly outlines some mathematical background relating to z-transforms and to polynomial and rational matrices. For more detail, refer for example to Chen (1999) or to Fuhrmann and Helmke (2015).

The unilateral z-transform

In employing frequency-domain methods to solve discrete-time initial-value problems, one generally applies the *unilateral*, or *one-sided* z-transform:

$$Y[z] := \mathcal{Z}\{y_t\} := \sum_{t=0}^{\infty} y_t z^{-t}, \ z \in \mathbb{C} \ . \tag{1}$$

Here, y_t may be scalar-, vector-, or matrix-valued.

A unilateral z-transform can be interpreted as a matrix of the transforms of the impulse responses of a causal system. If it is a rational function (or a rational matrix[1]) it must therefore be *proper*: the numerator (of any of its elements) should have a degree no greater than that of the denominator.

The transform is obviously linear; other fundamental properties are summarized below.

Convergence

Suppose that every element $(y_t)_{ij}$ of the matrix y_t satisfies $|(y_t)_{ij}| \leq K\alpha^t$, for some positive $K, \alpha \in \mathbb{R}$. Then the z-transform of y_t converges wherever $|z| > \alpha$.

When all of its elements satisfy such inequalities, y_t is said to be of *exponential order*. Thus, polynomials and exponentials are of exponential order, as are sums, products, and convolutions of functions of exponential order.

[1]See the next subsection.

Inversion integral

The time-domain function y_t is determined by $Y[z]$ via the following contour integral:

$$y_t = \frac{1}{2\pi i} \oint Y[z] z^{t-1} dz \ , \ \forall t \in \mathbb{Z},$$

where the integration is performed in the counterclockwise direction around a closed contour within the region of convergence of the z-transform.

If $Y[z]$ is a proper rational function (or a proper rational matrix – see below), the inverse transform vanishes for negative t.

In practice, inversion is often performed by other means than a direct evaluation of the above integral.

Left-shift rule

The transform, by definition, ignores any nonzero values of y_t for negative values of t. It therefore always yields a transform whose inverse vanishes for negative values of t. This feature is reflected in the standard rule for left shifts of time-domain functions:

$$\mathcal{Z}\{y_{t+1}\} := \sum_{t=0}^{\infty} y_{t+1} z^{-t} = z \sum_{t=0}^{\infty} y_{t+1} z^{-(t+1)} = z[Y[z] - y_0] \tag{2}$$

The transform of the shifted sequence y_{t+1} is obtained by simply multiplying the transform of the unshifted sequence y_t by z – after annihilating the first element of the sequence, so that its left-shifted version vanishes for negative indices.

More generally, we have, by repeated application of the above,

$$\mathcal{Z}\{y_{t+\tau}\} = z^{\tau} \left[Y[z] - \sum_{k=0}^{\tau-1} z^{-k} y_k \right] \ , \forall \tau \geq 1.$$

We illustrate the definition and the shift operation by finding the unilateral z-transform of the sequence that is R^t for nonnegative t, and zero otherwise. Apply a left shift after subtracting $R^0 = I$, which of course yields the same result as multiplying each term of the exponential sequence by R. The z-transform $R[z]$ of the original sequence therefore satisfies:

$$z[R[z] - I] = RR[z] \ . \tag{3}$$

Solving, we find

$$R[z] = [zI - R]^{-1} zI = [I - Rz^{-1}]^{-1} \ .$$

(because the matrix polynomial $[zI - R]$ is regular – see the next section). The sum converges, and the z-transform exists, if and only if $|z|$ is greater than the spectral radius of the matrix – that is, the largest modulus of any eigenvalue.

Applying (3), we obtain an equality that is invoked implicitly in our calculations:

$$z[[I - Rz^{-1}]^{-1} - I] = R[I - Rz^{-1}]^{-1}$$

Right-shift rule

The fundamental right-shift rule is as follows.

$$\mathcal{Z}\{y_{t-1}\} = \sum_{t=0}^{\infty} y_{t-1} z^{-t} = z^{-1}[\sum_{\tau=0}^{\infty} y_\tau z^{-\tau} + zy_{-1}] = z^{-1}[Y[z] + zy_{-1}] \ .$$

By repeated application, we find, more generally,

$$\mathcal{Z}\{y_{t-\tau}\} = z^{-\tau} \left[Y[z] + \sum_{k=1}^{\tau} z^k y_{-k} \right] \ , \forall \tau \geq 1.$$

Time-domain convolution

$$\mathcal{Z}\{\sum_{\tau=0}^{t} h_{t-\tau} x_\tau\} = H[z]X[z] \ .$$

If $y_t = \sum_{\tau=0}^{t} h_{t-\tau} x_\tau$, then the algebraic z-domain counterpart equation, $Y[z] = H[z]X[z]$, can be displayed graphically in a ‘block diagram’ as

$$\xrightarrow{X[z]} \boxed{H[z]} \xrightarrow{Y[z]} \qquad \text{or} \qquad \xrightarrow{x_t} \boxed{H[z]} \xrightarrow{y_t}$$

Polynomial and rational matrices

An $n\times m$ *matrix polynomial* of degree d is a polynomial with $n\times m$ matrix coefficients:

$$P[z] = z^d A_d + z^{d-1} A_{d-1} + \ldots\ + A_0\ ,\ (A_d \neq 0)\ .$$

Equivalently, an $n \times m$ *polynomial matrix* of degree d is a matrix whose entries are polynomials of maximum degree d.

A square matrix polynomial ($n \times n$) is *regular* if its determinant is not the zero polynomial. An eigenvalue is a value λ of z such that there exist nonzero vectors x and y for which

$$P[\lambda]x = 0 = y^\top P[\lambda]\ .$$

The vectors x and y are respectively right and left *eigenvectors* associated with λ. An $n \times n$ regular matrix polynomial has nd eigenvalues, of which, in general, some are *infinite* (lying at the point at infinity on the Riemann sphere). If the leading coefficient A_d is nonsingular, then all nd eigenvalues are finite. if A_d is singular, then every zero eigenvalue of A_d gives rise to a zero eigenvalue of the *reverse* polynomial,

$$z^d[(z^{-1})^d A_d + (z^{-1})^{d-1} A_{d-1} +\ \ldots\ + A_0] = z^d A_0 + z^{d-1} A_1 +\ \ldots\ + A_d$$

The finite, nonzero eigenvalues of $P[z]$ are the reciprocals of those of the reverse polynomial; the infinite eigenvalues of $P[z]$ correspond to the zero eigenvalues of the reverse polynomial – that is, to the zero eigenvalues of A_d. The degree r of $\mathrm{d}etP[z]$ equals the number of finite eigenvalues of $P[z]$, counting multiplicities; the number of infinite eigenvalues of $P[z]$ is $nd - r$.

A *left (resp., right) matrix fraction description* (MFD) is a representation of an $n \times m$ *rational matrix* – a matrix of rational functions – in the form $D(z)^{-1}N(z)$ (resp., $N[z]D[z]^{-1}$), where $D[z]$ is a regular $n \times n$ (resp., $m \times m$) polynomial matrix (viewed here as a rational matrix, and hence possessing an inverse) and $N[z]$ an $n\times m$ polynomial matrix.

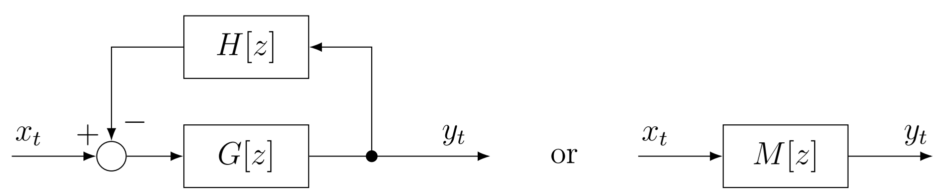
Another common representation of a rational matrix has the form $D_1[z]^{-1}N[z]D_2[z]^{-1}$, where $D_1[z]$ is regular and $n \times n$, $D_2[z]$ is regular and $m \times m$ and $N[z]$ is $n \times m$.

A matrix of rational functions is *proper* (respectively, *strictly proper*) if each of its entries is proper (resp., strictly proper) – that is, if, for each rational-function entry, the degree of the numerator polynomial is no greater than (resp., strictly less than) that of the denominator polynomial.

If z is interpreted as a left-shift operator (in accordance with the previous subsection), each of the rational-function entries represents a time-domain recurrence, with the numerator polynomial acting on an exogenous variable and the denominator on an endogenous variable, then properness implies nonanticipation: the endogenous variable does not depend on future values of the exogenous variable. The converse also holds.[2]

Both properness and strict properness of rational matrices are preserved under addition, subtraction and multiplication. Moreover, multiplication of a proper matrix by a strictly proper one yields a strictly proper matrix.

By straightforward algebra, a ("negative") feedback interconnection of two proper rational matrices $G[z]$ (in the forward path) and $H[z]$ (in the feedback channel), such that the product $G[z]H[z]$ is strictly proper, can be represented by the following block diagrams:



where

$$M[z] := [I + G[z]H[z]]^{-1}G[z] \ .$$

The rational matrix $[I+G[z]H[z]]^{-1}$ is proper. It follows that $M[z]$ is proper; indeed, if $G[z]$ is strictly proper, then so is $M[z]$.

[2]Indeed, an alternative definition of strict properness is that the Laurent expansion of the matrix contains no nonnegative powers of z (Fuhrmann and Helmke, 2015): accordingly, we adopt for convenience the convention that an identically zero-valued matrix is strictly proper.

References

Chen, Chi-Tsong (1999) *Linear System Theory and Design*: Oxford University Press, 3rd edition.

Fuhrmann, P.A. and U. Helmke (2015) *The Mathematics of Networks of Linear Systems*: Springer.