Problems

Kevin G. Hare

March 10, 2017

This document attempts to describe some of my mathematical interests. Along the way it will list some problems that come up in the area, and suggest some problems that I think would be interesting to look at.

Problems on Pisot and Salem numbers

A Pisot number is a real algebraic integer $q > 1$ all of whose conjugates are strictly less than 1 in absolute value. Probably the simplest example is $\tau = 1.618\cdots$ the root of $x^2 - x - 1$. The conjugate is $\tau' = -0.618$ which satisfies $|\tau'| < 1$. Pisot numbers have a long and rich history. They were initially studied by Thue in 1912, [73] but became popularized by Pisot [64]. Since then, they have found applications in beta-expansions, tilings of surfaces (cut and project quasi-crystals), fractals, Mahler measure questions, Bernoulli convolutions, Garsia's entropy, to name just a few. A Salem number is similar, with all of q's conjugates less than or equal to 1 in absolute value, with at least one conjugate equal to 1 in absolute value. Salem numbers have had a somewhat shorter, but equally impressive, history. Introduced by Salem in 1945 [67] to help in the study of the set of Pisot numbers, the set of Salem numbers are intimitely connected with deep questions on Mahler measure. Lehmer's polynomials, which gives rise to the smallest known Salem number was found in a different context much earlier in 1933 [51]. Below I have expanded upon some of the areas of research relating to Pisot and Salem numbers, as well as my interests in these areas.

Beta-expansion

Instead of looking at a number in base 10, we can look at it in base $\beta \in \mathbb{R}$. Choosing $\beta \in \mathbb{R}$ has the potential to dramatically alter the properties of such expansions. As an example, in base 10, with digits $0, 1, \cdots, 9$, we have that $1 = 0.9999\cdots$. If instead we use base $\tau = 1.618...$ the root of $x^2 - x - 1$, and digits $0, 1$ we have multiple ways of representing 1, for example $1 = 0.11$, $1 = 0.1011$, $1 = 0.101010\cdots$, and $1 = 0.0111\cdots$. It can be shown with this case, that there are in fact an infinite number of ways of representing 1. This is the main premise behind the study of beta-expansions. These problems were initiated by Rényi in 1957, [66]. He was interested in looking at this type of expansion, and how it related to

Ergodic theory. (He did it as one of a number of examples.) The idea was then expanded and popularized by Parry in 1960, [62]. These problems have applications to Ergodic theory, Bernoulli convolutions, and Rauzy fractals [48, 41]. In [2], we created tools that would not only compute the beta-expansions, but would conjecture, formulate, and then prove results for entire families of algebraic integers, which drastically extended our knowledge of the structure of these objects. This has led to three other papers [33, 44, 61] one alone, one with a former Master's student and the last with a former USRA. In the joint paper with my former Masters student [44], a classification of beta-expansions of some regular Pisot numbers is given. Here regular Pisot numbers are a special class of Pisot numbers that account for "most" Pisot numbers less than 2. It seems likely that a complete classification should be achievable, giving rise to the first problem. In [61] a complete classification of beta-expansions for regular Pisot numbers in $(1, 2)$ is given. A related problem, that was partially answered in [44] is

Problem 1 *Given a sequence of Pisot numbers $q_n \to q^*$, where q^* is either a Pisot or a Salem number, what can be said about the sequence of beta-expansions of the q_n, and the beta-expansion of q^*.*

This problem could make an interesting group undergraduate research project, a masters project, or a component of a PhD program. It would require some computational experimentation.

Most results on beta-expansions are given in the case when we choose the *greedy* beta-expansion. There are multiple different types of beta-expansions though, the second most common being the *lazy* beta-expansions. This leads to the obvious question

Problem 2 *Which results on beta-expansions are still true if we look at the lazy beta-expansions? And/or what are the equivalent statements?*

Depending on the exact question asked, this could range anywhere from an undergraduate project to a PhD program.

The next problem is a long standing conjecture of David Boyd's relating to the beta-expansions of Salem numbers. See for instance [14, 15]. We say that a beta-expansion is eventually periodic if it can be written as $0.a_1a_2 \cdots a_k(a_{k+1}a_{k+2} \cdots a_n)^\infty$. This is the equivalent idea to a rational number in an integer base. As an example, with $\beta = \tau$, the golden ratio, we have that $1 = 0.(10)^\infty$ is a periodic expansion, but if $\beta = \sqrt{3}$, then the expansion $1 = 0.1100101000 \cdots$ is not periodic. (This is not immediately obvious, but is not overly difficult to prove.) If β is a degree 4 Salem number, then it is known that 1 will have an eventually periodic beta-expansions. What happens for higher degree Salem numbers is still not fully understood.

Problem 3 *Is it true that: "If β is a degree 6 Salem number, then the beta-expansion of 1 is periodic. Moreover, there exists degree 8 Salem numbers where this is not true."?*

It is not clear what techniques would be required to prove this results, nor what level of sophistication would be needed. As such it would probably not make a good question for an undergraduate, or graduate student to work on. It is still an interesting problem, and is one that I will continue to think about as a long term project.

Garsia's Entropy as it relates to beta-expansions

A problem that is related to beta-expanions, but is somewhat more related to analysis and less to number theory involves Garsia's entropy.

Let μ_β denote the *Bernoulli convolution* parameterized by β on $I_\beta := [0, 1/(\beta-1)]$, i.e.,

$$\mu_\beta(E) = \mathbb{P}\left\{(a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k \beta^{-k} \in E\right\}$$

for any Borel set $E \subset I_\beta$, where $\mathbb{P}$ is the product measure on $\{0,1\}^{\mathbb{N}}$ with $\mathbb{P}(a_1 = 0) = \mathbb{P}(a_1 = 1) = 1/2$. Since $\beta < 2$, it is obvious that $\operatorname{supp}(\mu_\beta) = I_\beta$.

Bernoulli convolutions have been studied for decades (see, e.g., Peres, Schlag and Solomyak [63] and Solomyak [71]), but there are still many open problems in this area. The most significant property of μ_β is the fact that it is either absolutely continuous or purely singular (see Jessen and Wintner [49]); however Erdős showed that the only family of βs for which it is known to be singular, are the Pisot numbers (see [21]).

Garsia [23] introduced a new notion associated with a Bernoulli convolution. Namely, put

$$D_n(\beta) = \left\{x \in I_\beta : x = \sum_{k=1}^{n} a_k \beta^{-k} \text{ with } a_k \in \{0,1\}\right\}$$

and for $x \in D_n(\beta)$,

$$p_n(x) = \#\left\{(a_1, \dots, a_n) \in \{0,1\}^n : x = \sum_{k=1}^{n} a_k \beta^{-k}\right\}. \tag{1}$$

Finally, put

$$H_\beta^{(n)} = -\sum_{x \in D_n(\beta)} \frac{p_n(x)}{2^n} \log \frac{p_n(x)}{2^n}$$

and

$$H_\beta = \lim_{n\to\infty} \frac{H_\beta^{(n)}}{n \log \beta}$$

(it was shown in [23] that the limit always exists). The value H_β is called *Garsia's entropy*.

If β is transcendental or algebraic but not satisfying an algebraic equation with coefficients $\{-1, 0, 1\}$, then all the sums $\sum_{k=1}^{n} a_k \beta^{-k}$ are distinct, whence $p_n(x) = 1$ for any $x \in D_n(\beta)$, and $H_\beta = \log 2 / \log \beta > 1$.

However, if β is Pisot, then it was shown in [23] that $H_\beta < 1$ – which means in particular that β does satisfy an equation with coefficients $\{0, \pm 1\}$. Furthermore, Garsia also proved that if $H_\beta < 1$, then μ_β is singular, but no non-Pisot β with this property has been found so far.

In 1991 Alexander and Zagier in [1] managed to evaluate H_β for the golden ratio $\beta = \tau$ with an astonishing accuracy. It turned out that H_τ is close to 1 – in fact $H_\tau \approx 0.9957$.

Grabner, Kirschenhofer and Tichy [26] extended this method to the multinacci numbers; in particular, $H_{\tau_3} \approx 0.9804$, $H_{\tau_4} \approx 0.9867$, etc. They also showed that H_{τ_m} is strictly increasing for $m \geq 3$, and $H_{\tau_m} \to 1$ as $m \to \infty$ exponentially fast.

It is shown in [41] that that $H_\beta > 0.81$ for all Pisot β and improve this bound for certain ranges of β. This was improved in [42] that $H_\beta > 0.82$ for all $\beta \in (1, 2)$. Further, if β is algebraic and $[\mathbb{Q}(\beta) : \mathbb{Q}(\beta^k)] = k$ for some $k \geq 2$ then $\dim(\mu_\beta) = 1$.

Problem 4 *Are there $\beta \in (1, 2)$ which are not Pisot numbers and for which μ_β is purely singular (or equivalently $H_\beta < 1$)?*

Problem 5 *Are there any other non-obvious families of $\beta \in (1, 2)$ for which we can say μ_β is absolutely continuous (or equivalently $H_\beta = 1$)?*

These two problems will involve a fair amount of experimentation to find candidates. It is not clear though what techniques could be used to prove that μ_β is either purely singlar or absolutely continuous. This problem would probably not make a good research project for graduate students unless they managed to make some sort of significant insight early into their research. (I.e. if they found a technique that can occasionally prove β is one or the other, it would be worth persuing ... but it is not clear how doable this is.)

Spectra of Pisot numbers

This problem is primarily concerned with the study of

$$\Lambda(q) = \{a_n q^n + \cdots + a_0 \mid a_i \in \{0, \pm 1\}, n \in \mathbb{N}\}$$

and the related constant

$$\ell(q) = \inf\{y \in \Lambda(q) \mid y > 0\}.$$

As an example, $\Lambda(2) = \mathbb{Z}$ and $\Lambda(\sqrt{2}) = \mathbb{Z}[\sqrt{2}]$, and so we get $\ell(2) = 1$ and $\ell(\sqrt{2}) = 0$. The set $\Lambda(q)$ is occasionally discrete in $\mathbb{R}$, and occasionally dense. These sets and the value of $\ell(q)$ were originally introduced by Erdős, in the 1990s; and a series of papers was written by numerous authors that examined these types of problems. See for instance [16, 20, 50], and the references contained within. The study of these sets came out of the study of beta-expansions, as properties of the spectra have direct applications to properties of the beta-expasions. When $1 < q < 2$ it can be shown that q is a Pisot number if and only if $\ell(q) > 0$ [22].

In some special cases $\ell(q)$ is known exactly.

Problem 6 *Let $q \in (1, 2)$ be a Pisot number with property X, then $\ell(q) = ...$*

Here the generality of property X would dictate the difficulty of the question, and would require a fair bit of experimentation. Some examples of possible properties X could be "degree 2", or *Perron*, or that the minimal polynomial has a special form, or that q is a special class of regular Pisot number. Again, depending on the specific question involved, this could make a nice undergraduate research project, group undergraduate research project, masters project, or a component of a PhD program.

The earliest result of mine in this area concerns the calculation of $\ell(q)$ for general Pisot numbers q. This is summarized in [9]. Before [9], values for $\ell(q)$ were known only for a few values of q, and the methods used were always ad hoc. This paper gave a systematic method to perform these calculations, and presented an algorithm that would compute these minimal values. Since then, I have written a number of papers in this area, both as joint projects with other researchers, and as singly authored papers [9, 10, 11, 24, 31]. There are a number of interesting questions concerning $\ell(q)$, $\Lambda(q)$, and their relationship with beta-expansions, or quasicrystals.

Quasicrystals

Quasicrystals are aperiodic tilings of $\mathbb{R}^n$ that use Pisot numbers as a key component in their construction. First discovered in 1984, quasicrystals have quickly become a very active area of research, [70]. They combine techniques from experimental physics, crystolography, combinatorics, number theory, and many other branches of math and physics. See for instance [4, 58]. Given the short period of time that the subject has been studied, there are still a huge number of properties about quasicrystals that are waiting to be discovered.

I have three papers that are related to this area. The first paper, [5], gives a complete classification of all Pisot numbers up to degree 5 that can be used in the construction of quasicrystals. The second, [36], looks at the possible structure in $\mathbb{R}$ that result from these Pisot numbers. In the last paper, [39], we show that if we use 0 and roots of unity in place of $\{0, \pm\}$ in the definition of spectra, we often get quasicrystal type structures.

Problem 7 *Better understand the relationship between beta-expansion, Spectra of Pisot numbers, and quasicrystals.*

This would make a nice expository Master's thesis.

Measures of finite type

Consider the iterated function system (IFS) consisting of the contractions $S_j : \mathbb{R} \to \mathbb{R}$, $j = 0, \ldots, m$, defined by

$$S_j(x) = \varrho x + d_j \tag{2}$$

where $0 < \varrho < 1$, $d_0 < d_1 < d_2 < \cdots < d_m$ and $m \geq 1$ is an integer. By the associated self-similar set, we mean the unique, non-empty, compact set K satisfying

$$K = \bigcup_{j=0}^{m} S_j(K).$$

Suppose p_j, $j = 0, \ldots, m$ are probabilities, i.e., $p_j > 0$ for all j and $\sum_{j=0}^{m} p_j = 1$. Our interest is in the self-similar measure μ associated to the family of contractions $\{S_j\}$ as above, which satisfies the identity

$$\mu = \sum_{j=0}^{m} p_j \mu \circ S_j^{-1}. \tag{3}$$

These measures are sometimes known as *equicontractive*, or *ϱ-equicontractive* if we want to emphasize the contraction factor ϱ. They are non-atomic, probability measures whose support is the self-similar set.

When the ratio of contraction is the inverse of a Pisot number, quite often these measures have a very rich, and very computable structure. We have looked at these in a series of papers [29, 28, 30].

For many of these measures, very precise things can be said about the local dimensions at points in this measure. Despite there, there are of interesting variations that can still be explored.

Problem 8 *How much of the structure of the IFS can be relaxed, while still allowing us to get some meaningful results.*

This is a very vague and somewhat open ended question. I suspect that there are many versions of this questions that will results in good USRA projects for years to come.

Mahler Measure Problems

The Mahler measure of a polynomial, measures the size of the roots bigger than 1. More precisely, if we have $p(x) = a_n x^n + \cdots + a_0 = a_n \prod (x - \alpha_i)$ an irreducible polynomial in $\mathbb{Z}[x]$, then we define the Mahler measure of $p(x)$ as

$$M(p) = |a_n| \prod \max(1, |\alpha_i|).$$

Here the Mahler measure of an algebraic number is the Mahler measure of its irreducible polynomial. In particular, the Mahler measure of a Pisot or Salem number is the Pisot or Salem number in question. One of the oldest questions involving Salem numbers is that of Lehmer, which roughly says "Is there a smallest Salem number? If so, what is it?". Moreover, Lehmer discovered the smallest known Salem number in 1933, without the aid of computers [51]. This is approximately 1.1763, the root of

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Currently this polynomial is also the smallest known Mahler measure that is strictly greater than 1. Numerous computer searches have been done since then with no better example found [13, 59]. In [12] we looked at the Mahler measure problem, in the restricted case of the polynomial having only ± 1 coefficients. This has spawned an active area of research looking at Mahler measure problems for polynomials in restricted classes. See for example [8]. Much like before, there are many variations of this problem that are quite doable, at any of the undergradate, masters, or a component of a PhD level. Examples are

Problem 9 *Is there a smallest Salem number with property X? If so, what is it?*

Problem 10 *Can we extend/modify the search for Salem numbers to search for Salem numbers with property X.*

The second question would require strong programming skills.

In [19], Dobrowolski gave a proof bounding below the Mahler measure based on the number of coefficients. The key lemma used for this result gave lower bounds on resultants of particular forms. In [40], we looked at this resultant in more detail, strengthening the result. It still remains to be seen if this stronger result maybe used to improve the lower bounds on the Mahler measure by any meaningful amount.

These problems motivate two problems to be considered.

Problem 11 *Given a restricted class of polynomials, what can be said about the maximum, or minimum Mahler measure of these polynomials?*

There are many variations that this problem can take. Many of these variations could make interesting components of a PhD program.

Problem 12 *Can the techniques of [40] be used to improve upon the lower bounds for non-trivial Mahler measures?*

This again is a problem that is approachable by a graduate student. Depending upon the scope, or direction of the research, this could make a good Master's thesis, or a good starting point for a PhD thesis.

Other problems of interest

Integer Chebyshev Problem

The Integer Chebyshev problem is concerned with finding $p(x) \in \mathbb{Z}[x]$ that is "small" in some sense on an interval I. By this we mean that it has a small supremum norm, normalized by the degree of the polynomial. This problem was originally introduced to find a "simpler" proof of the prime number theorem. Unfortunately the proposed technique failed (and provably so), but it was quickly seen to be an interesting problem in its own right. See for instance [27, 65]. I have two papers in this area. The first, [43] involves looking at a related case, when $p(x) \in \mathbb{Z}[x]$ and $p(x)$ a monic polynomial. Exact solutions are given for some intervals for the monic case, which is a notable contrast to the non-monic case where no exact solutions are known. In the second, [36], we examine the lower bounds for a generic interval. This requires the construction of an infinite sequence of polynomials with specific properties. There are many variations of this problem that are poorly understood. One such problem is

Problem 13 *What can be said about the 2-dimensional analog of this problem?*

Or alternately

Problem 14 *What can be said if we restrict $p(x)$ to be in $\mathbb{Z}[x]$ with property X.*

This was done in the case where property X meant that $p(x)$ was monic, but we could forsee many other interesting variations.

Most techniques used to find examples of good polynomials use the Simplex method. There are a number of semi-infinite programming techniques that have the potential to improve the results significantly.

Problem 15 *How can better optimization techniques be used to improve the bounds?*

Either of these problems could make a nice starting point for a Master's or PhD thesis, depending upon the scope of the actual question, and the background of the student. It would probably be too ambitious for an undergraduate project.

Odd Perfect numbers

A perfect number N satisfies the property that N is a sum of all of its proper divisors. The first example is 6, where the proper divisors are 1, 2 and 3, and $6 = 1+2+3$. There is a huge history to this problem, being studied by Euclid (∼300 BCE), Mersenne (1588-1648) and Euler (1707-1783) among others. Despite its long history, there are still many unanswered questions concerning them. To date, there are no known odd perfect numbers, but a proof is not known for existence, or non-existence. This is probably my most accessible area of research, and has resulted in the largest number of unsolicited communications with amateur mathematicians. Primarily this research involves devising algorithms for computational verification on the non-existences of odd perfect numbers. Examples of possible bounds would be things such as "If an odd perfect number exists, it must be bigger than M, have more than K factors, have a prime factor bigger than C", etc. Here M, K, and C are constantly being increased, as a result of improved algorithms, and computational power. My two main papers in this area are [32, 34]. There are a number of variations of this problem that would make interesting projects for graduate, or undergraduate students.

Problem 16 *Modify these techniques to look at Tri-Perfect numbers, Perfect polynomials, etc.*

Fekete Polynomials & Barker Sequences

In 2001, Borwein, Choi, and Yazdani looked at an extremal property of a class of polynomial with ±1 coefficients, [7]. Their key result was:

Theorem 1 (Borwein, Choi, Yazdani, 2001) *Let $f(z) = \pm z \pm z^2 \pm \cdots \pm z^{N-1}$, and ζ a primitive N root of unity. If N is an odd positive integer then*

$$\max_i |f(\zeta^i)| \geq \sqrt{N}$$

with equality if and only if N is an odd prime.

Moreover, if equality holds, they gave an explicit construction for $f(z)$. In this paper, we look at the case when N is even. In particular, we investigate the following

Conjecture 1 *Let $f(z)$ and ζ be as above. If $N > 2$ is an even positive integer then*

$$\max_i |f(\zeta^i)| \geq \sqrt{N+1}$$

with equality if and only if $N+1$ is a power of an odd prime.

This conjecture was made after extensive computations. Partial results towards proving this conjecture are given in [45].

Problem 17 *What can be said for other N?*

It is unclear how difficult either of these problems are, and as such they would not make good projects for students.

We say a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is a *Littlewood polynomial* if $a_k = \pm 1$ for $0 \leq k \leq n$. Let $p(z)p(1/z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_{-n} z^{-n}$. It is easy to show that $c_0 = n+1$. We say that $p(z)$ is a *Barker polynomial* if $|c_k| \leq 1$ for $k \neq 0$. There are only 8 known Barker polynomials (normalized to have $a_n = a_{n-1} = 1$). There are many results known about the existence and non-existence of Barker polynomials for various degrees. This paper deals with the infinite case, when $f(z) = \pm 1 \pm z \pm z^2 \pm \cdots$ is a power series with ± 1 coefficients, where $f(z)f(-z) = 1 + z^2 + z^4 + z^6 + \cdots$. We give a complete description of all Barker series in [35]. There are two problesm taht are suggested by this.

Problem 18 *Extend the non existance proof for additional Barker polynomials.*

Problem 19 *Can we classify $f(z)f(-z) = G(z)$ for generic rational functions.*

Again, it is not clear how difficult these are, and as such they would not make good projects for students.

Sum of digits function

We denote by $s_q(n)$ the sum of the q-ary digits of n. In recent years, much effort has been made to get a better understanding of the distribution properties of s_q regarding certain subsequences of the positive integers. See for example [18, 53, 54].

Particular attention has been given to the ratio

$$\frac{s_q(p(n))}{s_q(n)}.$$

It is known for example that this ratio gets arbitarily close to 0, 1 and infinity. This was initially proved for $p(n) = n^2$ and $q = 2$ by [57, 72] and later generalized in [37, 38]. My PhD student showed [68] that $\frac{s_q(n^a)}{s_q(a)}$ can take any rational value.

An interesting various of this question is:

Problem 20 *Can we say anything about the ratio* $\frac{s_q(p_1(n))}{s_q(p_2(n))}$ *polynomials* $p_1(n)$ *and* $p_2(n)$*?*

Problem 21 *What is the "average" value of this ratio?*

The first of these questions would make interesting Masters project or thesis (depending on the exact question asked), or a nice component of a PhD thesis. The last question is probably too ambitions for a Masters thesis, but, if combined with other problems would be a nice component of a PhD thesis.

Past Projects

Here I've listed some of the past projects done under my supervision. I've really only gone up to 2011 — I need to redo this section

- Undergraduate Projects
  - *Optimization methods and the Integer Chebyshev Problems.* The idea of this project was to use Newton's method, or other optimization techniques to find good solutions for the Integer Chebyshev Problem. This is a project that is worth revisiting, possibly in collaboration with a student/professor with a good knowledge of optimization techniques.
  - *Roots of polynomials with* $\{0, \pm 1\}$ *coefficients.* Consider the closure of all the roots of polynomials with coefficients $\{0, 1\}$. A somewhat surprising result is that this set is path connected. The idea of this project was to extend the results of [60] to a wider variety and class of polynomials. This project has been extended again by a student of Simon Kristensen (Denmark).
  - *Solutions to* $\mathbb{Z}[q] = \mathbb{Z}[r]$ *with* r *fixed,* q *Pisot.* This problem was original motivated by questions that came up in the research of Quasicrystals in [5, 6], and asked, for fixed r, how many solutions, and what types of solutions would you get, looking for Pisot numbers q where $\mathbb{Z}[q] = \mathbb{Z}[r]$. This required a fair bit of Galois Theory.
  - *Maximal Tiles sets of cut & project Quasicrystals.* This project was the first application of an algorithm found in [36].
  - *Tri-perfect numbers* This looked at a variation of Odd Perfect Numbers, that of tri-Perfect numbers where $\sigma(n) = 3n$. This modified existing techniques for searching for odd perfect numbers to attempt to find odd tri-perfect numbers. This project is worth revisiting either as is, or as a variation.
  - *Quasicrystals and* C^**-Algebras* I wasn't properly qualified to supervise this project. It was interesting, and the student was strong, so it worked out fine, but it is not a topic I would look at again.

- *Solutions to $m^2 = aa \cdots aba \cdots a$ base c.* There are examples of numbers m such that m^2 looks reasonably nice, such as $21^2 = 441$ or $212^2 = 44944$. The question was, can these methods to be used for numbers is bases other than 10. There was a lot of nice results discovered, but it was somewhat incomplete as, for some basis, the techniques used to solve various diophantine equations that come up, quickly became much more involved that originally expect. This project would be worth revisiting with a more serious student with a strong background in p-adic analysis and diophantine equations. This was based upon the work of [25].
- *Graph Theory, Pisot and Salem numbers.* This project involved constructing Salem numbers, and Pisot numbers from eigenvalues of incidence matrices of graphs. This was mostly based on the works [55, 56]. Some of the questions investigates involved, what happens if the graph allows directed edges, or multi-edges. Variations of this problem would be worth revisiting in the future.
- *Rational periodic points of polynomials* Consider the polynomial $f(x) = x^2 - 1$. We easily see that $f(f(-1)) = -1$. We call -1 a *periodic point of order* 2. A conjecture of B. Poonen, bridging number theory and dynamical systems, suggests that there is no quadratic polynomial with rational coefficients with a rational periodic point (under iteration of the polynomial) of any period greater than 3. There are a few results in the literature lending credence to the claim, and recently B. Hutz and P. Ingram have conducted some computations to support the claim. The purpose of this project was o investigate the analogous problem for polynomials of higher degree, largely through computational means.
- *Beta expansions of Pisot numbers* This resulted in the publication [61].
- *A multi-dimensional analogue of Cobham's theorem for fractals.* This resulted in the publication [17].
- *Measures of finite type* This was discussed earlier in an earlier section.

• Masters Projects

- *Coefficients of Cyclotomic polynomials.* This project followed the work of [3], expanding on it, and in some cases offering some refinements. The main question was, "how big can the coffeience be of a cyclotomic polynomials $\Phi_{pqr}(x)$ for distinct primes $p < q < r$".
- *Ruth-Aaron Numbers.* This thesis looked at solutions to $d(n) = d(n+1)$, $\omega(n) = \omega(n+1)$ and $S(n) = S(n+1)$ where d, ω, S are all multiplicative functions. These followed the works of [46, 47, 69].
- *Pisot numbers.* This was a survey paper about Pisot numbers, following no particular paper, but instead synthesising many properties that were known about them.
- *Vector Rational Number Reconstruction.* Let v_i be a set of vectors in $\mathbb{R}^n$. This project was to find a sequence of rationals such that $\sum a_i v_i \approx \mathbf{0}$, where $\mathbf{0}$ is the zero vector. It was closely related to the LLL algorithm [52].

  - *Elliptic curve cryptograpy*
  - *Quadratic Sieve*
  - *The ω function and generalizations*
  - *On Greedy and Lazy Beta-expansions*
  - *Additive properties of β-integers for Quadratic Pisot numbers*

- PhD Projects

  - *Prouhet-Tarry-Escott problems over $\mathbb{Z}[i]$.* This is an old problem with a new twist. The original problem was to find non-trivial sets of integers solutions $\{a_1, \cdots, a_k\}$ and $\{b_1, \cdots, b_k\}$ to the simultaneous equations

$$\begin{aligned} a_1^1 + a_2^1 + \cdots + a_k^1 &= b_1^1 + b_2^1 + \cdots + b_k^1 \\ a_1^2 + a_2^2 + \cdots + a_k^2 &= b_1^2 + b_2^2 + \cdots + b_k^2 \\ &\vdots \\ a_1^n + a_2^n + \cdots + a_k^n &= b_1^n + b_2^n + \cdots + b_k^n \end{aligned}$$

    There is nothing special about the integers in this problem at this project involves looking at the equivalent results in $\mathbb{Z}[i]$, or various other rings.

References

[1] J. C. Alexander and Don Zagier, *The entropy of a certain infinitely convolved Bernoulli measure*, J. London Math. Soc. (2) **44** (1991), no. 1, 121–134. MR MR1122974 (92g:28035)

[2] Jean-Paul Allouche, Christiane Frougny, and Kevin G. Hare, *On univoque Pisot numbers*, Math. Comp. **76** (2007), no. 259, 1639–1660 (electronic). MR MR2299792

[3] Gennady Bachman, *On the coefficients of ternary cyclotomic polynomials*, J. Number Theory **100** (2003), no. 1, 104–116. MR MR1971249 (2004a:11020)

[4] Damien Barache, Bernard Champagne, and Jean-Pierre Gazeau, *Pisot-cyclotomic quasilattices and their symmetry semigroups*, Quasicrystals and discrete geometry (Toronto, ON, 1995), Amer. Math. Soc., Providence, RI, 1998, pp. 15–66. MR 99g:52028

[5] Jason P. Bell and Kevin G. Hare, *A classification of (some) Pisot-cyclotomic numbers*, J. Number Theory **115** (2005), no. 2, 215–229. MR MR2180499

[6] ———, *On $\mathbb{Z}$-modules of algebraic integers*, Canad. J. Math. **61** (2009), no. 2, 264–281. MR MR2504015

[7] Peter Borwein, Kwok-Kwong Stephen Choi, and Soroosh Yazdani, *An extremal property of Fekete polynomials*, Proc. Amer. Math. Soc. **129** (2001), no. 1, 19–27 (electronic). MR MR1784013 (2001j:11061)

[8] Peter Borwein, Edward Dobrowolski, and Michael J. Mossinghoff, *Lehmer's problem for polynomials with odd coefficients*, Ann. of Math. (2) **166** (2007), no. 2, 347–366. MR MR2373144 (2008j:11153)

[9] Peter Borwein and Kevin G. Hare, *Some computations on the spectra of Pisot and Salem numbers*, Math. Comp. **71** (2002), no. 238, 767–780 (electronic). MR 2003a:11135

[10] ______, *General forms for minimal spectral values for a class of quadratic Pisot numbers*, Bull. London Math. Soc. **35** (2003), no. 1, 47–54. MR 2003i:11154

[11] ______, *Non-trivial quadratic approximations to zero of a family of cubic Pisot numbers*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 4767–4779 (electronic). MR 1 997 583

[12] Peter Borwein, Kevin G. Hare, and Michael J. Mossinghoff, *The Mahler measure of polynomials with odd coefficients*, Bull. London Math. Soc. **36** (2004), no. 3, 332–338. MR 2 038 720

[13] David W. Boyd, *Reciprocal polynomials having small measure. II*, Math. Comp. **53** (1989), no. 187, 355–357, S1–S5. MR 89m:30013

[14] ______, *Salem numbers of degree four have periodic expansions*, Théorie des nombres (Quebec, PQ, 1987), de Gruyter, Berlin, 1989, pp. 57–64. MR MR1024551 (90j:11071)

[15] ______, *On the beta expansion for Salem numbers of degree* 6, Math. Comp. **65** (1996), no. 214, 861–875, $S29$–$S31$. MR MR1333306 (96g:11091)

[16] Yann Bugeaud, *On a property of Pisot numbers and related questions*, Acta Math. Hungar. **73** (1996), no. 1-2, 33–39. MR 98c:11113

[17] Davy Ho-Yuen Chan and Kevin G. Hare, *A multi-dimensional analogue of Cobham's theorem for fractals*, Proc. Amer. Math. Soc. **142** (2014), no. 2, 449–456. MR 3133987

[18] Cécile Dartyge and Gérald Tenenbaum, *Congruences de sommes de chiffres de valeurs polynomiales*, Bull. London Math. Soc. **38** (2006), no. 1, 61–69. MR MR2201604 (2006k:11039)

[19] Edward Dobrowolski, *Mahler's measure of a polynomial in function of the number of its coefficients*, Canad. Math. Bull. **34** (1991), no. 2, 186–195. MR 92f:11138

[20] P. Erdős, I. Joó, and F. J. Schnitzer, *On Pisot numbers*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **39** (1996), 95–99 (1997). MR 98d:11127

[21] Paul Erdös, *On a family of symmetric Bernoulli convolutions*, Amer. J. Math. **61** (1939), 974–976. MR MR0000311 (1,52a)

[22] De-Jun Feng, *On the topology of polynomials with bounded integer coefficients*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 1, 181–193. MR 3438383

[23] Adriano M. Garsia, *Entropy and singularity of infinite convolutions*, Pacific J. Math. **13** (1963), 1159–1169. MR MR0156945 (28 #187)

[24] David Garth and Kevin G. Hare, *Comments on the spectra of Pisot numbers*, J. Number Theory **121** (2006), no. 2, 187–203.

[25] Alexandru Gica and Laurenţiu Panaitopol, *On Obláth's problem*, J. Integer Seq. **6** (2003), no. 3, Article 03.3.5, 12 pp. (electronic). MR MR2046405 (2004k:11014)

[26] Peter J. Grabner, Peter Kirschenhofer, and Robert F. Tichy, *Combinatorial and arithmetical properties of linear numeration systems*, Combinatorica **22** (2002), no. 2, 245–267, Special issue: Paul Erdős and his mathematics.

[27] Laurent Habsieger and Bruno Salvy, *On integer Chebyshev polynomials*, Math. Comp. **66** (1997), no. 218, 763–770. MR 97f:11053

[28] Kathryn E. Hare, Kevin G. Hare, and Kevin R. Matthews, *Local dimensions of measures of finite type on the torus*, (manuscript), (arXiv:1607.03515), 28 pages.

[29] ———, *Local dimensions of measures of finite type*, J. Fractal Geom. (to appear), (arXiv:1504.00510), 40 pages.

[30] Kathryn E. Hare, Kevin G. Hare, and Michael Ka Shing Ng, *Local dimensions of measures of finite type ii - measures without full support and with non-regular probabilities*, (arXiv:1603.02244), 41 pages.

[31] Kevin G. Hare, *The structure of the spectra of Pisot numbers*, J. Number Theory **105** (2004), no. 2, 262–274. MR 2 040 158

[32] ———, *More on the total number of prime factors of an odd perfect number*, Math. Comp. **74** (2005), no. 250, 1003–1008 (electronic). MR MR2114661

[33] ———, *Beta-expansions of Pisot and Salem numbers*, Computer algebra 2006, World Sci. Publ., Hackensack, NJ, 2007, pp. 67–84. MR MR2427721

[34] ———, *New techniques for bounds on the total number of prime factors of an odd perfect number*, Math. Comp. **76** (2007), 2241–2248 (electronic).

[35] ———, *Infinite Barker series*, J. Number Theory **129** (2009), no. 12, 2991–2999.

[36] ———, *Tiles in quasicrystals with quartic irrationality*, Math. Comp. **78** (2009), no. 265, 405–420. MR MR2448713

[37] Kevin G. Hare, Shanta Laishram, and Thomas Stoll, *Stolarsky's conjecture and the sum of digits of polynomial values*, Proc. Amer. Math. Soc. **139** (2011), no. 1, 39–49. MR 2729069 (2011j:11012)

[38] ———, *The sum of digits of n and n^2*, Int. J. Number Theory **7** (2011), no. 7, 1737–1752. MR 2854212 (2012k:11157)

[39] Kevin G. Hare, Zuzana Masáková, and Tomáˇ s Vávra, *On the spectra of Pisot-cyclotomic numbers*, (arXiv:1612.09285), 28 pages.

[40] Kevin G. Hare, David McKinnon, and Chris Sinclair, *Patterns and periodicity in a family of resultants*, J. Théor. Nombres Bordeaux **21** (2009), no. 1, 215–234.

[41] Kevin G. Hare and Nikita Sidorov, *A lower bound for Garsia's entropy for certain Bernoulli convolutions*, LMS J. Comput. Math. **13** (2010), 130–143. MR 2638985

[42] ———, *A lower bound for the dimension of bernoulli convolutions*, Exp. Math. (to appear), (arXiv:1606.02131), 8 pages.

[43] Kevin G. Hare and Chris J. Smyth, *The monic integer transfinite diameter*, Math. Comp. **75** (2006), no. 256, 1997–2019 (electronic), *Corrigendum to "The monic integer transfinite diameter"*, Math. Comp. (2008). MR MR2240646

[44] Kevin G. Hare and David Tweedle, *Beta-expansions for infinite families of Pisot and Salem numbers*, J. Number Theory **128** (2008), no. 9, 2756–2765. MR MR2444222

[45] Kevin G. Hare and Soroosh Yazdani, *Fekete-like polynomials*, J. Number Theory **130** (2010), no. 10, 2198–2213. MR 2660887

[46] D. R. Heath-Brown, *The divisor function at consecutive integers*, Mathematika **31** (1984), no. 1, 141–149. MR MR762186 (86c:11071)

[47] Douglas E. Iannucci and Alexia S. Mintos, *On consecutive integer pairs with the same sum of distinct prime divisors*, Integers **5** (2005), no. 1, A12, 9 pp. (electronic). MR MR2192230 (2006h:11005)

[48] Shunji Ito and Yuki Sano, *On periodic β-expansions of Pisot numbers and Rauzy fractals*, Osaka J. Math. **38** (2001), no. 2, 349–368. MR 1 833 625

[49] Børge Jessen and Aurel Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. **38** (1935), no. 1, 48–88. MR MR1501802

[50] V. Komornik, P. Loreti, and M. Pedicini, *An approximation property of Pisot numbers*, J. Number Theory **80** (2000), no. 2, 218–237. MR 2000k:11116

[51] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. (2) **34** (1933), no. 3, 461–479. MR MR1503118

[52] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. **261** (1982), no. 4, 515–534. MR 84a:12002

[53] Christian Mauduit and Joël Rivat, *La somme des chiffres des carrés*, Acta Math. **203** (2009), no. 1, 107–148. MR MR2545827

[54] ______, *Sur un probl eme de gelfond: la somme des chiffres des nombres premiers*, Ann. of Math. (2) (to appear).

[55] J. F. McKee, P. Rowlinson, and C. J. Smyth, *Salem numbers and Pisot numbers from stars*, Number theory in progress, Vol. 1 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 309–319. MR MR1689512 (2000d:11127)

[56] James McKee and Chris Smyth, *Salem numbers, Pisot numbers, Mahler measure, and graphs*, Experiment. Math. **14** (2005), no. 2, 211–229. MR MR2169524 (2006d:11127)

[57] Giuseppe Melfi, *On simultaneous binary expansions of n and n^2*, J. Number Theory **111** (2005), no. 2, 248–256. MR MR2130110 (2006g:11016)

[58] R. V. Moody, *Model sets: a survey*, From Quasicrystals in More Complex Systems (2000), 145–166, ed. F. Axel, F. Dénoyer and J. P. Gazeau.

[59] Michael J. Mossinghoff, *Polynomials with small Mahler measure*, Math. Comp. **67** (1998), no. 224, 1697–1705, S11–S14. MR 99a:11119

[60] A. M. Odlyzko and B. Poonen, *Zeros of polynomials with $0, 1$ coefficients*, Enseign. Math. (2) **39** (1993), no. 3-4, 317–348. MR 95b:11026

[61] Maysum Panju, *Beta expansions for regular Pisot numbers*, J. Integer Seq. **14** (2011), no. 6, Article 11.6.4, 22. MR 2811220

[62] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416. MR MR0142719 (26 #288)

[63] Yuval Peres, Wilhelm Schlag, and Boris Solomyak, *Sixty years of Bernoulli convolutions*, Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), Progr. Probab., vol. 46, Birkhäuser, Basel, 2000, pp. 39–65. MR 2001m:42020

[64] Charles Pisot, *La répartition modulo 1 et les nombres algébriques*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **7** (1938), no. 3-4, 205–248. MR MR1556807

[65] Igor E. Pritsker, *Chebyshev polynomials with integer coefficients*, Analytic and geometric inequalities and applications, Math. Appl., vol. 478, Kluwer Acad. Publ., Dordrecht, 1999, pp. 335–348. MR 2001h:30007

[66] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar **8** (1957), 477–493. MR MR0097374 (20 #3843)

[67] R. Salem, *Power series with integral coefficients*, Duke Math. J. **12** (1945), 153–172. MR 6,206b

[68] J. C. Saunders, *Sums of digits in q-ary expansions*, Int. J. Number Theory **11** (2015), no. 2, 593–611. MR 3325437

[69] J.-C. Schlage-Puchta, *The equation $\omega(n) = \omega(n+1)$*, Mathematika **50** (2003), no. 1-2, 99–101 (2005). MR MR2136354 (2005k:11198)

[70] D Shechtman, I Blech, D Gratias, and Cahn J.W., *Metallic phase with long-range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), no. 20, 1951–1953.

[71] Boris Solomyak, *Notes on Bernoulli convolutions*, Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 2004, pp. 207–230. MR MR2112107 (2005i:26026)

[72] Kenneth B. Stolarsky, *The binary digits of a power*, Proc. Amer. Math. Soc. **71** (1978), no. 1, 1–5. MR MR495823 (80b:10020)

[73] Axel Thue, *Uber eine Eigenshaft, die keine transcendenteGrosse haben kann*, Videnskapsselskapets Skrifter. I Mat.-naturv. Klasse, Kristiania (1912), no. 20.