Control Lyapunov Functions and Periodic Orbits
based on the works of Ames, Galloway, Sreenath, and Grizzle

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November 23, 2016
Control System

\[
\begin{align*}
\dot{x} &= f(x, z) + g(x, z)u \\
\dot{z} &= q(x, z)
\end{align*}
\]  

(1)

$x \in X$ – controlled output states

$z \in Z$ – uncontrolled states

$u \in U$ – admissible control values

$f, g, q$ are locally Lipschitz continuous.

$f(0, z) = 0 \implies x = 0$ defines invariant surface $Z$.

$Z$ is called the invariant zero dynamics manifold.
ES-CLF

Definition

For the system (1), a continuously differentiable function $V : X \to \mathbb{R}$ is an exponentially stabilizing control Lyapunov function (ES-CLF) if $\exists c_1, c_2, c_3 > 0$ such that

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

$$\inf_{u \in U} [L_f V(x, z) + L_g V(x, z)u + c_3 V(x)] \leq 0$$

$\forall (x, z) \in X \times Z$, where $L_f V(x, z)$ is the lie derivative of $V(x)$ along $f(x, z)$:

$$L_f V(x, z) = \frac{dV(x)}{dx} \cdot f(x, z)$$
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For $u \in K = \{ u \in U : L_f V(x, z) + L_g V(x, z)u + c_3 V(x) \leq 0 \}$:

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \|x(0)\| e^{-\frac{c_3}{2}t} \xrightarrow{t \rightarrow \infty} 0$$
Periodic Orbits

The flow $\phi_t$ is periodic with period $T > 0$ and fixed point $(x^*, z^*)$ if $\phi_T(x^*, z^*) = (x^*, z^*)$.

The associated periodic orbit is given by $O = \{\phi_t(x^*, z^*) \in X \times Z : 0 \leq t \leq T\}$. 

By invariance of $Z$, a periodic orbit $O_{\mathcal{Z}}$ in the zero dynamics corresponds to a periodic orbit $O$ in the full dynamics through the canonical embedding $i_0(z) = (0, z)$. i.e., $O = i_0(O_{\mathcal{Z}})$ is a periodic orbit in the full dynamics.

Question: If $O_{\mathcal{Z}}$ is an exponentially stable periodic orbit in the zero dynamics, can we say that $O$ is exponentially stable in the full dynamics?
The flow $\phi_t$ is periodic with period $T > 0$ and fixed point $(x^*, z^*)$ if $\phi_T(x^*, z^*) = (x^*, z^*)$.

The associated periodic orbit is given by $\mathcal{O} = \{\phi_t(x^*, z^*) \in X \times Z : 0 \leq t \leq T\}$.

By invariance of $Z$, a periodic orbit in $\mathcal{O}_Z$ in the zero dynamics corresponds to a periodic orbit $\mathcal{O}$ in the full dynamics through the canonical embedding $i_0(z) = (0, z)$.

i.e., $\mathcal{O} = i_0(\mathcal{O}_Z)$ is a periodic orbit in the full dynamics.
Periodic Orbits

The flow $\phi_t$ is periodic with period $T > 0$ and fixed point $(x^*, z^*)$ if $\phi_T(x^*, z^*) = (x^*, z^*)$. The associated periodic orbit is given by $O = \{\phi_t(x^*, z^*) \in X \times Z : 0 \leq t \leq T\}$.

By invariance of $Z$, a periodic orbit in $O_Z$ in the zero dynamics corresponds to a periodic orbit $O$ in the full dynamics through the canonical embedding $i_0(z) = (0, z)$. i.e., $O = i_0(O_Z)$ is a periodic orbit in the full dynamics.

**Question:** If $O_Z$ is an exponentially stable periodic orbit in the zero dynamics, can we say that $O$ is exponentially stable in the full dynamics?
Exponentially Stable Periodic Orbits

\[
\begin{aligned}
\dot{x} &= f(x, z) + g(x, z)u(x, z) \\
\dot{z} &= q(x, z)
\end{aligned}
\]  

(1)

Theorem

For system (1), let \( O_Z \) be an exponentially stable periodic orbit for the zero dynamics \( \dot{z} = q(0, z) \). If there exists an ES-CLF \( V(x) \), then for all locally Lipschitz continuous feedbacks \( u(x, z) \in K(x, z) \), \( O = i_0(O_Z) \) is an exponentially stable periodic orbit of (1).
Extension to Hybrid Systems

Some systems are better modelled with hybrid dynamics (continuous & discrete).

\[ \mathcal{H}C = \begin{cases} 
\dot{x} = f(x, z) + g(x, z)u \\
\dot{z} = q(x, z), (x, z) \in D \setminus S \\
\Delta X(x^-, z^-), (x, z) \in S \\
\Delta Z(x^-, z^-), (x, z) \in S 
\end{cases} \]

\( D \subset X \times Z \) is closed
\( S \subset D \) is a submanifold of \( D \), called the \textit{switching surface}.
Figure: Trajectories of $\mathcal{HC}$. The bold trajectory is a hybrid periodic orbit in $Z$. [1]
Switching surface has discrete dynamics → acts as “reset map”

Need a stronger notion of stability to counter this, called: rapidly exponentially stabilizing control Lyapunov function (RES-CLF).

A similar theorem holds:

**Theorem**

If there exists an exponentially stable periodic orbit $O_Z$ of the hybrid zero dynamics, and if there exists a RES-CLF $V_\epsilon$ for the continuous dynamics of $HC$, then $\exists \bar{\epsilon} > 0$ such that $\forall 0 < \epsilon < \bar{\epsilon}$ and for all Lipschitz continuous $u = u_\epsilon(x, z) \in K_\epsilon(x, z)$, $O = i_0(O_Z)$ is an exponentially stable hybrid periodic orbit of $HC$. 
Aaron D. Ames, Kevin Galloway, Koushil Sreenath, and Jessy W. Grizzle.
Rapidly Exponentially Stabilizing Control Lyapunov Functions and Hybrid Zero Dynamics.