Geometric Tracking Control of a Quadrotor UAV on SE(3)

T. Lee, M. Leok, N.H. McClamroch

Presented by Luc Larocque
Department of Applied Mathematics
University of Waterloo

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Outline

Setup

$\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$	inertial reference frame
$\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$	body-fixed frame
$m \in \mathbb{R}$	total mass of quadrotor
$J \in \mathbb{R}^{3\times 3}$	inertia matrix w.r.t. body-fixed frame
$R \in \mathsf{SO}(3)$	rotation matrix (body to inertial frame)
$\Omega \in \mathbb{R}^3$	angular velocity in the body-fixed frame

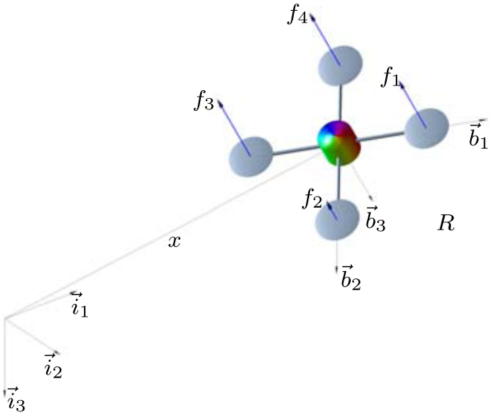

Fig. 1. Quadrotor model

Setup

$x \in \mathbb{R}^3$	location of the center of mass in the inertial frame
$v \in \mathbb{R}^3$	velocity of the center of mass in the inertial frame
$d \in \mathbb{R}$	distance from the CoM to each rotor
$f_i \in \mathbb{R}$	thrust generated by the i-th rotor along $-\vec{b}_3$ axis
$\tau_i \in \mathbb{R}$	torque generated by the i-th rotor about $\vec{b}_3$ axis
$f \in \mathbb{R}$	total thrust, i.e., $f = \sum_{i=1}^{4} f_i$ in $-\vec{b}_3$ direction
$M \in \mathbb{R}^3$	total moment in the body-fixed frame

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$$-fRe_3 \in \mathbb{R}^3$$

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- Write total thrust f and moments M_i as:

$$\begin{bmatrix} f \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -c_{\tau f} & c_{\tau f} & -c_{\tau f} & c_{\tau f} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

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- For $d, c_{\tau f} > 0$, the above matrix is invertible.
  Use f, M_1, M_2, M_3 as control inputs, and solve for each f_i.

Equations of Motion

$$\dot{x} = v \tag{1}$$

$$m\dot{v} = mge_3 - fRe_3 \tag{2}$$

$$\dot{R} = R\hat{\Omega} \tag{3}$$

$$J\dot{\Omega} + \Omega \times J\Omega = M \tag{4}$$

where the *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by the condition that $\hat{x}y = x \times y, \;\; \forall x, y \in \mathbb{R}^3$

Outline

Goal: track prescribed trajectory of the flat output variables

$$x_d(t) \quad \text{and} \quad \vec{b}_{1_d}(t),$$

where $x_d(t) \in \mathbb{R}^3$ is the desired position of the CoM,
and $\vec{b}_{1_d}(t)$ is the desired direction of the first body-fixed axis.

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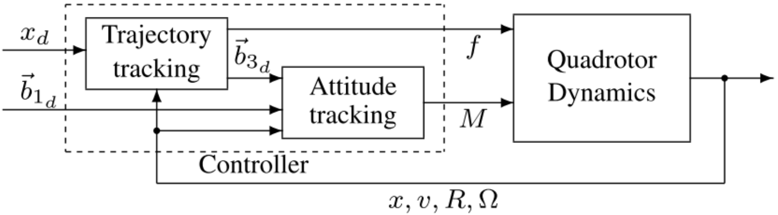

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Define tracking errors:

$$e_x = x - x_d \tag{5}$$

$$e_v = v - v_d \tag{6}$$

$$e_R = \frac{1}{2}\left(R_d^T R - R^T R_d\right)^\vee \tag{7}$$

$$e_\Omega = \Omega - R^T R_d \Omega_d \tag{8}$$

where the *vee map* $(.)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of the hat map.

Given smooth tracking commands $x_d(t)$, $\vec{b}_{1_d}(t)$, and positive constants k_x, k_v, k_R, k_Ω, define the desired direction of the third body-fixed axis:

$$b_{3_d} = \frac{-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d}{\|-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d\|} \tag{9}$$

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We may now write the desired attitude (rotation) matrix:

$$R_d = [\vec{b}_{2_d} \times \vec{b}_{3_d}, \vec{b}_{2_d}, \vec{b}_{3_d}] \tag{11}$$

We choose the control inputs f, M to be:

$$f = -\left(-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d\right) \cdot Re_3 \tag{12}$$

$$\begin{aligned} M = &- k_R e_R - k_\Omega e_\Omega + \Omega \times J\Omega \\ &- J(\hat{\Omega} R^T R_d \Omega_d - R^T R_d \dot{\Omega}_d) \end{aligned} \tag{13}$$

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- exponentially stabilizes the zero equilibrium of the attitude tracking error
- translational tracking error converges to zero provided the attitude tracking error is zero
- $f = c(e_x, e_v, \ddot{x}_d)\vec{b}_{3_d} \cdot \vec{b}_3 \quad \rightarrow |f|$ smaller when $\vec{b}_3 \neq \vec{b}_{3_d}$

Proposition 1 (Exponential Stability of Attitude Dynamics)

Consider the control moment M defined in (13) for any positive constants k_R, k_Ω. Suppose that the initial condition satisfies

$$\Psi(R(0), R_d(0)) < 2 \tag{14}$$

$$\|e_\Omega(0)\|^2 < \frac{2}{\lambda_{max}(J)} k_R(2 - \Psi(R(0), R_d(0))) \tag{15}$$

Then **the zero equilibrium of the attitude tracking error** e_R, e_Ω **is exponentially stable**. Furthermore, there exist constants $\alpha_2, \beta_2 > 0$ such that

$$\Psi(R(t), R_d(t)) \leq \min\{2, \alpha_2 e^{-\beta_2 t}\}. \tag{16}$$

Proposition 2 (Exponential Stability of Complete Dynamics)

Consider control force f and moment M as defined in (12), (13). Suppose the initial condition satisfies $\Psi(R(0), R_d(0)) \leq \psi_1 < 1$.

Then for any positive constants k_x, k_v, we can carefully choose k_R, k_Ω (details skipped) such that **the zero equilibrium of the tracking errors of the complete dynamics is exponentially stable**. The region of attraction is characterized by

$$\Psi(R(0), R_d(0)) \leq \psi_1 < 1 \tag{17}$$

$$\|e_\Omega(0)\|^2 < \frac{2}{\lambda_{max}(J)} k_R(1 - \Psi(R(0), R_d(0))) \tag{18}$$

Proposition 3 (Almost Global Exponential Attractiveness of Complete Dynamics)

Consider a control system designed according to Proposition 2. Suppose the initial condition satisfies

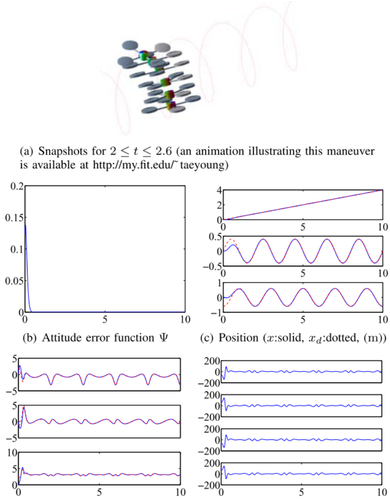
$$\Psi(R(0), R_d(0)) \leq \psi_1 < 2 \tag{19}$$

$$\|e_\Omega(0)\|^2 < \frac{2}{\lambda_{max}(J)} k_R(2 - \Psi(R(0), R_d(0))) \tag{20}$$

Then the zero equilibrium of the tracking errors of the complete dynamics is exponentially attractive.

Outline

1. Quadrotor Dynamics Model
2. Geometric Tracking Control
   - Tracking Errors
   - Tracking Controller
   - Propositions
3. Simulations



(a) Snapshots for $2 \leq t \leq 2.6$ (an animation illustrating this maneuver is available at http://my.fit.edu/˜taeyoung)

(b) Attitude error function Ψ

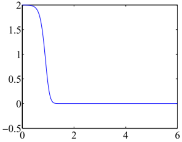
(c) Position (x:solid, x_d:dotted, (m))

(d) Angular velocity (Ω:solid, Ω_d:dotted, (rad/sec))

(e) Thrust of each rotor (N)

Fig. 4. Case I: following an elliptic helix (horizontal axes represent simulation time in seconds)

(a) Snapshots for $0.5 \leq t \leq 4$ (Snapshots are shifted forward to represent the evolution of time. In reality, the quadrotor is flipped at a fixed position. An animation is available at http://my.fit.edu/~taeyoung)



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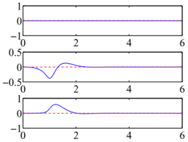
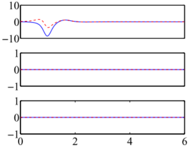
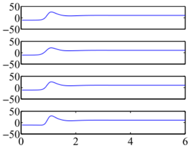


(c) Position (x:solid, x_d:dotted, (m))



(d) Angular velocity (Ω:solid, Ω_d:dotted, (rad/sec))



(e) Thrust of each rotor (N)

Fig. 5. Case II: recovering from an initially upside down attitude (horizontal axes represent simulation time in seconds)