An Intersection between Drawings and Graph Theory

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Abstract

Given a graph G, we can draw this graph on the plane in many different ways. A very common question to ask when it comes to drawings of graphs, is if the graph is planar. In other words, can we draw a graph where no two edges cross. A simple question, not so easily answered. So we ask an even more difficult question by generalizing. Given G, what is the minimum amount of crossings necessary to draw G on the plane. This is the Crossing Number $CR(G)$. In this paper, we discuss definitions, results, conjectures, and some applications of this crossing number. This paper was heavily inspired by the topics discussed in the wonderful survey by Székely [24]. We omit some things mentioned in that survey, but provide more proofs and dive deeper into the things we discuss.

1 Introduction and Definitions

A graph G is a set of vertices $V(G)$, and a set of edges $E(G)$ which are pairs of vertices in G. A graph can be represented pictorially by representing the vertices as dots or circles, and edges as curves connecting a pair of vertices. A graph G is *planar* if there exists a drawing of G where no edges cross. See Figure 1.

To define specifically what a crossing is, we need to disallow some types of drawings. Székely [24] provides a great description of the definition, which we outline here.

For a drawing of a graph G, every two edges must have finitely many points of intersection, and if edges intersect, the edges must not touch tangentially. Lastly, we do not consider drawings with at least three edges that share the same intersection point. If we had such a drawing, we could manipulate the edges within a small ball centered around the intersection point, so that we remove this intersection point, however all the edges will still cross each other edge once.

With this in mind, a crossing in a drawing of a graph G is an intersection point of two edges. We define the *crossing number* $CR(G)$ as the minimum amount of crossings in a drawing of G. A drawing is *optimal* if the drawing has $CR(G)$ crossings.

Of course a drawing can be very "bad"; we can create a drawing of any graph that has an arbitrarily large amount of crossings, and edges are allowed to intersect any other edges as many times as we would like, as long as it is finite. It turns out however, we can limit the amount of drawings we need to look at. An optimal drawing of G satisfies:

- any two edges leaving the same vertex do not cross
- any two edges do not cross more than once

There are much fewer of these "nice" drawings, and it is much easier to work with drawings with these restrictions.

We will outline the proof of the second statement, as the first applies the exact same method. Consider a drawing where two edges cross at least twice. Let these edges be defined by curves p and q, and let two of the intersection points be u and v. Then we can dissect p into segments containing only u, both u and v, and only v. Call these p_1, p_2, and p_3 respectively. We can dissect q in the same way. Now we define new curves:

$$p' = p_1 \cup q_2 \cup p_3 \qquad \text{and} \qquad q' = q_1 \cup p_2 \cup q_3$$

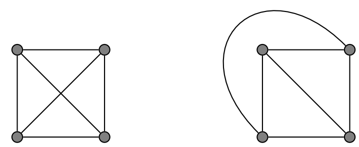

Figure 1: The complete graph on 4 vertices K_4 drawn in two different ways. The left contains one crossing, and the right is a planar drawing.

Replacing p and q with p' and q' respectively, leaves us with a drawing of G except with these two edges touching tangentially. We can then remove this tangential intersection, leaving us with a drawing that has less crossings then the original. Therefore, the original drawing cannot be optimal.

Definitions now cleared up, we can now see that if G is planar, then $CR(G) = 0$ as there is a drawing with no crossings. In this sense, the crossing number can be considered as a measure of how far a graph is from being planar. This and its applications makes it an interesting property to study.

2 Preliminary Results on Crossing Numbers

2.1 Euler's Formula and the Crossing Lemma

A classical result on planar graphs is Euler's Formula, which tells us the following: If G is planar and connected then,

$$|V(G)| - |E(G)| + \text{Reg}(G) = 2$$

where $\text{Reg}(G)$ is the number of regions in the graph. (A region being the area enclosed by a border of edges of G) Every edge of G is adjacent to at most two regions, and a region has at least three edges in its border. So $2|E(G)| \geq 3\text{Reg}(G)$, and so, plugging this into Euler's formula, we get a useful inequality, $|E(G)| \leq 3|V(G)| - 6$.

Now we can get our first bound on $CR(G)$. Let G be a graph, it has $CR(G)$ crossings, so if we remove an edge for each crossing, we can get a planar graph, and then we can apply the above inequality, giving us $CR(G) \geq |E(G)| - 3|V(G)| + 6$

Next we move onto the Crossing Lemma, this theorem relies on probability and random graphs.

Theorem 2.1: Let G be a graph with $4|V(G)| \leq |E(G)|$, then

$$CR(G) \geq \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2}$$

Proof: Let H be a subgraph of G where each vertex of G remains in H with probability p. Now we try to upper and lower bound the expected value of $CR(H)$. Using the inequality above we have,

$$\mathbb{E}[CR(H)] \geq \mathbb{E}[|E(H)|] - 3\mathbb{E}[|V(H)|]$$

A vertex survives with probability p, so $\mathbb{E}[|V(H)|] = p|V(G)|$. An edge survives if both ends survive, so $\mathbb{E}[|E(H)|] = p^2|E(G)|$, and a crossing survives if the ends of two edges survives, so $\mathbb{E}[CR(H)] \leq p^4 CR(G)$. Combining and simplifying we get:

$$CR(G) \geq \frac{|E(G)|}{p^2} - \frac{3|V(G)|}{p^3}$$

Maximizing p, $p = \frac{4|V(G)|}{|E(G)|} \leq 1$ by the assumption. Plugging in this p gives us the desired result. $\square$

2.2 Kuratowski's Theorem and different Crossing Numbers

Kuratowski's Theorem is a very important theorem that completely characterizes planar graphs. It tells us that a graph G is planar if and only if G does not contain a subdivision of K_5 (the complete

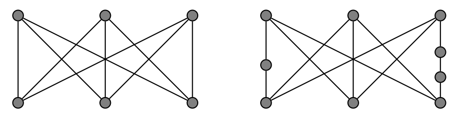

Figure 2: $K_{3,3}$ on the left, and a subdivision of it on the right.

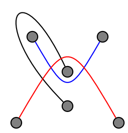

Figure 3: If we apply the "re-stitching" argument on the red an blue edges, the resulting graph would have less regular crossings, but the same amount of pairwise crossings, meaning it might still be an optimal drawing for the pairwise crossing number.

graph on 5 vertices) or $K_{3,3}$ (the complete bipartite graph on 3 and 3 vertices). You can think of a subdivision of G as a graph where we replace the edges of G with paths of varying lengths. See Figure 2. These graphs are minimally non-planar, as if you remove any vertex or edge from the graph it becomes planar. Therefore, it is easy to see that $CR(K_5) = 1$ and $CR(K_{3,3}) = 1$.

One could ask the question, what if we relax the definition of a crossing or restrict the drawings? What would happen to the crossing number? Well, there are four other crossing numbers that we will introduce here:

- CR-PAIR(G): The *pairwise crossing number* only counts pairs of edges that cross, not the points of intersection. So if two edges cross more than once, it still only counts as one crossing.
- CR-ODD(G): The *odd crossing number* only counts pairs of edges that cross an odd number of times.
- CR-IODD(G): The *independent odd crossing number* only counts pairs of edges that cross an odd number of times, and the edges must not share an end.
- CR-RECT(G): The *rectilinear crossing number* considers crossings as in the original definition, but forces the drawings considered to have the curves representing edges be straight lines.

Looking at the definitions above, one might think that the first three definitions are pointless, as we proved in Section 1, that a drawing with edges that cross multiple times, or edges that share an end that cross are not optimal. It turns out that the proof we used does not hold when considering the other types of crossing numbers. See Figure 3 for a counter example.

Clearly, we have the following:

$$\text{CR-IODD}(G) \leq \text{CR-ODD}(G) \leq \text{CR-PAIR}(G) \leq \text{CR}(G) \leq \text{CR-RECT}(G)$$

Bienstock and Dean [2] constructed graphs with $\text{CR}(G) = 4$, with arbitrarily large CR-RECT(G), so we can see that some graphs will always need curved edges in order to have minimal crossings. However the interesting thing is that there still is no single counterexample to show that $\text{CR-IODD}(G) < \text{CR}(G)$. It has even been proved by Chojnacki [3] that $\text{CR-IODD}(K_5) = 1$ and $\text{CR-IODD}(K_{3,3}) = 1$. Despite all restrictions that the other crossing numbers remove, it seems like an optimal drawing will always be "nice" as described in Section 1, but no proof of this has been shown.

2.3 Crossing Numbers of $CR(K_t)$

Along with paths and cycles, one of the most common families of graphs to examine are the complete graphs K_n. Despite this, the crossing number of these graphs is not exactly known. However we can say somethings about the crossing number.

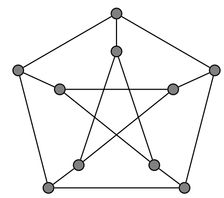

Figure 4: The Peterson graph, in terms of its generalization, $G = G(5,2)$

Theorem 3.1 (Lemma 1 in [15]):

$$CR(K_n) \geq \frac{1}{120} n(n-1)(n-2)(n-3)$$

Proof: We prove this using a recursive relation based on $|V(G)| = n$ for $n \geq 5$.

Base Case: $CR(K_5) = 1$ by Kuratowski, and $\frac{1}{120} n(n-1)(n-2)(n-3) = 1$ when $n = 5$.

Recursive Step: Let $n \geq 2$ and K_n be drawn in the plane optimally, i.e. with $CR(K_n)$ crossings. If we delete a vertex from the graph, we get a drawing of a K_{n-1} graph. Also for a specific crossing of K_n, this crossing remains if we do not remove one of the four vertices belonging to one of the edges creating the crossing. Counting all the crossings in all possible removals of a vertex, we can see that:

$$(n-4) CR(K_n) = \sum_{v \in V(G)} |\{\text{Crossings in } K_n \setminus v\}| \geq n CR(K_{n-1})$$

Now we have the recursive relation $CR(K_n) \geq \frac{n}{n-4} CR(K_{n-1})$, which we can apply until we reach $CR(K_5)$, which is 1 by the base case. And so, by canceling like terms in the fraction we get the desired result. □

As the most amount of crossings can technically happen only when every edge crosses each other edge, we now have that $CR(K_n) = \Theta(n^4)$. The current conjecture for the exact number by Guy [11] is $CR(K_n) = \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. This conjecture has been verified for $n \leq 10$ by Guy, and $n = 11, 12$ by Pan and Richter [19]. There is a drawing that provides this crossing number; split the n vertices evenly and place them on the two circular rims of a cylinder. Connect all the vertices on the top disc with straight lines, similarly with the bottom disc. Now we connect the remaining vertices together by traveling along the length of the cylinder. Using stereographic projection, we can then move this drawing to the plane.

3 Crossing Numbers of Families of Graphs

While we do not know the crossing numbers of the complete graphs, there are still some families of graphs for which we know exactly what the crossing numbers are.

3.1 Peterson Graphs

The *generalized Peterson graph* $G = G(n,k)$ has $V(G) = \{u_i, v_i \,|\, 1 \leq i \leq n\}$ and $E(G) = \{u_i v_i,\, u_i u_{i+1},\, v_i v_{i+k} \;|\, 1 \leq i \leq n\}$. Addition is done modular n. See Figure 4 for an example.

Exoo et al. [8] showed that $CR(G(n,2)) = 0$ for $n = 3$ or n even, $CR(G(n,2)) = 2$ for $n = 5$ and lastly, $CR(G(n,2)) = 3$ for $n \geq 7$ and odd. This result is attained by noticing that for $n \geq 7$, $G(7,2)$ is a minor of $G(n,2)$, so it boils down to just checking the crossing number of $G(7,2)$. More results from Fiorini [9] show that $G(9,3) = 2$, $CR(3k,3) = k$ for $k \geq 4$, $CR(3k+2,3) = k+2$ and $CR(G(4k,4)) = 2k$.

3.2 Product of Graphs

The Cartesian Product of two graphs G and H is the graph defined by $V(G \times H) = \{(u,v) \,|\, u \in V(G),\, v \in V(H)\}$ and $E(G \times H) = \{(u_1,v_1)(u_2,v_2) \,|\, u_1 u_2 \in E(G) \text{ or } v_1 v_2 \in E(H)\}$. To quickly imagine what $G \times H$ would look like, you can imagine replacing each vertex of in a drawing of G with an entire copy of the graph H. Then connect vertices of two different H subgraphs if G had a edge between the two vertices that the H subgraphs replaced.

The most commonly looked at graphs for products are S_n, C_n, and P_n, which represent a star, cycle and path on n vertices respectively. For G with $|V(G)| = 4$, Beineke and Ringeisen [1] determined $CR(G \times C_n)$, while Klešč [13, 14] determined $CR(G \times P_n)$ and $CR(G \times S_n)$, as well as $CR(G \times P_n)$ for $|V(G)| = 5$.

Lastly, there is a conjecture of Harary et al. [12] for the crossing number on the product of cycles:

$$CR(C_n \times C_m) = n(m-2) \qquad \text{for } n \geq m \geq 3$$

4 Probabilistic Results

The crossing number of random graphs and the behaviour of it as a function of $|V(G)|$ is also something that is extensively studied. As an example, here we display a proof of a result from Spencer and Tóth [21].

Theorem 4.1: Let $c > 1$ be any real number and $G = G(n, p)$ be a random graph, with $p = \frac{c}{n}$, then

$$\liminf_{n \to \infty} \frac{\mathbb{E}[CR(G)]}{(\binom{n}{2}p)^2} > 0$$

In other words, we want to show that $\mathbb{E}[CR(G)] > \delta n^2$ for some δ dependant on c. The proof will proceed in the following way: given a random graph with certain probability p, we will view it as the union of two random graphs with smaller probability. Then one of these graphs will have a large enough connected component almost surely, and then we will show that this must mean G must have a large bisection width, and therefore a large crossing number. The definition of this "bisection width" follows, as well as some results that will do most of the work for this proof.

The *bisection width* of a graph $b(G)$ is the minimal number of edges that need to be removed from G, so that the remaining vertices can be split into two groups, T and B s.t. $\frac{2}{3}|V(G)| \geq |T|$, $\frac{1}{3}|V(G)| \leq |B|$, and there is no edge going from a vertex in T to a vertex in B. The following results (given without proof) are what allow us to bridge the gap between crossing number and bisection width.

Lemma 4.2 (Pach, Shahrokhi, and Szegedy [18]):

$$b(G) \leq 10\sqrt{CR(G)} + 2\sqrt{\sum_{v \in V(G)} \deg(v)}$$

Theorem 4.3 (Spencer and Tóth [21]): Let $c > 1$ be any real number and let $G = G(n, p)$ be a random graph with $p = \frac{c}{n}$. Let T be a tree defined on $V(G)$ then,

$$b(T \cup G) = \Omega(n)$$

In other words, there exists δ dependant on c s.t. $\mathbb{P}(b(T \cup G) \leq \delta n)$ goes to zero as n goes to infinity.

Theorem 4.4 (Erdös and Rényi [6]): Let $G = G(n, p)$ be a random graph with $p = \frac{1+\epsilon}{n}$ with $\epsilon > 0$, then there exists, almost surely, a "giant" connected component of G with size $\sim kn$, $k = k(\epsilon)$. All other components have size $O(ln(n))$ and $\lim_{\epsilon \to 0} k = 0$

Proof of 4.1: Let $c = 1 + \epsilon$ for some $\epsilon > 0$, take $c_1 = \frac{c+1}{2}$ and $a = \frac{c-1}{2}$. Next take $p = \frac{c}{n}, p_1 = \frac{c_1}{n}$ and set p_2 such that $p = p_1 + p_2 - p_1 p_2$ holds. Then $p_2 \sim \frac{a}{n}$. Now given $G = G(n, p)$ we can view it as the union of two random graphs $G_1 = G(n, p_1)$ and $G_2 = G(n, p_2)$ thanks to the relationship between p, p_1 and p_2.

Now since $p_1 = \frac{1+\frac{\epsilon}{2}}{n} > \frac{1}{n}$ we can apply 4.3 to G_1 and find a giant component on $X \subset V(G)$, with $|X| \sim kn$, for a constant k. As $G_1|_X$ is connected, we can remove edges until we are left with a tree

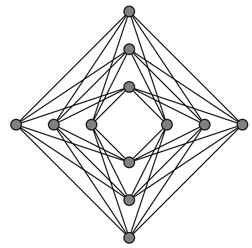

Figure 5: The conjectured optimal drawing for $K_{6,6}$

T. We can now apply 4.4, giving us that $b(T \cup (G_2|_X)) \geq \delta(kn)$, and so $b(T \cup G_2|_X) = \Omega(n)$. If we add edges to a graph, its bisection width can only increase, this gives us:

$$b(T \cup (G_2|_X)) \leq b((G_1 \cup G_2)|_X) = b(G|_X) \leq b(G)$$

So $b(G) = \Omega(n)$. Finally, applying Lemma 4.2 we can see that $CR(G)$ is bounded below by the square of $b(G)$ and so $CR(G) = \Omega(n^2)$ as desired. □

5 Applications

Paul Turán [25] was one of the first people to introduce the concept of crossing numbers, however its origin was not theoretical in nature. This concept arose while working in a forced labour camp during World War 2. They were tasked making bricks, and placing them on trucks, which would then deliver them to warehouses elsewhere. However, sometimes when a truck would go through a crossing, bricks would fall off, causing problems for everyone.

So naturally, Turán then wondered, what would be the minimal amount of crossings given n kilns, and m warehouse that bricks must be delivered to. This is known as Turán's Brick Factory problem and in the notation we have been using, the question is, what is $CR(K_{n,m})$?

Like K_n, the crossing number is not known for the complete bipartite graph, however Zarankiewicz's conjecture states,

$$CR(K_{n,m}) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$$

There is a drawing with this amount of crossings. Consider a drawing on $\mathbb{R}^2$ where we place the n vertices along the y-axis, on the integers $\lfloor \frac{n}{2} \rfloor$ to $\lceil \frac{n}{2} \rceil$, and we do the same for the m vertices, this time along the x-axis. Then connect all the all vertices on the y-axis with those on the x-axis. See Figure 5 for an example.

This story was just a way of introducing how the crossing number can be useful in many practical applications. Another classic application is in the creation of circuits in electronics. Finding configurations with minimal crossings could help minimize area, or possibly increase the number of transistors on a chip.

However with respect to mathematical application, a very important proof of the Szemeredi-Trotter Theorem comes from the crossing number.

5.1 Szemeredi-Trotter Theorem

Let $I(n, m)$ be the maximum number of intersections between a set of points P and a set of lines L in $\mathbb{R}^2$, such that $|P| = n$ and $|L| = m$. The Szemeredi-Trotter theorem is an upper bound on this number, and its original proof was quite difficult [23]. The new proof by Székely was a very slick proof, and a nice application of the crossing number.

Theorem 5.1 (Székely [22]):

$$I(n, m) \leq 4n^{\frac{2}{3}}m^{\frac{2}{3}} + 4n + m$$

Proof: We will define a graph G using the point and line sets P and L. Let $V(G) = P$, and we connect two vertices if they are sequential intersections on the same line. In other words, if $x_1, x_2, \ldots, x_m \in P$

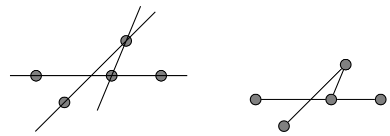

Figure 6: On the left, an example of a set of points P and lines L intersecting. On the right, the graph that would be constructed from the left in the proof of Theorem 5.1.

are points of intersection with the line $l \in L$ ordered in the direction parallel to l, then $x_i x_{i+1} \in E(G)$ $\forall i \in [m-1]$. See Figure 6 for an example of this construction.

We have that $|V(G)| = n$ and $|E(G)| = I(n,m) - m$ as having k points on a line will only create $k-1$ edges in G. If $|E(G)| < 4|V(G)|$, then we get that $I(n,m) - m < 4n \implies I(n,m) < 4n + m$. On the other hand, if $|E(G)| \geq 4|V(G)|$, we can now apply the Theorem 2.1, and the fact that $CR(G) \leq m^2$ to get:

$$\frac{1}{64}\frac{(I(n,m)-m)^3}{n^2} \leq CR(G) \leq m^2$$

Moving terms around we get $I(n,m) \leq 4n^{\frac{2}{3}}m^{\frac{2}{3}} + m$. Combining both cases together gives us the desired result. □

6 Miscellaneous

6.1 Graph Minors

A graph H is a *minor* of G if H can be obtained by repeatedly removing edges, vertices or contracting edges. Contracting an edge is essentially combining two adjacent vertices into one, so that this "combined" vertex maintains all the connections that the two ends had. The main question one will ask in Graph Minor theory is, what properties does the graph G have, if it does not have certain minors?

An example of this can be seen from the Kuratowski theorem, K_5 and $K_{3,3}$ are minors of their subdivisions, so then if G doesn't have K_5 or $K_{3,3}$ as a minor, then it must be planar.

One could ask, what minors make a graph have a certain amount of crossings? This is what Robertson and Seymour [20] asked and answered for $CR(G) = 1$. They showed that a graph is singly crossing if it does not contain any out of a list of 41 graphs as a minor.

6.2 Biplanar Crossing Number

A graph G is *biplanar* if there exists two planar subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$. The biplanar crossing number is an attempt to look at crossings in the same vein as biplanar graphs. The *biplanar crossing number* $CR_2(G)$ is the sum of $CR(G_1)$ and $CR(G_2)$ minimized over all G_1 G_2 such that $G = G_1 \cup G_2$. In other words,

$$CR_2(G) := \min_{G = G_1 \cup G_2} (CR(G_1) + CR(G_2))$$

Of course, for biplanar graphs $CR_2(G) = 0$. In general however, the biplanar crossing number is harder to work with than the original crossing number, and the crossing number was not easy to begin with. For example, while subdivision did not change the regular crossing number, a subdivision may change the biplanar crossing number. Informally speaking, a subdivided edge can be "shared" among G_1 and G_2, leaving gaps for other edges, that could have potentially crossed, to pass through. Despite this, we do still have some results to share from Sýkora et al [5, 4].

They showed that $CR_2(G) \leq \frac{3}{8}CR(G)$. Sadly, one cannot get an upper bound in terms of only $CR_2(G)$. There are examples of graphs with large crossing numbers ($\Theta(|E(G)|^2)$) but very small biplanar crossing numbers ($\Theta(\frac{|E(G)|^3}{|V(G)|})$) where $\frac{m}{n} \geq c$ for some constant c. As well, for $n \geq 12$, they showed that:

$$CR_2(K_{5,n}) = \lfloor \frac{n}{12} \rfloor (n - 6\lfloor \frac{n}{12} \rfloor - 6)$$

While for $n < 12$ $K_{5,n} = 0$ as they are biplanar.

6.3 Computational Complexity

Determining $CR(G)$ was found to be NP-Complete, as shown by Garey and Johnson [10]. Pach and Tóth [17], found the same for CR-ODD(G), and CR-PAIR(G) was determined to be NP-Hard. Testing that $CR(G) \leq k$ for a fixed k can be tested in polynomial time. It uses the fact that a graph can be tested to be planar in polynomial time, by adding at most k vertices to G and then testing planarity.

In terms of approximation algorithms, Leighton and Rao [16] shared an algorithm which approximates $|V(G)| + CR(G)$ within a factor of $(\log n)^4$. This was improved by Evan et al. [7] to a factor of $(\log n)^3$.

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