Extremal functions of kC_4 and kK_4^-

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Abstract

For a simple graph H, let the extremal function of H, $c(H)$, be the supremum of $\frac{|E(G)|}{|V(G)|}$, for all G not containing H as a minor. We show that when H is k disjoint copies of K_4^- then $c(H) = 2k - \frac{1}{2}$

1 Introduction

Let $v(G) := |V(G)|$ and $e(G) := |E(G)|$. The following statement provided by Csóka et al.[1] conjectures what the value of the extremal function would be when H is a disjoint union of cycles:

Conjecture 1. *Let H be a 2-regular graph with odd(H) odd components, then*

$$c(H) = \frac{v(H) + odd(H)}{2} - 1,$$

unless $H = C_{2l}$, in which case $c(H) = \frac{2l-1}{2}$, or $H = kC_4$ in which case $c(H) = 2k - \frac{1}{2}$

In the same paper, it is shown that the conjectured extremal function for the case where H is a union of disjoint odd cycles, is indeed true. It is given by the equality of the lower bound, $c(H) \geq \tau(H) - 1$ and the upper bound which was the main result of the paper:

Theorem 2. *Let H be a disjoint union of cycles. Then*

$$c(H) \leq \frac{v(H) + comp(H)}{2} - 1$$

The original intent for this paper was to prove for the case when $H = kC_4$, a disjoint union of k cycles of length 4, that the conjecture is indeed true. However, during the course of our research we discovered a result due to Wang [2] which one can use to derive this result.

Theorem 3. *Let G be a graph which does not contain kC_4 as a minor, then*

$$e(G) \leq (2k - \frac{1}{2})(v(G) - 1)$$

consequently, $c(kC_4) \leq 2k - \frac{1}{2}$

We will provide this derivation, as well as our independent proof, as it also gives us the extremal function for kK_4^- instead, with very little modification. In fact, the inequality describing these graphs was the same.

Theorem 4. *Let G be a graph which does not contain kK_4^- as a minor, then*

$$e(G) \leq (2k - \frac{1}{2})(v(G) - 1)$$

consequently, $c(kK_4^-) \leq 2k - \frac{1}{2}$

We later look at the behaviour of $\text{ex}(n, kC_4)$ when $n \in [4k, 5k-1]$, where $ex(n, H)$ is the maximum amount of edges in a graph with $v(G) = n$ with no H minor. While we have an extreme example that is what gives us the value of the extremal function, these examples are not optimal on graphs with few vertices.

2 Derivation of $c(kC_4)$

Before showing Wang's result, we must first define some sets of graphs. We define sets of graphs Γ_k for $k \geq 3$. $G \in \Gamma_k$ if $V(G) = V(G_1 \cup G_2)$ s.t. $G_1 \cong K_{2k+1}$ and either $G_2 \cong K_{2k+1}$ and there is at most one edge connecting G_1 and G_2, or $G_2 \cong K_{2k}$ and the only other edges are from a single vertex of G_1 which is adjacent to all vertices in G_2.

We now define sets of graphs $\Sigma_{k,n}$ for $n \geq 4k+1$. G is in this set if $v(G) = n$ and $V(G) = A \cup B$ s.t. $|A| = 2k-1$, $G[A]$ is complete, $G[B]$ is a maximal matching, and all edges between A and B are present.

Lastly let F_9 be defined as the following: $V(F_9) = \{a_1, a_2, a_3, a_4\} \cup \{x_1, x_2, x_3, x_4, x_5\}$, $\{x_1a_1, x_1a_2, x_2a_3, x_2a_4, x_4a_2, x_4a_3, x_5a_1, x_5a_4\} \subset E(G)$ and $a_1a_2a_3a_4a_1$, $x_1x_2x_3x_1$ and $x_3x_4x_5x_3$ are cycles in F_9.

Something important to note, is that $\Sigma_{k,n}$ are the graphs used to give the lower bound for the extremal function of kC_4 and kK_4^-. They do not contain kC_4, this is because, any 4-cycle must use at least 2 vertices in A, but $|A| = 2k-1$. $G[A]$ is complete, all edges are present between A and B, and $G[B]$ is a maximal matching, which means it contains $\lfloor\frac{|B|}{2}\rfloor$ edges. This gives us $\binom{2k-1}{2} + (2k-1)(n-2k+1) + \lfloor\frac{n-2k+1}{2}\rfloor$ edges. Dividing by n and taking the limit, gives us our lower bound of $2k - \frac{1}{2}$.

Theorem 5. *Let k and n be two integers with $k \geq 2$ and $n \geq 4k$. If G is a graph of order $n \geq 4k$ and the minimum degree of G is at least $2k$, then G contains k disjoint cycles of length at least 4 if and only if $G \ncong F_9$ or $G \notin \Gamma_k \cup \Sigma_{k,n}$.*

We begin by first stating a lemma which was the guiding force in our results:

Lemma 6. *Let G be a minor-minimal graph such that $e(G) > cv(G) - d$, then every edge in G must be part of at least $\lfloor c \rfloor$ triangles.*

Proof. Let G' be the graph G with one edge, e, contracted. Then $e(G') \leq cv(G') - d$, and so we have,

$$e(G) - e(G') > cv(G) - d - (c(v(G)-1) - d) = c.$$

G lost $> c$ edges when e was contracted, e itself was one of these edges, the remaining edges must have come from neighbors that both ends of e had in common. The ends of e and one of these common neighbors form a triangle, giving us that e must be part of $> (c-1)$ triangles, or, rounding to the nearest integer, $\geq \lfloor c \rfloor$ triangles. □

Proof of Theorem 3. Suppose not, we have some graph G that is minor-minimal such that $e(G) > (2k - \frac{1}{2})v(G) - (2k - \frac{1}{2})$ and it contains no kC_4 minor. First thing to notice is that $v(G) \geq 4k$, since K_{4k-1} does not satisfy the inequality and so no smaller graph can satisfy it. Now we can also apply lemma 6, which tells us that every edge is part of $2k-1$ triangles. This gives us that every vertex has $deg(v) \geq 2k$. Therefore, by Theorem 5, $G \cong F_9$ or $G \in \Gamma_k \cup \Sigma_{k,n}$ as long as $k \geq 2$. The case where $k = 1$ is proved at the end.

Case $G \cong F_9$: This is only possible if $k = 2$, so we need to verify $e(G) > \frac{7}{2}(v(G)-1)$ only. But as F_9 has 9 vertices and only 18 edges, this gives us $18 > 28$, a contradiction.

Case $G \in \Sigma_{k,n}$: In this case, A is complete, all edges are present between A and B, and B is a maximal matching, which means it contains $\lfloor\frac{|B|}{2}\rfloor$ edges.

$$\begin{aligned} \binom{2k-1}{2} + (2k-1)(n-2k+1) + \lfloor\frac{n-2k+1}{2}\rfloor &> (2k - \frac{1}{2})n - (2k - \frac{1}{2}) \\ (2k^2 - 3k + 1) + (2k-1)n - (4k^2 - 4k + 1) + \lfloor\frac{n-2k+1}{2}\rfloor &> (2k - \frac{1}{2})n - (2k - \frac{1}{2}) \\ -2k^2 + 3k - \frac{1}{2} &> \frac{n}{2} - \lfloor\frac{n-2k+1}{2}\rfloor \end{aligned} \tag{1}$$

We get a contradiction as the left side is negative and the right side is positive for $k \geq 2$.

Case $G \in \Gamma_k$: When $v(G_2) = 2k+1$ we have,

$$\begin{aligned} 2\binom{2k+1}{2} + 1 &> (2k - \frac{1}{2})(4k+1) \\ -4k^2 + 2k + \frac{3}{2} &> 0 \end{aligned} \tag{2}$$

or when $v(G_2) = 2k$ we have,

$$\begin{aligned}\binom{2k+1}{2} + \binom{2k}{2} + 2k &> (2k - \frac{1}{2})(4k) \\ -4k^2 + 4k &> 0\end{aligned} \tag{3}$$

Both give us contradictions for $k \geq 3$.

Now what remains is the case when $k = 1$, this is solved with an argument based on connectivity.

G is disconnected: Take one connected component C of G. We have

$$e(G) = e(G \setminus C) + e(C) \leq \frac{3}{2}(v(G \setminus C) + v(C)) - 3 = \frac{3}{2}v(G) - 3 < e(G),$$

a contradiction.

G is connected and not 2-connected: In this case we can find a separation (G_1, G_2) of G of order 1.

$$e(G) = e(G_1) + e(G_2) \leq \frac{3}{2}(v(G_1) + v(G_2)) - 3 = \frac{3}{2}(v(G) + 1) - 3 = \frac{3}{2}v(G) - \frac{3}{2} < e(G),$$

another contradiction.

G is 2-connected: In this case we can use 6, and see that we can find a triangle in G. Now, using 2-connectivity, we can take a vertex outside the triangle and see that there must be two vertex disjoint paths connecting it to two different vertices on the triangle. Contracting these paths gives us a K_4^-, giving us a contradiction.

□

3 Extremal function of $c(kK_4^-)$

Given that the graphs $\Sigma_{k,n}$ give us the lower bound of $2k - \frac{1}{2}$, all that remains is to show the upper bound given by Theorem 4.

Proof of Theorem 4. We proceed by induction on k,

Base Case ($k = 1$): Suppose not, let G be a minor-minimal graph with no K_4^- minor, and with $e(G) > \frac{3}{2}v(G) - \frac{3}{2}$. We split the proof into cases based on connectivity:

G is disconnected: Take one connected component C of G. We have

$$e(G) = e(G \setminus C) + e(C) \leq \frac{3}{2}(v(G \setminus C) + v(C)) - 3 = \frac{3}{2}v(G) - 3 < e(G),$$

a contradiction.

G is connected and not 2-connected: In this case we can find a separation (G_1, G_2) of G of order 1.

$$e(G) = e(G_1) + e(G_2) \leq \frac{3}{2}(v(G_1) + v(G_2)) - 3 = \frac{3}{2}(v(G) + 1) - 3 = \frac{3}{2}v(G) - \frac{3}{2} < e(G),$$

another contradiction.

G is 2-connected: In this case we can use 6, and see that we can find a triangle in G. Now, using 2-connectivity, we can take a vertex outside the triangle and see that there must be two vertex disjoint paths connecting it to two different vertices on the triangle. Contracting these paths gives us a K_4^-, giving us a contradiction.

Inductive step: Suppose not, assume that we have a minor-minimal G such that $e(G) > (2k - \frac{1}{2})v(G) - (2k - \frac{1}{2})$. We can notice the following about G:

- $v(G) \geq 4k$ as K_{4k-1} can not be G, and so anything with fewer vertices also could not be G.
- By 6, we know that every edge is part of at least $2k - 1$ triangles.
- From the last point we can see that $\forall v \in V(G)\ deg(v) \geq 2k$.
- $e(G) \leq (2k - \frac{1}{2})v(G) - (2k - \frac{3}{2})$, otherwise we would be able to remove an edge from G, contradicting minor-minimality.

Let u and v be adjacent vertices, then there are at least two triangles using this edge, and therefore, a K_4^- subgraph. If we were to remove these four vertices, and the resulting graph G' would have $e(G') > (2k - \frac{5}{2})(v(G) - 4) - (2k - \frac{5}{2})$, then it would have a $(k-1)K_4^-$ minor, by the induction hypothesis, disjoint from the K_4^- we deleted. This shows that G has a kK_4^- minor, a contradiction. So if we delete any K_4^- from G, there is $\leq (2k - \frac{5}{2})(v(G) - 4) - (2k - \frac{5}{2})$ edges remaining.

Let S denote the vertices of a K_4^- in G, then we have the following:

$$\begin{aligned}\sum_{v \in V(G)} deg(v) &= 2e(G[V(G) \setminus S]) + 2\sum_{v \in S} deg(v) - 2e(G[S]) \\ &\leq (4k-5)(v(G) - 4) - (4k-5) + 2\sum_{v \in S} deg(v) - 10 \\ &= (4k-5)v(G) - 20k + 15 + 2\sum_{v \in S} deg(v)\end{aligned}$$

On the other hand, we have $2e(G) > (4k-1)v(G) - (4k-1)$, or $2e(G) \geq (4k-1)v(G) - (4k-1) + 1$. Combining and simplifying these we get the following inequality:

$$\begin{aligned}(4k-1)v(G) - 4k + 2 &\leq (4k-5)v(G) - 20k + 15 + 2\sum_{v \in S} deg(v) \\ 2v(G) + 8k - \frac{13}{2} &\leq \sum_{v \in S} deg(v)\end{aligned} \tag{4}$$

Now let $u \in V(G)$ be a vertex with minimum degree d, then $2k \leq d \leq 4k - 2$. Otherwise, if $d \geq 4k - 1$ then,

$$4k - 1 \leq \frac{1}{v(G)} \sum_{v \in V(G)} deg(v) \leq \frac{2e(G)}{v(G)} \leq (4k-1) - \frac{4k-3}{v(G)} < 4k - 1$$

a contradiction. We consider $N(u)$, the neighborhood of u. For every $v \in N(u)$, there must be $2k - 1$ triangles using the edge uv. The end of these triangles must also be in $N(u)$, so this gives us that $deg_{G[N(u)]}(v) \geq 2k - 1$ for every $v \in N(u)$. As $|N(u)| \leq 4k - 2$, by Dirac's theorem, $G[N(u)]$ contains a Hamiltonian cycle.

We would like to combine equation 4 with a matching of size k which we are guaranteed from the Hamiltonian cycle, and show that the degree of the vertices in the matching are large enough to greedily construct a kK_4^- minor. We will split this into two cases.

Case 1: For every edge in the matching, one of the ends of the $2k - 1$ triangles using this edge (other than u) has degree $< v(G) - 1$.

Let v and w be the vertices of one of the edges in the matching, and let x be the vertex with $deg(x) \leq v(G) - 2$ which forms a triangle with vw. We have that (u, v, w, x) form a K_4^- subgraph. Using equation 4,

$$\begin{aligned}deg(v) + deg(w) &\geq 2v(G) + 8k - \frac{13}{2} - deg(u) - deg(x) \\ &\geq 2v(G) + 8k - \frac{13}{2} - (4k-2) - (v(G) - 2) = v(G) + 4k - \frac{5}{2} \\ &= (v(G) - 2) + 2 + (4k - \frac{5}{2})\end{aligned} \tag{5}$$

This tells us that the ends of each edge in a matching, have at least $4k - \frac{5}{2}$ shared neighbors. We can round this to see that they in fact have $\geq 4k - 2$ shared neighbors.

We now construct the kK_4^- greedily, for every edge in the matching, we pick two shared neighbors of the ends of this edge. This gives us a K_4^-. For the k^{th} edge, there are $\geq (4k-2) - 4(k-1) = 2$ vertices left to choose. We have successfully constructed a kK_4^- minor in G. Contradiction.

Case 2: There is at least one edge in the matching where all of the ends of the $2k - 1$ triangles using this edge (other than u) have $deg(x) = v(G) - 1$.

Let v and w be the vertices of one of the edges in the matching, let X be the set of $2k - 2$ vertices with $deg(x) = v(G) - 1$ which form triangles with vw. Lastly, let $x \in X$.

Repeating the same arguments used in equation 5, except this time we have $deg(x) = v(G) - 1$, we get that v and w share $\geq 4k - 3$ neighbors. We greedily pick two of these shared neighbors, excluding any of the vertices in X. This gives us our first K_4^-, the remaining are gotten by taking a complete matching on the vertices of X and for each edge, greedily picking two shared neighbors. This is possible as, $(v(G) - 2) - 4(k - 1) \geq 2$. We have obtained a kK_4^- minor, contradiction.

□

This gives our upper bound, and so $c(kK_4^-) = 2k - \frac{1}{2}$

4 Exact edge inequality for kC_4 on exactly $4k$ vertices

Consider the following graph: we begin with a K_{4k-1} and for every two additional vertices needed, we form a triangle between an existing vertex and the two we wish to add. If there is only one remaining vertex, we connect it to exactly one other vertex. This graph does not have kC_4 minor, it only contains a $(k-1)C_4$ within the K_{4k-1}, and the rest of the graph consists of triangles, which at most share one vertex, so there cannot be a C_4 within them.

This graph has $e(G) = \binom{4k-1}{2} + \frac{3(v(G)-(4k-1))}{2}$ if $v(G)$ is odd, and $e(G) = \binom{4k-1}{2} + \frac{3(v(G)-4k)}{2} + 1$ if $v(G)$ is even. This example contains more edges than the vertex cover extreme example, when $v(G) \in [4k, 5k-1]$. The following proof shows that this is optimal for $v(G) = 4k$.

Theorem 7. *Let G have $v(G) = 4k$ and no kC_4 minor, then $e(G) \leq \binom{4k-1}{2} + 1 = 8k^2 - 6k + 2$*

Proof. Base Case ($k = 1$): Suppose not, then there exists a graph G on 4 vertices, with 5 edges and no C_4. As this graph is one edge removed from a K_4, that must mean that there are two complete matchings in G. These edges form a C_4 subgraph.

Inductive step: Suppose not, let G be a graph with $v(G) = 4k$, $e(G) \geq 8k^2 - 6k + 3$, and no kC_4 minor. It will be much more simple and clear to consider the amount of missing edges, which in this case is $\binom{4k}{2} - e(G) \leq 4k - 3$.

Claim 1. *Every two adjacent vertices have at least two shared neighbors.*

Proof. Suppose not, if there exists adjacent vertices u and v that had no shared neighbors, then at minimum, there must be $4k - 2$ missing edges, which is too many.

If these two vertices instead had one shared neighbor, say w, then at minimum there must be $4k-3$ missing edges, meaning all other edges are present. At least one of u or v have another neighbor x, then (u, v, w, x) form a C_4. $G \setminus \{u, v, w, x\}$ is complete from what was noticed before, and so it contains the remaining $(k-1)C_4$, contradiction. □

Let $\overline{G}$ be the complement of G, let $d := deg_{\overline{G}}(u)$ where u is a vertex with maximum degree.

Case $d \geq 4$: Let v be adjacent to u, and w and x be their shared neighbors in G. $G \setminus \{u, v, w, x\}$ has at most $4(k-1) - 3$ edges missing from being $K_{4(k-1)}$. This is because deleting u removes d non-edges. So $e(G \setminus \{u, v, w, x\}) \geq \binom{4(k-1)}{2} - 4(k-1) + 3 = \binom{4(k-1)-1}{2} + 2$, and so $G \setminus \{u, v, w, x\}$ must contain a $(k-1)C_4$, by the induction hypothesis.

Notice that it is not necessary to have $d \geq 4$ to arrive at this conclusion. All that is needed is for u, v, w, and x to be adjacent to 4 unique non-edges in total. For the remaining cases we focus on $\overline{G}$.

Case $d = 3$: Since u is already adjacent to 3 edges, this must mean that every vertex adjacent to u must have $deg_{\overline{G}}(v) = 0$, otherwise the C_4 involving u and v would be adjacent to at least 4 non-edges, and then we would fall into the previous case.

The three vertices adjacent to u in $\overline{G}$ must all have $deg_{\overline{G}}(v) = 1$, if it was any greater, then v, u, and any two of the other vertices with degree 0 would create a C_4 in G, which would be adjacent to 4 non-edges. We have,

$$\sum_{v \in V(G)} deg_{\overline{G}}(v) = 3 + 1 + 1 + 1 = 7 = 8k - 6 = 2e(\overline{G})$$

This is not true for any valid k.

Case $d = 2$: Like the previous case, we can see that any of the vertices not already adjacent to u must have $deg_{\overline{G}}(v) \leq 1$. Now if one of these vertices does indeed have degree, then by the claim above, this vertex and u must have two shared neighbors in G. These two vertices must then have degree 0 in $\overline{G}$, otherwise these four vertices make a C_4, and deleting it will also delete four non-edges.

This means we have at least two vertices in $\overline{G}$ with degree 0. This gives us,

$$\sum_{v \in V(G)} deg_{\overline{G}}(v) \leq (4k-5)+2+2+2 = 4k+1,$$

but this is less than $8k-6$, for all valid k.

Case $d \leq 1$: The sum of all the degrees in $\overline{G}$ is at most $4k$, which is always less than $8k-6$ for all valid k.

□

References

[1] Endre Csóka, Irene Lo, Sergey Norin, Hehui Wu, and Liana Yepremyan. The extremal function for disconnected minors. 2015.

[2] Hong Wang. An extension of the corr´adi-hajnal theorem. *Australasian Journal of Combinatorics*, 54:59–84, 2012.