Trapped internal waves over topography: Non-Boussinesq effects, symmetry breaking and downstream recovery jumps

Nancy Soontiens, Marek Stastna, and Michael L. Waite
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

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It is well-known that in certain parameter regimes, the steady flow of a density stratified fluid over topography can yield large amplitude internal waves. We discuss an embedded boundary method to solve the Dubreil-Jacotin-Long (DJL) equation for steady-state, supercritical flows over topography in an inviscid, stratified fluid. The DJL equation is equivalent to the full set of stratified steady Euler equations and thus the waves we compute are exact nonlinear solutions. The numerical method presented yields far better scaling with increase in grid size than other iterative methods that have been used to solve this equation, and this in turn allows for a more thorough exploration of parameter space. For waves under the Boussinesq approximation, we contrast the properties of trapped waves over hill-like and valley-like topography, finding that the symmetry of freely propagating solitary waves when the stratification is reflected across the middepth is not present for trapped waves. We extend the derivation of the DJL equation to the non-Boussinesq case and discuss the effect of the new, non-Boussinesq terms on the structure of the trapped waves, finding that the sharp transition between large and small amplitude waves observed under the Boussinesq approximation is much more gradual when the Boussinesq approximation is relaxed. Finally, we demonstrate the existence of asymmetric steady states over hill-like topography where the flow is subcritical upstream of the topography but transitions to supercritical somewhere over the hill. Waves in this new class of exact solutions are related to so-called downstream recovery jumps predicted on the basis of hydrostatic (shallow water) theories, but when breaking does not occur the recovery jump does not stop propagating downstream and an asymmetric state across the topography maximum is reached for long times. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4811404]

I. INTRODUCTION

Density stratification is an essential property of the dynamics of oceans, lakes, and atmospheres. Topography can act as a forcing mechanism for internal waves in a stratified fluid, which has important consequences for the momentum and energy budgets. There are many instances of large internal waves observed in oceans and lakes, which have motivated fundamental questions about their generation, evolution, and dissipation. A varied and rich body of literature has investigated topographic waves from many different perspectives, including experimental, analytical, and numerical methods (see Refs. 4 and 5 for an overview).

Steady stratified flows and exact internal solitary waves have been studied using the Dubreil-Jacotin-Long (DJL) equation, a mathematical description of the streamline displacement in the form of a single, scalar, and strongly nonlinear elliptic partial differential equation that is equivalent to the full set of stratified Euler equations. Historically, this equation has often been studied for a...
fluid with a constant buoyancy frequency,\textsuperscript{4,7} in which case it linearizes. Turkington et al.\textsuperscript{8} developed a variational algorithm to solve the DJL equation for an arbitrary density profile. This equation has also been used to study mountain waves in the atmosphere\textsuperscript{7} as well as various situations in natural waters.\textsuperscript{9–12} Ocean and lake stratifications typically include a layer of rapid density change, called a pycnocline, which separates regions of nearly constant density and regions over which the density changes quickly. With these types of stratifications, the DJL equation is known to yield very large trapped disturbances over topography for background speeds close to the conjugate flow speed.\textsuperscript{12–14} In this article, we continue the investigation of large trapped states by comparing properties of trapped waves over elevation and depression topography for a variety of inflow speeds as well as density stratification strengths that require relaxation of the Boussinesq approximation.

We are mainly interested in steady, supercritical flows for which no upstream propagating waves exist. While the condition of criticality is often presented in terms of the speed of linear waves expressed through the Froude number,\textsuperscript{15} this characterization loses its meaning when large amplitude internal waves (such as internal solitary waves) are considered, since these are well-known to have propagation speeds that are larger than the largest linear wave speed. It has been shown in the literature that the condition of super criticality is satisfied when the background flow speed, $U_0$, is greater than the conjugate flow speed, $c_j$.\textsuperscript{13} The conjugate flow theory is a better choice for a definition of sub and super criticality since it accounts for nonlinear effects. The term conjugate flow is due to Benjamin\textsuperscript{16} and describes two horizontally uniform states connected by a wave front of permanent form. The transition region is not computed by the conjugate flow theory. The relevant formulation of conjugate flow theory for a continuously stratified fluid, with and without the Boussinesq approximation, has been presented by Lamb and Wan.\textsuperscript{17} These authors provide a detailed method for calculating conjugate flow speeds for a general density stratification, and some of their methodology is applied in our work below.

A key issue in the study of large amplitude internal waves is what mathematical model to apply. The first-order in amplitude and dispersion theory, leading to the Korteweg deVries (KdV) equation, which dates back to Benney’s work in 1966,\textsuperscript{18} is probably the most widely known. However, a variety of other theories have been developed. These include higher order extensions using Benney’s formalism,\textsuperscript{19} as well as models that allow for strong nonlinearity but only weak dispersion.\textsuperscript{20} The weakly nonlinear description of flows over topography dates back to the seminal papers by Djordevic and Redekopp\textsuperscript{21} and Grimshaw and Smyth,\textsuperscript{15} who developed the forced KdV (fKdV) equation for the upstream response. The numerical solutions of the fKdV equation and its extensions have provided good qualitative predictions for the types of response generated by flow over topography in various regimes, but again, the actual structure of the wave-induced flows is not particularly well represented by the approximate theory when the waves are large.\textsuperscript{13} Moreover there is no a priori bound that would tell an experimentalist or field scientist when the approximate theory can be expected to give reliable results. It is worth noting that there is a parameter regime in which approximate theory can consider large amplitude waves, and that is the rather exotic case of a nearly linear stratification profile, treated using an integro-differential equation.\textsuperscript{22} While all these models make consistent physical approximations (e.g., they assume that the fluid is inviscid), they also make mathematical assumptions, which do not come with clear a priori error bounds. It is well-known that approximate theories generally have problems quantitatively matching the vertical structure of wave-induced velocities and other physically relevant parameters.\textsuperscript{23}

DJL theory takes a different approach. It does so by treating only certain classes of solutions (e.g., steady flows, solitary waves), but treating those solutions without approximation. Solutions of the DJL equation have been verified to match internal solitary waves and large amplitude trapped waves in numerical simulations essentially exactly.\textsuperscript{12,13,24} This is true both for freely propagating waves, and flows over topography. Exact DJL theory is thus an important and powerful tool for both exploring new wave phenomena and interpreting the results of numerical models. Indeed, for free internal solitary waves DJL theory has been used to determine limiting amplitudes, and to classify these into three conditions: (1) the breaking limit, which describes the onset of isopycnal overturning, (2) the shear-instability limit, for which waves become shear unstable,\textsuperscript{26} and (3) the conjugate flow limit, for which waves broaden and flatten.\textsuperscript{17,27} Similar limiting amplitudes for breaking waves have been discovered in laboratory experiments and theoretical computations.
of layered fluids. Indeed, for the case of solitary wave broadening it has been shown that the weakly nonlinear Gardner equation only gives a qualitative description of the limiting amplitude, while the conjugate flow description (which neglects dispersion but not nonlinearity) yields excellent quantitative results. The concept of conjugate flows has also been used to understand the broadening and flattening of internal solitary waves in some oceanic observations.

The recognition that the conjugate flow speed provides a strict upper bound on upstream propagating internal solitary waves led to discovery of a parameter regime in which very large trapped waves were observed over the topography (well above the theoretical upper bound on internal solitary wave amplitudes). Apart from being a strongly nonlinear, yet laminar exact solution of the stratified Euler equations that is both generated and remains stable in time-dependent numerical simulation, these trapped waves have also been examined using DJL framework and have a variety of unexpected properties, including hysteresis.

Steady states for subcritical and supercritical flows over topography (using the more traditional definition of criticality) have been previously studied in the context of shallow water theory. One of these steady states includes a downstream recovery jump when the flow upstream of the topography is subcritical but transitions to supercritical as it passes over the topography (i.e., a finite amplitude topography effect). A form of the downstream recovery jump was also observed by Stastna et al. in rotating continuously stratified flows. Similar features are presented in this article for nonrotating continuously stratified subcritical flows.

In this article, we present a novel numerical method to solve the DJL equation in an efficient manner, and use it to extend past studies of exact, fully nonlinear waves over topography in three ways: First, we consider how the polarity of the topography affects the type of trapped wave that occurs, finding that the largest trapped waves are generated for the case of a pycnocline above the middepth and a valley-like topography. These waves can be up to 100% larger than the biggest waves found over hill topography. Second, we derive a non-Boussinesq form of the DJL equation and use it to study the effect of a change in density stratification strength, finding that the sudden transition from large amplitude waves to small amplitude waves that is observed under the Boussinesq approximation as the inflow speed is increased does not occur when the Boussinesq approximation is relaxed and the top to bottom density jump is on the order of 10% (or larger). Third, we present time-dependent numerical simulations that demonstrate the existence of large amplitude, asymmetric states for the case when the inflow is subcritical with respect to the conjugate flow speed but achieves super criticality over the hill topography. We confirm that these states are solutions of the DJL equation, and hence present an entirely new class of exact, fully nonlinear internal waves. An overview of the results, along with future research directions, is provided in the final section.

II. METHODS

A. Equations

In this section, we present the DJL equations for Boussinesq and non-Boussinesq fluids with constant background velocity $U_0$. A short derivation for the non-Boussinesq equation is also shown, which closely parallels the derivation for a Boussinesq fluid with vertically varying background current presented by Soontiens et al., but is different in a few key respects.

1. Boussinesq

Given a constant background velocity, $U_0$, the steady-state Euler equations of motion for an inviscid, stratified fluid under the Boussinesq approximation can be reduced to a single equation for the isopycnal displacement, $\eta$. This equation is commonly referred to as the DJL equation and, along with boundary conditions, is given by

$$\nabla^2 \eta + \frac{N^2(z - \eta)}{U_0^2} \eta = 0,$$

$$\eta = 0 \text{ at } z = H,$$  \hspace{1cm} (1)
FIG. 1. A single density contour is drawn over the topography $h(x)$. The isopycnal displacement $\eta$ is defined as the distance the density contour is displaced from its far upstream value. For waves of depression, $\eta$ is negative and for waves of elevation it is positive.

$$\eta = h(x) \text{ at } z = h(x),$$

$$\eta \to 0 \text{ as } x \to \pm \infty \text{ imposed at } x = \pm L,$$

where

$$N_B^2(z) = -\frac{g}{\rho_0} \frac{d}{dz} \bar{\rho}(z),$$

is the squared buoyancy frequency and $h(x)$ is the bottom topography. In addition, $\bar{\rho}(z)$ is the undisturbed density profile and $\rho_0$ is a constant reference density. The boundary conditions are determined by requiring that the upper boundary $z = H$ and lower boundary $z = h(x)$ are both streamlines. In addition, for supercritical flows in the steady-state limit, we require that the isopycnal displacement tends to zero far away from the topography. A cartoon image representing the definition of $\eta$ is provided in Fig. 1.

2. Non-Boussinesq

A short derivation of the non-Boussinesq form of the DJL equation with constant background velocity is presented below. A similar derivation for the Boussinesq case with non-constant background current is provided in Ref. 12. The non-Boussinesq case with constant background velocity yields a simpler derivation; however, care must be taken when dealing with the non-Boussinesq description of the momentum equations. The derivation begins with a statement of the steady-state 2D Euler equations of motion for an inviscid, incompressible fluid (subscripts denote partial derivatives):

$$\rho (uu_x + uw_z) = -p_x,$$

$$\rho (uw_x + wu_z) = -p_z - \rho g,$$

$$u_x + w_z = 0,$$

$$u\rho_x + w\rho_z = 0.$$  \hspace{1cm} (5)

In these equations, $u$ and $w$ are the horizontal and vertical velocities, respectively; $p$ is the pressure; $\rho$ is the density; and $g$ is the acceleration due to gravity.

As in the Boussinesq case, Eq. (4) defines a stream function $\psi$, where $u = \psi_z$ and $w = -\psi_x$. Then, Eq. (5) can be rewritten in terms of the Jacobian operator,

$$J(\rho, \psi) = 0,$$
where $J(a, b) = a_2 b_1 - a_1 b_2$. Since streamlines connect any point to the far field, the density has the form $\rho(x, z) = \bar{\rho}(z - \eta)$, where $\bar{\rho}(z)$ is the density profile in the far field, upstream and $\eta(x, z)$ is the isopycnal displacement. This statement is valid when flow properties far upstream are not affected by the wave disturbance, and no closed streamlines (e.g., wave overturning) occur. The result is that the stream function can be written as

$$\psi = U_0(z - \eta)$$

with the details of the calculation available in Ref. 12.

Next, working with the momentum Eqs. (2) and (3), we rewrite in terms of the stream function and take the curl in order to eliminate the pressure terms. After some algebraic manipulations and the use of the Jacobian operator, the result is

$$\rho J(\nabla^2 \psi, \psi) + \rho_x J(\psi_x, \psi) + \rho_z J(\psi_z, \psi) = -J(gz, \rho),$$

where $\nabla^2 \psi = \psi_{xx} + \psi_{zz}$. Compared to the Boussinesq case, the second and third terms on the left-hand side of this equation are due to the non-Boussinesq effects, and hence new.

This equation can be simplified to obtain

$$J \left( -U_0^2 \nabla^2 \bar{\rho}(z - \eta) + \frac{U_0^2}{2} \eta_z \frac{d \bar{\rho}}{dz} (z - \eta) + \frac{U_0^2}{2} \eta_z \eta_z - 2 \frac{d \bar{\rho}}{dz} (z - \eta) + g \frac{d \bar{\rho}}{dz} (z - \eta) \eta, z - \eta \right) = 0,$$

which implies that

$$-U_0^2 \nabla^2 \eta \bar{\rho}(z - \eta) + \frac{U_0^2}{2} \eta_z \frac{d \bar{\rho}}{dz} (z - \eta) + \frac{U_0^2}{2} \eta_z \eta_z - 2 \frac{d \bar{\rho}}{dz} (z - \eta) + g \frac{d \bar{\rho}}{dz} (z - \eta) \eta = G(z - \eta),$$

where $G(\cdot)$ is a function with continuous partial derivatives. The behaviour as $x \to -\infty$ requires that $\eta \to 0$, along with its derivatives, which gives $G(z) = 0$. This leads to the DJL equation for a non-Boussinesq fluid with constant background velocity, presented below with boundary conditions:

$$\nabla^2 \eta + \frac{N_{NB}^2(z - \eta)}{U_0^2} \eta + \frac{N_{NB}^2(z - \eta)}{2g} (\eta_z^2 + \eta_z (\eta_z - 2)) = 0,$$

$$\eta = 0 \text{ at } z = H,$$

$$\eta = h(x) \text{ at } z = h(x),$$

$$\eta \to 0 \text{ as } x \to \pm \infty \text{ imposed at } x = \pm L,$$

where

$$N_{NB}^2(z) = -\frac{g}{\bar{\rho}(z)} \frac{d \bar{\rho}}{dz}(z).$$

Comparing to the Boussinesq equation, the main differences are the revised definition of the buoyancy frequency $N_{NB}(z)$ and the additional nonlinear terms involving $\eta_z$ and $\eta_z$. In addition, a connection can be made with Lamb and Wan’s17 equation for non-Boussinesq conjugate flows. The equation governing conjugate flows take $\eta_z(z)$, or a function of $z$ only, and can be recovered from (6) by neglecting the dependence on $x$.

B. Numerical methods

For both the Boussinesq and non-Boussinesq cases, the DJL equations are nonlinear and have no known analytical solutions. However, numerical methods have had success at finding solutions. Soontiens et al.12 used an iterative mapped method to solve the DJL equation for a Boussinesq fluid with a non-constant (vertically varying) background current. The form of the DJL equation in this case is very similar to Eqs. (1) and (6); hence, this numerical method can be easily adapted to solve
these equations. The method involves mapping the physical domain to a rectangular, computational domain using a Chebyshev, pseudospectral discretization of the derivatives and Laplacian operator (details provided in Ref. 12). The pseudospectral discretization allows for high accuracy with only moderate grid sizes; however, the mapping introduces a complicated cross-derivative term in the Laplacian operator, thus the simplicity in the form of Eq. (1) is lost. In addition, the matrix yielded from this procedure is full, which leads to a reduction in computational efficiency for both direct and iterative solution techniques of linear systems that include the Laplacian. A more efficient numerical method is attainable.

In order to improve computational efficiency, we consider an embedded boundary method to solve these types of equations. Laprise and Peltier7 applied this procedure to the DJL equation for flow over bell-shaped topography in an idealized atmosphere with constant buoyancy frequency, which is a linear problem. We have extended this method to consider flow over topography with non-constant buoyancy frequency and a rigid lid upper boundary condition. In this method, instead of mapping the physical domain to a rectangular computational domain, the problem is solved on a rectangular box with the bottom topography $z = h(x)$ embedded in this box, and an a priori unknown boundary condition at $z = h_{\text{bot}}$ computed iteratively, where $h_{\text{bot}}$ denotes the bottom of the rectangular domain. For elevation topography $h_{\text{bot}} = 0$, and for depression topography $h_{\text{bot}}$ represents the lowest point of the topography as depicted in Fig. 1. The details of this method are described in Ref. 7. It should be noted that the term embedded boundary or immersed boundary method is often used in computational fluid dynamics to represent boundaries that are deformed by the flow.36 In this context, we are representing a solid, stationary bottom topography and the method is best described by the iterative procedure discussed below.

The desired boundary condition, $\eta(x, z = h(x)) = h(x)$, is achieved by iteratively setting an appropriate condition on $\eta(x, z = h_{\text{bot}})$ using the following algorithm:7

$$\eta^{N+1}(x, z = h_{\text{bot}}) = \eta^{N}(x, z = h_{\text{bot}}) - \text{err}^{N},$$

$$\text{err}^{N} = \eta^{N}(x, z = h(x)) - h(x),$$

where $\eta^{N}$ represents $\eta(x, z)$ at the $N$th iteration. The iteration is initialized with

$$\eta^{0}(x, z = h_{\text{bot}}) = h(x).$$

The term $\text{err}^{N}$ represents the error between the current value of $\eta(x, z = h(x))$ and the actual boundary condition described in Eq. (6). The term $\eta^{N}(x, z = h(x))$ is calculated using a cubic spline interpolation. The iteration is concluded when a suitable error on the lower boundary condition is reached (typically up to $10^{-7}H$).

Next, the Laprise and Peltier7 method must be adapted in order to implement the rigid lid upper boundary condition. We perform a Fourier decomposition in $x$, hence we enforce periodic boundary conditions at $x = \pm L$. In the vertical, we use a Chebyshev discretization with the upper boundary Dirichlet condition, $\eta(x, z = H) = 0$ enforced. The differential equations described by (1) and (6) are then solved numerically using the pseudo-time stepping algorithm described in Ref. 12. The Laplacian term is treated implicitly and the other terms are treated explicitly. This method shows a marked increase in computational efficiency compared to the mapped domain approach, as shown in Fig. 2. Here, we display the computation times from the embedded and mapped methods for several grid sizes and two different waves (one large amplitude and one small). It is clear that the embedded method performs better than the mapped method, especially for large grid sizes. In fact, it was not possible to utilize the mapped method for the 256 by 149 grid size used in most of the results due to the memory restrictions of a typical workstation. Although the implementation of each method could be optimized further (for example by parallelizing the computation for the individual wave numbers), the embedded method clearly scales better with grid size.

In addition to the steady results of the DJL equation, we conduct several numerical simulations of the time-dependent Euler equations under the Boussinesq approximation in order to compute steady states in situations where the DJL theory is formally not valid (i.e., subcritical flows).
Euler equations are solved numerically using a second-order projection method with variable time step and terrain-following vertical coordinates.\textsuperscript{37, 38} Free slip boundary conditions are set at the upper and lower boundaries and inflow boundary conditions are set at the left (upstream) side of the domain. Waves are allowed to propagate out of the right (downstream) side of the domain with grid-stretching close to the far right boundary. Since the flow is subcritical, an upstream propagating mode is possible in these simulations. However, in the simulations the amplitude of the upstream propagating wave is insignificantly small, for the range of parameters discussed below.

C. Description of experiments

We consider supercritical flows for both Boussinesq and non-Boussinesq fluids with constant background velocity $U_0$. In this context, the term supercritical means that both the linear and nonlinear disturbances are unable to propagate upstream. This condition is satisfied when $U_0/c_j > 1$, where $c_j$ is the conjugate flow speed. The conjugate flow speed is determined by solving an eigenvalue problem with auxiliary conditions as detailed by Lamb and Wan.\textsuperscript{17} The authors use a shooting method for solving the conjugate flow eigenvalue problem for both the Boussinesq and non-Boussinesq cases. We follow a similar method, seeding the shooting method with the linear long wave speed $c_{lw}$.

We have examined a number of cases, with different bottom topographies and density stratifications. In each case we have considered a general form of the bottom topography $h(x)$ and background density $\bar{\rho}(z)$ given by

$$h(x) = h_0 \text{sech}^2(x/w_d),$$

$$\bar{\rho}(z) = \rho_0 \left( 1 - \Delta \rho \tanh \left( \frac{z - z_0}{\delta_z} \right) \right).$$
In these equations, \( h_0 \) is the amplitude of the topography and \( w_d \) is a parameter that sets the topographic width. We consider both elevation \((h_0 > 0)\) and depression \((h_0 < 0)\) topography and take \(|h_0| = 0.1H\) in both cases. The domain spans from \( x = [-L, L] \), \( z = [0, H] \) for elevation topography and \( z = [-|h_0|, H] \) for depression topography. In addition, the width of the topography is set as \( w_d = 0.225L \) for all simulations, noting that previous work by Soontiens \textit{et al.}\(^\text{12}\) has examined several variations on the topographic width, finding multiple states for narrow topography. Results are non-dimensionalized by the height, \( H \), either the linear long wave speed, \( c_{ljw} \) or the conjugate flow speed \( c_j \), and the advective timescale \( H/c_{ljw} \) for time dependent simulations.

The density profiles we have chosen represent a pycnocline stratification centered at \( z_0 \). The parameter \( z_0 \) is set case by case, as indicated in the description of each experiment. The other parameters are \( \rho_0 = 1 \) and \( \delta_z = 0.1H \), which represent the reference density and pycnocline thickness, respectively. The parameter \( \Delta \rho \) represents the change in density across the pycnocline and is set to \( \Delta \rho = 0.02 \) for Boussinesq cases. Several values of \( \Delta \rho \) are considered for the non-Boussinesq comparisons.

Three sets of experiments will be presented. First, we investigate the difference between trapped waves generated over elevation topography and depression topography by considering density stratifications with the pycnocline centered above and below the middepth. This problem is motivated by the symmetry across the middepth for free travelling waves of elevation and depression.\(^\text{17}\) With this motivation, we set out to determine if a corresponding symmetry exists for trapped waves over topography.

The next set of experiments aims to classify the non-Boussinesq effects on trapped waves. Solutions to the Boussinesq and non-Boussinesq equations are computed for several density stratifications and for several background speeds \( U_0 \). The goal is to determine at what stratification and background velocity the Boussinesq approximation breaks down, and how this breakdown is manifested.

In a third set of experiments, we perform several numerical simulations of the time-dependent Euler equations with the goal of determining steady states of subcritical flows for which \( U_0 < c_j \) and upstream propagating modes are attainable. We integrate the Euler equations of motion until a steady state, if possible, is reached. We consider two cases that include isolated hill topography and double-hill topography. In both cases, the flow is subcritical far upstream of the hill. We subsequently confirm that this new, asymmetric type of trapped wave can be computed as a solution to the DJL equation.

III. RESULTS

A. Elevation vs. depression topography

Travelling waves of elevation and depression under the Boussinesq approximation satisfy a symmetry condition about the middepth\(^\text{17}\) (see Appendix A for details on this symmetry property). We set out to determine to what extent a corresponding symmetry exists for trapped waves over topography. In order to accomplish this goal, we compare trapped waves of elevation and depression for a pycnocline centered above and below the middepth in Fig. 3. In these examples, the pycnocline is centered at \( z_0 = 0.75H \) or \( z_0 = 0.25H \). For both of these stratifications the conjugate flow speed is \( c_j = 1.17c_{ljw} \) and we have chosen the background speed to be \( U_0 = 1.06c_j \).

Solutions with depression topography are shown in Figs. 3(a) and 3(b). The most striking feature is the very large trapped wave of depression in Fig. 3(a), which is similar in structure and amplitude to waves found by Soontiens \textit{et al.}\(^\text{12}\) for non-constant background currents. With the pycnocline centered above the middepth, trapped waves of depression can have amplitudes much larger than the conjugate flow amplitude for \( U_0 \) greater than, and close to, the conjugate flow speed \( c_j \). This is in contrast to freely propagating waves of depression, which have a limiting amplitude as the propagation speed approaches the conjugate flow speed.\(^\text{17}\) In contrast, when the pycnocline is centered below the middepth (Fig. 3(b)), the amplitude of the trapped waves of depression is not very large since the pycnocline is too close to the bottom boundary to allow large trapped disturbances to form.
FIG. 3. Plots of the density field for trapped depression waves (top) and elevation waves (bottom). On the left (a) and (c), the pycnocline is centered at $z_0 = 0.75H$ and on the right (b) and (d), it is centered at $z_0 = 0.25H$. In these cases, $U_0 = 1.06c_j$. The thick black line represents the bottom topography.

Waves of elevation do not exhibit such a large contrast between the stratifications centred above and below the middepth (see Figs. 3(c) and 3(d)). In Fig. 3(c), the above middepth solution showcases a small amplitude wave since the pycnocline is too close to the top boundary for large amplitudes. The below middepth solution exhibits a larger wave, but it is not nearly as large as the corresponding large wave of depression. There is thus a fundamental difference between trapped waves of depression and elevation with regards to large amplitude solutions, and this is in sharp contrast to the case of freely propagating internal solitary waves.

This observation can be explained by considering the effects of the topography on the depth of the water column. For elevation (depression) topography, the depth of the domain is decreased (increased) directly over the topography. This local change in depth suggests that a local conjugate flow speed ($c_{j_{\text{loc}}}$) over the center of the topography can be defined, and that this local conjugate flow speed will change in different manner for the cases of elevation and depression topography. A similar concept was discussed in terms of a local Froude number for single-layer homogeneous flow over topography by Baines.\textsuperscript{4} His calculations are framed in terms of the local depth and local horizontal velocity to define a local Froude number (based on linear wave speed), which is used to determine a maximum topographic height for which the flow is either supercritical or subcritical everywhere. Here, we consider changes in the conjugate flow speed due to the local water column depth, in the framework of fully nonlinear theory in a continuously stratified fluid.

In Table I, we provide several calculations of the conjugate flow speed in a domain whose depth is given by the water column depth over the center of the topography. We consider each of the stratifications and topography profiles presented in Fig. 3.

<table>
<thead>
<tr>
<th>Topography</th>
<th>$z_0$</th>
<th>$H_{\text{loc}}$</th>
<th>$c_{j_{\text{loc}}}/c_{j_{\text{tw}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depression</td>
<td>0.75H</td>
<td>1.1H</td>
<td>1.24</td>
</tr>
<tr>
<td>Depression</td>
<td>0.25H</td>
<td>1.1H</td>
<td>1.25</td>
</tr>
<tr>
<td>Elevation</td>
<td>0.75H</td>
<td>0.9H</td>
<td>1.1</td>
</tr>
<tr>
<td>Elevation</td>
<td>0.25H</td>
<td>0.9H</td>
<td>1.09</td>
</tr>
</tbody>
</table>
The effects of the topography on the local conjugate flow speed are clear. An increase in the water column depth (depression topography) causes an increase in the local conjugate flow speed. By contrast, the elevation topography demonstrates a decrease in the local conjugate flow speed. This change in local conjugate flow speed can explain the very large disturbance over the depression topography seen in Fig. 3(a). Away from the topography the background flow is $U_0 = 1.06c_j$ and the flow is in a supercritical state. However, over the topography $c_{jloc}$ has increased, which implies that locally the flow is less supercritical and, in some cases, may even transition into the subcritical regime. As the flow becomes less supercritical, there is a greater propensity for waves to travel upstream which leads to the very large disturbance in Fig. 3(a). For elevation topography, the local conjugate flow speed decreases so that the flow becomes more supercritical and the trapped waves do not become as large. It should be noted that other features, such as topographic width, could affect the structure of the resulting waves. Previously, Soontiens et al. examined similar trapped wave solutions for depression topography with several different widths and found that multiple states could exist for very narrow topography. The results presented here consider wide obstacles and multiple states are not expected.

Next, we examine the large amplitude cases for both elevation and depression waves in more detail in order to determine when large trapped solutions may occur. As suggested by Fig. 3, large amplitude trapped waves of depression can occur when the pycnocline is centered above the middepth and large amplitude waves of elevation can occur when the pycnocline is centered below the middepth. This has been confirmed through a series of experiments with several background speeds $U_0$ and pycnocline stratifications.

A closer examination of the properties of these large waves is presented in Fig. 4. Here, we plot the maximum value of $|\eta|$ as a function of the background speed $U_0$ for both elevation and depression waves. The elevation waves are generated over hill topography with pycnocline centered at $z_0 = 0.4H$ and the depression waves are generated over hole topography with pycnocline centered at $z_0 = 0.6H$. In both cases the conjugate flow speed is $c_j = 1.02c_{jw}$. In general, the depression waves reach larger amplitudes than the elevation waves for most background speeds. In addition, both elevation and depression waves experience a growth in amplitude as $U_0$ approaches the conjugate flow speed. This observation can be explained by the form of Eq. (1). Notice that the nonlinear term, $N^2(z + \eta)\eta/U_0^2$, becomes smaller as $U_0$ becomes larger. Hence, nonlinear effects are less important for large $U_0$, explaining the reduction in amplitude for large $U_0/c_j$. For large enough $U_0$, $(U_0/c_j > 1.5)$, the extreme value of $\eta$ is $0.1H$ and occurs along the bottom boundary.

![Diagram of wave properties as a function of the background velocity $U_0$.](image)

**Fig. 4.** Diagram of wave properties as a function of the background velocity $U_0$. (a) Maximum value of $|\eta|$ (scaled by $1/H$) for waves over hole topography with $z_0 = 0.6H$ and hill topography with $z_0 = 0.4H$. (b) A measure of the wave width for the corresponding cases in panel (a). The wave width (scaled by $1/H$) is measured as twice the distance between the wave center and location where the wave induced surface velocities reach half of their extreme value. In both cases the conjugate flow speed is $c_j = 1.02c_{jw}$. 
In addition to wave growth, we also observe a broadening of waves as $U_0$ approaches $c_j$ since the nonlinear effects are more important for small $U_0$ (Fig. 4(b)). The depression waves tend to be broader than the elevation waves for $U_0$ close to $c_j$. However, for $U_0 \geq 1.2c_j$, the elevation waves are broader. Both elevation waves and depression waves tend to approach a minimum width for large $U_0/c_j$, something that is apparent from the DJL equations since in the large $U_0$ limit the DJL equation tends to a Laplace’s equation.

### B. Non-Boussinesq effects

Next, we examine non-Boussinesq effects by comparing solutions of the non-Boussinesq DJL Eq. (6) to solutions of the Boussinesq DJL Eq. (1). Mathematically, two changes occur when modifying the problem from Boussinesq to non-Boussinesq: (1) the definition of the buoyancy frequency changes, and (2) additional nonlinear terms appear in the DJL equation. A detailed analysis of the separate influences of these two changes from a mathematical point of view is provided in Appendix B.

To explore the differences between Boussinesq and non-Boussinesq solutions, we consider solutions to the DJL Eqs. (1) and (6) using a density profile with $z_0 = 0.75H$, $\Delta \rho = 0.05$, depression topography, and several different background velocities. For this stratification, the non-Boussinesq conjugate flow speed is $c_j = 1.16c_l w$ where $c_l w$ is the non-Boussinesq linear long wave speed. The conjugate flow speed derived from the Boussinesq equation is almost identical for this density stratification. For this fairly small density difference one would expect the Boussinesq approximation to be good, yet we find some interesting differences between Boussinesq and non-Boussinesq cases. These results are compared in Fig. 5 where we plot the maximum value of $|\eta|$ as a function of background speed $U_0$. The Boussinesq and non-Boussinesq wave amplitudes are of similar magnitude for $U_0 < 1.17c_j$ and $U_0 > 1.29c_j$. The main difference between the two cases occurs for moderate values of $U_0$. The non-Boussinesq solutions experience a more gradual transition from large to small amplitude waves as $U_0$ increases and thus exhibit larger amplitudes for moderate values of $U_0$.

The shapes of the width versus $U_0$ curves for each of the cases are similar, although there are some differences mainly around the transition velocities from large to small waves (Fig. 5(b)). The overall trend is that large amplitudes are associated with broader waves. However, around the transition to smaller amplitude the waves exhibit a narrowing trend followed by an abrupt broadening. This is exhibited by a cusp-like form of the width curve. For small waves (large $U_0$), the limiting width approaches the width of the topography. This is observed in both cases, although

![Fig. 5.](image-url)
the location of the cusp occurs at a slightly higher velocity in the non-Boussinesq case. This sharp transition and cusp-like behaviour was not noted in the depression waves of Fig. 4 with \( z_0 = 0.6H \) and \( \Delta \rho = 0.02 \).

Next, we examine the wave-induced velocities for waves generated with \( U_0 = 1.23c_J \) for each of these cases. This choice of \( U_0 \) represents one of the velocities in the range of transition between large and small amplitude waves. In Fig. 6(a), normalized wave-induced surface velocities reveal the horizontal structure of the waves. At this value of \( U_0 \), the wave in the Boussinesq case is slightly broader than the wave in the non-Boussinesq case. Panel (b) displays the vertical structure of the waves through the center of the domain, scaled by the conjugate flow speed, which allows for comparison of velocity magnitudes. This plot reveals that the non-Boussinesq case yields significantly larger wave-induced velocities than the Boussinesq cases, even for modest \( \Delta \rho = 0.05 \).

Finally, we compare Boussinesq and non-Boussinesq solutions for several different density stratifications in Fig. 7. The goal is to determine where the Boussinesq approximation breaks down, in terms of the density stratification and the background speed. All of the stratifications use \( z_0 = 0.75H \) but the density change is modified between cases, taking \( \Delta \rho = [0.05, 0.1, 0.15, 0.2] \). Interestingly, very large waves (small \( U_0 \)) do not vary significantly between the Boussinesq and non-Boussinesq solutions. For example, even for a large density change, \( \Delta \rho = 0.2 \), the amplitude of the non-Boussinesq wave has decreased by only 3.1% when compared to the Boussinesq wave for \( U_0 = 1.15c_J \). Additionally, the very small waves (large \( U_0 \)) also do not vary significantly between non-Boussinesq and Boussinesq results. Rather, the biggest deviations between the Boussinesq and non-Boussinesq solutions occur during the transition from large to small amplitude waves as \( U_0 \) increases. The transition occurs more gradually in the non-Boussinesq cases and becomes even more gradual for higher \( \Delta \rho \). This behaviour can be explained by the additional nonlinear terms in the non-Boussinesq DJL Eq. (6). As \( U_0 \) increases the \( N^2(z - \eta)\eta / U_0^2 \) term becomes less important, and in the Boussinesq limit the problem becomes close to solving Laplace’s equation for large enough \( U_0 \). However, the non-Boussinesq equation includes additional nonlinear terms that do not depend on \( U_0 \). For large \( \Delta \rho \), and hence larger \( N^2\eta(z) \), these nonlinear terms are more pronounced, explaining the more gradual transition to small amplitude states as \( U_0 \) increases. Hence, the non-Boussinesq effects are most important for large density differences, particularly in the region of transition from large to small amplitude waves. For the largest density jump, \( \Delta \rho = 0.2 \), the non-Boussinesq wave amplitude increases by 120% when \( U_0 = 1.25c_J \). Even for a moderate density jump, \( \Delta \rho = 0.1 \), the non-Boussinesq wave is 68% larger. Hence, for \( \Delta \rho \gtrsim 0.1 \) the Boussinesq approximation is not trustworthy for moderate values of \( U_0 \).
FIG. 7. Comparison of non-Boussinesq (solid) and Boussinesq (dashed) maximum $|\eta|$ (scaled by $1/H$) for several background speeds and stratifications. (a) $\Delta \rho = 0.05$, (b) $\Delta \rho = 0.1$, (c) $\Delta \rho = 0.15$, and (d) $\Delta \rho = 0.2$.

C. Subcritical flows

In the above discussion, we have considered inflows that were larger than the conjugate flow speed, $c_j$. Such flows preclude the upstream propagation of both linear and nonlinear waves. However, there has been some suggestion in the literature\textsuperscript{32, 33, 35} that it is possible for a finite amplitude, steady wave train to form in the lee of topography for flows over a hill for inflows that are subcritical away from the hill, but reach supercritical values somewhere over the hill. This subcritical to supercritical transition can lead to a downstream recovery jump (DRJ) and possibly a steady state that, unlike the supercritical cases discussed above, is horizontally asymmetric across the crest of the topography.

In Fig. 8, we show the density fields from time-dependent, nearly impulsively started simulations. These simulations are performed with a pycnocline stratification centered at $z_0 = 0.75H$ and density jump $\Delta \rho = 0.02$. The Boussinesq approximation is used for all simulations. It can be seen that for all $U_0$ values shown, a large amplitude wave forms downstream of the hill, and that the crest of this wave overshoots the middepth. For early times, it can be seen that this wave is terminated on its downstream end by a DRJ. However, the DRJ does not restore the pycnocline to its far upstream location, instead leading to an overshoot of the upstream pycnocline centre and a long, dispersive wave train (not visible in the figure). It is this wave train that eventually returns the pycnocline to its upstream state. This occurs far downstream of the topography. Indeed, the location of the DRJ gradually shifts downstream, and in all nonbreaking cases we found that the DRJ does not stop moving downstream. Thus the DRJ is not a part of the steady state reached by the fluid. Instead the steady state consists of the wave visible between $x = -20$ and $x = 20$ in Figs. 8(b), 8(d), and 8(f). In the most subcritical case shown, spatially growing shear instability sets in (visible for $x > 36$ in Fig. 8(g)) and leads to a rapid breakdown of the DRJ. We hypothesize that a DRJ that undergoes a turbulent breakdown will reach a quasi-steady state, in analogy with solutions of the shallow water equations studied by Esler et al.\textsuperscript{32} However, two dimensional simulations cannot be used to investigate this hypothesis. Moreover, the final location of the turbulent DRJ would likely be sensitive to the manner in which the flow was accelerated, since a slower acceleration would mean more time spent in the subcritical regime.

Similar types of disturbances generated by flow over topography have been discovered using weakly nonlinear theories such as the KdV equation.\textsuperscript{4, 15} The solutions presented here are derived from simulations of the fully nonlinear equations of motion for a continuously stratified fluid, and hence, represent situations that are not fully described by weakly nonlinear theory. Indeed, the
amplitude of the downstream depression is quite large, up to three times the topographic amplitude, and hence simply does not fit the small amplitude assumption of the KdV equation.

When more than one topographic hill occurs, the downstream hill can be expected to generate different waves during the adjustment process. This is demonstrated in Fig. 9, which shows the shaded horizontal velocity as well as three isopycnals in white at various times $t = [745, 993.3, 1987, 2483] t_d$, where $t_d = H/c_j$. It can be seen that for early times (Figs. 9(a) and 9(b)) a much smaller amplitude wave forms downstream of the second hill, in comparison with the wave generated between the two hills. This is because the DRJ from the first hill leads to a pycnocline that overshoots its far upstream height. When the wave generated behind the upstream hill extends far enough in the downstream direction so that the flow over the second hill is modified, the second downstream wave is “freed” and propagates upstream, decreasing in amplitude as it goes (see Fig. 9(b), the wave is near the limiting wave amplitude given by the conjugate flow to the far upstream state). When this wave reaches the second hill, it cannot propagate further upstream and it grows (see Fig. 9(c)) and begins to broaden. The long time state is thus an asymmetric state with the wave amplitude unchanged (apart from the region in the immediate vicinity of the second hill).

These results suggest that the symmetric, trapped waves computed theoretically from the DJL equation above are not the only possible steady states. Indeed, for inflows that are formally subcritical with respect to the conjugate flow speed, an asymmetric steady state is possible. In fact, the solution techniques described above can be modified to find these asymmetric steady states from the DJL equation. By modifying the horizontal boundary conditions of the DJL equation to $\eta(-L, z) = 0$ and $\eta_x(L, z) = 0$, we were able to find asymmetric steady states from the DJL equations for background speeds $U_0 < c_j$. Although the flow is formally subcritical, no upstream propagating modes existed for the inflow speeds tested. In order to produce the asymmetric states, the numerical solver was initialized with an asymmetric initial guess and the iterations were continued until a suitably low error norm was achieved. This technique may prove useful in finding other steady states for transcritical flows in a fully nonlinear framework.

An example of asymmetric DJL solutions is displayed in Fig. 10(a). This solution corresponds to the wave in Fig. 8(b), the long time solution to the numerical simulation with $U_0 = 0.96 c_j$. Both of
FIG. 9. Horizontal velocity fields with three white isopycnals superimposed for the double hill case. The background speed is $U_0 = 0.96c_j$. The results are presented at non-dimensional times (a) $t^* = 745$, (b) $t^* = 993.3$, (c) $t^* = 1987$, and (d) $t^* = 2483$. The scaling time is taken to be $t_d = H/c_j$.

these plots show similar features, including a slight elevation of the pycnocline above the topography, followed by a large depression feature in the lee. Additional wave properties for several background velocities are displayed in Fig. 10(b). This plot shows that the amplitude of the depression increases as the flow becomes more subcritical ($U_0$ decreases), which matches the trend observed in the numerical simulations of Fig. 8. Additionally, the displacement over the topography crest does not vary significantly between cases. Although not shown here, the waves undergo a steepening trend as $U_0$ decreases, similar to what is observed in Fig. 8.

IV. DISCUSSION

We have presented an embedded boundary method for solving the DJL equation in supercritical flows over topography with constant background current and non-uniform stratification. This solution technique is based on an iterative method by Laprise and Peltier\cite{7} for solving the DJL equation in flows.
with constant buoyancy frequency in the atmosphere, but has been adapted for general stratifications and rigid lid upper boundary conditions. This method shows a marked increase in computational efficiency when compared with previous mapped methods for solving the DJL equation with bottom topography and provides sufficient accuracy in implementing the bottom boundary condition.

This improved numerical method is used to investigate supercritical trapped waves over elevation and depression topography. These investigations have revealed several important differences between these two types of topography. Waves generated over depression topography can reach far larger amplitudes than waves over elevation topography, though large amplitude waves (more than 20% of the water column) are generated in both cases when the background speed approaches the conjugate flow speed. However, the waves over depression topography can achieve much larger states. A reasonable explanation of this fact is the concept of the local conjugate flow speed directly over the topography, which can locally change the criticality of the flow. Over depression topography, the domain depth increases which results in an increase in the local conjugate flow speed and a less supercritical state (i.e., $U_0/c_{jloc}$ decreases). The opposite occurs for elevation topography and the flow becomes more supercritical directly over the topography. Since the largest amplitude waves are observed for flows with $U_0/c_j \approx 1$, the depression topography allows for larger amplitude trapped waves. This result highlights a major difference between elevation and depression topography with regards to trapped internal waves, and generalizes the notion of the local Froude number (based on the linear long wave speed) as discussed by Baines.

Furthermore, we have discussed differences between trapped internal waves and freely propagating internal waves of elevation and depression. Under the Boussinesq approximation, freely propagating waves satisfy a symmetry property when the wave is reflected about the middepth (see Appendix A). A freely propagating wave reflected about the middepth also satisfies the DJL equation, hence, there is a symmetry between freely propagating waves of elevation and depression. This symmetry does not carry over to waves trapped over topography. As described above, trapped waves of depression exhibit larger amplitudes than trapped waves of elevation, and so, no symmetry condition exists.

We have derived the DJL equation under non-Boussinesq conditions and compared Boussinesq and non-Boussinesq results for large waves over depression topography. We found that the transition from large amplitude waves to small amplitude waves as $U_0$ increases occurs more gradually for the non-Boussinesq case. This effect is most noticeable in the cases with the largest density jumps, but can be significant even for modest density differences. Solutions with density changes beyond $\Delta \rho \gtrsim 0.1$ are not accurately described by the Boussinesq approximation, particularly for background speeds in the region which transitions from large to small amplitude waves. Interestingly, very large waves under the Boussinesq approximation do not deviate largely from their non-Boussinesq counterparts. Rather, it is in the transition from large to small waves where the Boussinesq approximation breaks down most significantly.

Finally, we have performed several numerical simulations of the Euler equations of motion in order to derive steady-state solutions that are asymmetric across the topography. We considered subcritical flow over isolated and double-hill topography and observed large amplitude waves in the lee of the topography. The large amplitude wave terminated in a downstream recovery jump, which overshooted the upstream pycnocline height. When breaking does not occur, the steady states for these simulations are not symmetric horizontally across the topography, in contrast to the symmetric DJL solutions for supercritical flows. However, nonbreaking subcritical steady states achieve flat isopycnals away from the topography, both upstream and downstream. We have outlined a method for computing these asymmetric states as solutions of the DJL equation (by modifying the horizontal boundary conditions, and choosing an appropriate initial guess for the iterative method).

The stability of the DJL solutions is an important point to address. The issues of stability for these solutions have been previously addressed by Soontiens et al., where DJL solutions were reproduced in simulations of the time-dependent Euler equations of motion. In those simulations, the solutions persisted for long periods of time and were determined to be stable. In some cases, this occurred even if a region of recirculation formed. The stability of the Boussinesq waves with elevation topography discussed above has been confirmed through numerical simulations in a similar fashion to Soontiens et al. The stability of the non-Boussinesq waves was not confirmed through...
time-dependent simulations, though there is no \textit{a priori} reason to expect the non-Boussinesq nature of the flow to destabilize the trapped waves. Additionally, the asymmetric states for subcritical flows were found to persist for long times in simulations, except where breaking occurs in the downstream portion, hence the DJL solutions with asymmetric states are expected to be stable. Indeed, as a general rule of thumb fully nonlinear waves are found to be highly stable, even in dynamically active situations.\textsuperscript{26}

This study emphasizes that the effects of strong nonlinearity are important in situations far different from the typical wave breaking scenario. Large-amplitude, nonbreaking, steady waves have been produced and analysed from a theory which represents the fully nonlinear Euler equations of motion, using both steady state calculations and time dependent numerical simulations. These large amplitude features cannot be studied thoroughly using classical weakly nonlinear theories such as the KdV equation and its extensions.

The results from this work are derived for an inviscid fluid, and hence exclude the effects of a boundary layer which could modify the observed steady states. Future work will consider steady states achieved from numerical simulations with the inclusion of viscous effects. In addition, this work provides a framework for producing fully nonlinear solutions for transcritical flows, a topic that has been examined previously in the context of weakly nonlinear theories such as the KdV equation (e.g., see Baines\textsuperscript{4}). In the future, it would be worthwhile to compare solutions generated through fully nonlinear and weakly nonlinear frameworks.

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APPENDIX A: SYMMETRY OF ELEVATION AND DEPRESSION WAVES

Freely travelling steady waves in an inviscid fluid under the Boussinesq approximation are described by the DJL equation written in a reference frame following the wave. For a freely travelling wave with propagation speed \(c\), this equation is given by

\[
\nabla^2 \eta + \frac{N^2(z - \eta)}{c^2} \eta = 0. \quad (A1)
\]

Such a wave, when reflected about the middepth, will also satisfy an equivalent form of the DJL equation. To see this, we construct a wave reflected about \(z = H/2\), defined as \(e(x, y) = -\eta(x, z)\) where \(y = H - z\), and show that it satisfies the DJL equation. Since \(\eta(x, z)\) is a solution to (A1), we can write this equation in terms of \(e(x, y)\) and write the Laplacian operator in the \((x, y)\) coordinate system. A simple calculation shows that \(\nabla^2 = \tilde{\nabla}^2\), where \(\tilde{\nabla}^2 = \partial_{xx} + \partial_{yy}\). Hence, the equivalent DJL equation is

\[
\tilde{\nabla}^2 e(x, y) + \frac{N^2(H - (y - e))}{c^2} e = 0.
\]

This equation reveals that a travelling wave reflected about the middepth also satisfies the DJL equation with the requirement that the buoyancy frequency profile is also reflected about the middepth. This result does not necessarily carry over to the topographic extension of the DJL equation because the change in coordinates would also result in a different set of boundary conditions, and hence modify the problem.

APPENDIX B: NON-BOUSSINESQ AND BOUSSINESQ COMPARISONS

Modifying the problem from a Boussinesq fluid to a non-Boussinesq fluid results in two changes in the mathematical description: (1) a change in the definition of the buoyancy frequency and (2) additional terms in the DJL equation due to the non-Boussinesq effects. To determine the effect of each of these changes, we solve the DJL equation in three circumstances for a given density
profile and topographic shape: (a) the non-Boussinesq equation with buoyancy frequency $N_{NB}(z)$, (b) the Boussinesq equation with buoyancy frequency $N_{NB}(z)$ (this would amount to changing the background density profile in the Boussinesq case), and (c) the Boussinesq equation with buoyancy frequency $N_B(z)$. We solve each of these problems for several background speeds $U_0$. Comparisons between the (a) and (b) indicate the effect of the non-Boussinesq terms in the DJL equation, while comparisons between (b) and (c) determine the effect of the revised definition of buoyancy frequency. Comparisons between (a) and (c) reveal the overall effects of the non-Boussinesq condition.

The solutions presented are computed for a density profile with $z_0 = 0.75H$, $\Delta \rho = 0.05$, and depression topography. For this case, the Boussinesq and non-Boussinesq conjugate flow speeds are nearly identical and take the value $c_j = 1.16c_{j0}$. First, the effects of the additional non-Boussinesq terms are determined by comparing cases (a) and (b). Examining the upper panels of Fig. 11, it is observed that these two cases are in close agreement, except in the transition region from large to small amplitude waves, where the non-Boussinesq waves are slightly larger. Furthermore, cases (b) and (c) match very closely for all values of $U_0$. In general, all three cases match well in for this stratification, although the fully non-Boussinesq case deviates slightly in the transition region. It appears that the additional nonlinear terms have the biggest effect on the non-Boussinesq results.

This can be investigated further by considering a stratification with a very large density jump $\Delta \rho = 0.2$ and $z_0 = 0.75H$. Although this may not be an accurate description of a physical ocean or lake stratification, it is perhaps more applicable to a laboratory situation using a gas for stratification. It is a
worthwhile exercise to examine the non-Boussinesq effects at this magnitude as it can provide further insights in determining where the Boussinesq approximation breaks down. For this stratification, the non-Boussinesq conjugate flow speed is \( c_{jB} = 1.14c_{wNB} \) and the Boussinesq conjugate flow speed is \( c_{jB} = 1.17c_{wB} \), where \( c_{wNB} \) and \( c_{wB} \) are the linear long wave speeds for the non-Boussinesq and Boussinesq cases, respectively.

Once again, cases (b) and (c) match very closely indicating that the alternative definition of \( N^2(z) \) does not have a significant impact on the solutions. As seen before, the fully non-Boussinesq case is very different from the other two cases throughout the transition from large waves to small waves. Additionally, the wave width behaviour of the non-Boussinesq case deviates from the Boussinesq cases, with a less noticeable cusp behaviour as the waves broaden. In summary, the main effect of the non-Boussinesq representation is in the gradual transition from large to small amplitude states. Mathematically, these effects are manifested in the additional terms from the non-Boussinesq representation.