Introduction to Intersection Theory

Convention: [reason postponed]

- base field $\Bbbk = \overline{\Bbbk}$ of char 0
- Schemes (sch) are separated (sep) of finite type (f.t.) $/\Bbbk$
- Varieties (var) are reduced irreducible (irred) schemes (sch)
- $\cap$ of schemes are always sch-theoretic $\cap$.
- subscheme means closed subscheme unless stated otherwise.

Kenote Qs

1) What's an AG analog of sing (co)homology ring $H_*(X;\mathbb{Z}) \overset{PD}{\cong} H^*(X;\mathbb{Z})$ $(A(X))$
2) What's $A(\mathbb{A}^n), A(\mathbb{P}^n)$?
3) What does $[Y]\cdot[Z] \in A(X)$ mean for sm X?

$$\begin{array}{c} [Y]\cdot[Z] \\ \shortparallel ? \\ [Y\cap Z] \end{array}$$

Chow Group (analog of $H_*(X;\mathbb{Z})$)

Def: A k-cycle in a scheme X is a formal fin sum $Z = \sum n_i Y_i$ w/ $n_i \in \mathbb{Z}$ and $Y_i \subset X$ k-dim'l subvars $\forall i$. More

generally, a <u>cycle</u> W in X is a sum
$W = \sum_k W_k$, where W_k is a k-cycle in X.

Given any subsch $Y \subset X$, $\langle Y \rangle := \sum \ell_i Y_i$
where $Y_{red} = \bigcup Y_i$ and $\ell_i = \text{mult}_{Y_i} Y$.

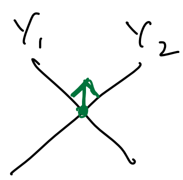
<u>Scheme facts</u>: $\forall\, U \subset X$ affine open chart,
$Y \cap U = V(I) \subset U$ w/ $\underset{\text{ideal}}{I} \subset \Bbbk[U] = H^0(\mathcal{O}_U)$

$\Rightarrow Y_{red} \cap U := V(\sqrt{I})$ reduced
radical of $I \subset \Bbbk[U]$

$\Rightarrow Y_{red} \cap U = \bigcup (Y_i \cap U)$ irred comps, s.t.
$\sqrt{I} = \bigcap I_{Y_i \cap U}$
reduced

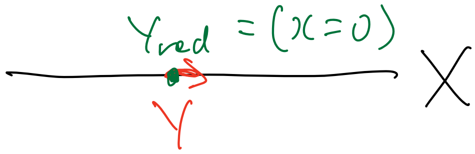
For $\text{mult}_{Y_i} Y$, it is
(generic pt of Y_i

$\text{length}_{\Bbbk(Y_i)} \mathcal{O}_{Y, \eta_{Y_i}} = \text{length}_{\Bbbk(Y_i)} \Bbbk[Y]_{I_{Y_i \cap U}} / I$

<u>Example</u>: $X = \mathbb{A}^2$, $Y =$ 

$\Rightarrow \langle Y \rangle = \langle Y_1 \rangle + \langle Y_2 \rangle$
$= \langle \times \rangle$

<u>Example</u>: $X = \mathbb{A}^1$, $Y = V(x^2)$ nonreduced subsch



$\mathrm{mult}_{Y_{red}} Y = \mathrm{length}\ \mathcal{O}_{Y,(x)} = 2$

$\Rightarrow \langle Y \rangle = 2 \cdot (x=0)$

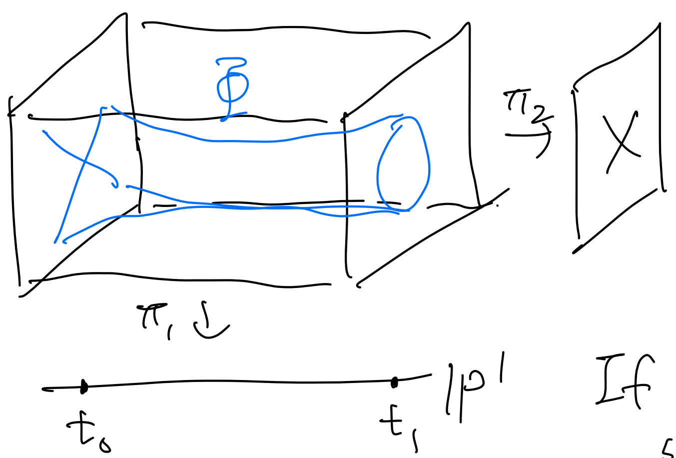
Def: The group of (alg) cycles on X is

$Z(X) = \bigoplus Z_k(X)$, $Z_k(X) :=$ gp of k-cycles

$\Rightarrow Z(X) = Z(X_{red})$

Def: The group of rational equivalence $\mathrm{Rat}(X) \subset Z(X)$ is generated by

$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$, where



$\Phi \subset \mathbb{P}^1 \times X$ subvar

$\downarrow \pi_1$

$\mathbb{P}^1$

image is not a pt

and $t_0, t_1 \in \mathbb{P}^1$.

If $\langle Y \rangle - \langle Z \rangle \in \mathrm{Rat}(X)$, say $\langle Y \rangle \sim \langle Z \rangle$.

Def: The Chow group of X is

$$A(X) := Z(X) / \mathrm{Rat}(X).$$

If $Y \subset X$ subsch, denote $[Y] \in A(X)$ its cycle class.

Prop: $A(X) = \bigoplus A_k(X)$ by dim.

Pf: Φ irred dim $k+1$ in $\mathbb{P}^1 \times X$

$\Rightarrow$ affine locally on $U \subset \Phi$ open,
$(\Phi \cap (\{t_0\} \times X)) \cap U = V(f)$ for some
$f \in \mathbb{k}[U] \Rightarrow V(f)$ pure codim 1 in Φ or empty

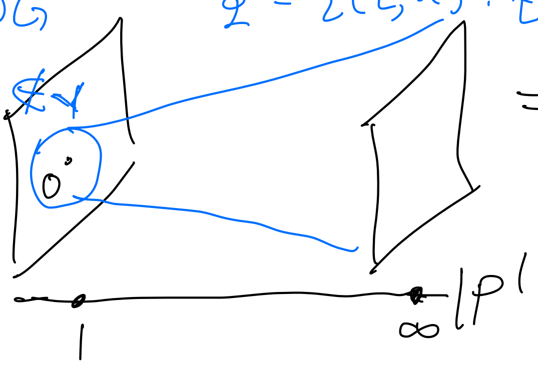
$\Rightarrow \dim(\Phi \cap (\{t_0\} \times X)) = k$ if nonempty

Same result for $\dim(\Phi \cap (\{t_1\} \times X))$. □

Convention: X pure dim $n \Rightarrow A(X) = \bigoplus A^k(X)$, where $A^k(X) := A_{n-k}(X)$. In this case,
$A^0(X) = \bigoplus \mathbb{Z}[X_i]$, $X = \bigcup X_i$

Example: $X = \mathbb{A}^n$. Then, $A^0(\mathbb{A}^n) = \mathbb{Z}\underbrace{[\mathbb{A}^n]}_{\text{fundamental class}}$,
$A^k(\mathbb{A}^n) = 0 \quad \forall k > 0$.

Pf: WLOG $\hookrightarrow 0 \notin Y$ $\quad \Phi = \{(t, x) : t^{-1}x \in Y\}$



$\Rightarrow \Phi \cap (\{\infty\} \times \mathbb{P}^1) = \emptyset$

$\Rightarrow \langle Y \rangle \in Rat(\mathbb{A}^n)$

□

Excision: $Y \xhookrightarrow{\text{closed}} X \xhookleftarrow{\text{open}} U = X \setminus Y$, then

$$A(Y) \to A(X) \to A(U) \to 0 \quad \text{is exact.}$$

as $W \subset Y$ $\qquad [Z] \mapsto [Z \cap U]$

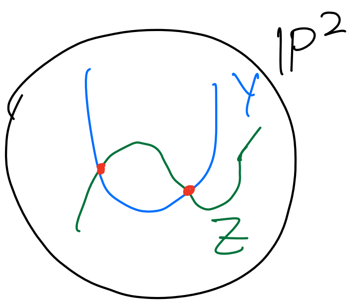
$[W] \mapsto [W]$ as $W \subset X$

Example: $X = \mathbb{P}^n$. Take affine stratification

$$\mathbb{P}^n \overset{\text{set}}{=} \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0$$

Excision $\Rightarrow$ $A(\mathbb{P}^n) \twoheadleftarrow \bigoplus \mathbb{Z}[H^k]$, $\quad H^k$ = codim k plane

A(X) as Chow Ring



If subvars $Y, Z \subset X$ are of codim k & l, then Y, Z are generically transverse if $Y \cap Z$ is gen reduced of pure codim $k+l$.

Define $[Y][Z] := [Y \cap Z]$ $\qquad$ equiv to

generically VB of cork $k+l$ $\to$ $\mathrm{Ker} \to T_Y \oplus T_Z \twoheadrightarrow T_X$

$\Rightarrow$ sample gen'l pt $p \in U \xhookrightarrow{\text{open}} Y \cap Z$ then

$\mathrm{Ker}_p = T_{Y,p} \cap T_{Z,p}$ w/ $T_{X,p} = T_{Y,p} + T_{Z,p}$

Next Time: $[Y][Z] \in A(X)$ in gen'l case.