Introduction to Intersection Theory II

base field $\mathbb{k} = \bar{\mathbb{k}}$ char 0

Kenote Qs

1) What does $[Y]\cdot[Z] \in A(X)$ mean for sm X? $\overset{?}{=} [Y \cap Z]$

   What about $\#(Y \cap Z)$ via $A(X)$?
2) $A(\mathbb{P}^n)$, $A(C)$ curve?
3) degree of pure dim k subsch of $\mathbb{P}^n$?

   degree of ($V_{2,2} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ Veronese?)

$A(X)$ as Chow Ring

Recall: X sm, $Y, Z \subset X$ gen transverse

$\Rightarrow [Y][Z] := [Y \cap Z]$

Thm-Def: If X is a sm qproj var, then $\exists!$ str of graded comm ring on $A(X) = \bigoplus A^k(X)$, called Chow ring, that extends above. (Too hard! - Fulton)

Conj for char $p>0$ ($\mathbb{Q}$-coeff is okay by de Jong)

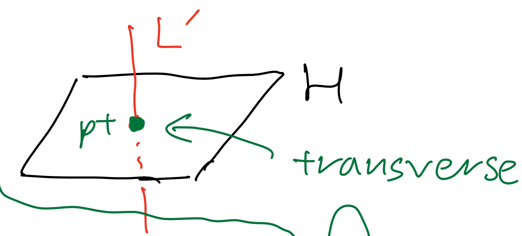
(non-alg closed is fine, but involves Gal action)

Example: For $\mathbb{P}^n$, L line, H hyppl, $H^k \cong \mathbb{P}^{n-k}$ codim k plane, then Excision on affine stratification of $\mathbb{P}^n \overset{\text{set}}{=} \bigsqcup_{k=0}^{n} \mathbb{A}^{n-k}$

$$\Rightarrow A(\mathbb{P}^n) \twoheadleftarrow \bigoplus_{k=0} \mathbb{Z}[\mathbb{A}^{n-k}]$$

$$[\overline{\mathbb{A}^{n-k}}] \longmapsfrom [\mathbb{A}^{n-k}]$$
$$\| $$
$$[H^k]$$

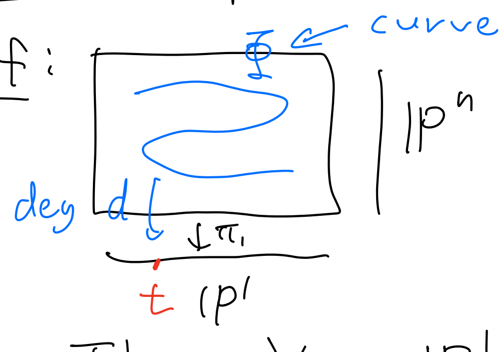
then $\underbrace{[L][H]=[pt]}_{(*)}$ b/c $L' \sim L$ $\forall L'$ line in $\mathbb{P}^n$ by above map &



(in fact, similar argument $\Rightarrow [H]^k = [H^k]$, $[H^k][H^{n-k}] = [pt]$)

Claim: $[pt] \neq 0$ in $A(\mathbb{P}^n)$

Pf:



$\pi_1 : \mathbb{P}^1 \times \mathbb{P}^n \twoheadrightarrow \mathbb{P}^1$ proper

$\Rightarrow \pi_1|_{\Phi} : \underset{\text{curve}}{\Phi} \twoheadrightarrow \mathbb{P}^1$ proper

$\Rightarrow$ fin deg d.

Then $\forall t \in \mathbb{P}^1$, π_1 surj deg d

$\Rightarrow \pi_1^{-1}(t) \cap \Phi \subset \{t\} \times \mathbb{P}^n$ is a dim 0 length d subscheme $\Rightarrow\ = \emptyset$ iff $\mathfrak{Z} = \emptyset$

$$0 \leftarrow A^n(\mathbb{P}^n) \twoheadleftarrow \mathbb{Z}[pt] \leftarrow \mathrm{Rat}_0(\mathbb{P}^n)$$

$$\begin{array}{ccc} & & \cup \\ 0 = d[pt] - d[pt] & \leftarrow\!\shortmid & \langle \pi_1^{-1}(t_1) \cap \mathfrak{Z} \rangle - \langle \pi_1^{-1}(t_2) \cap \mathfrak{Z} \rangle \end{array}$$

$\Rightarrow A^n(\mathbb{P}^n) \cong \mathbb{Z}[pt]$ $\square$

Thus, $[L] \neq 0$, $[H] \neq 0$ in $A(\mathbb{P}^n)$ by $(*)$.

$\Rightarrow$ extend so that $A(\mathbb{P}^n) \cong \mathbb{Z}[H]/[H]^{n+1}$.

<u>Def</u>: $\deg X \in \mathbb{N}$, where $X \subset \mathbb{P}^n$ pure dim k subsch, satisfies $[X] = (\deg X) \cdot [H]^{n-k}$.

$\Downarrow$

<u>(Baby) Bezout's Thm for $\mathbb{P}^2$</u>: If $C, D \subset \mathbb{P}^2$ curves that $\cap$ transversely, then

$\#(C \cap D) := \deg(C \cap D) = \deg(C) \cdot \deg(D)$. $\square$

[there's higher dim analog]

<u>Note</u>: In gen'l, affine stratification $X \overset{set}{=} \sqcup Y_i$

$\Rightarrow A(X) \overset{gp}{\cong} \bigoplus \mathbb{Z}[\overline{Y_i}]$ by Totaro '14.

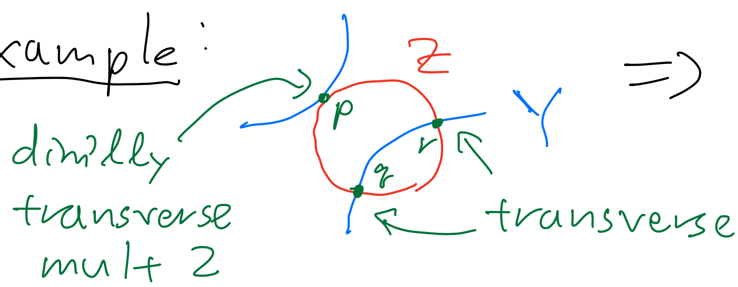
Rmk: Given $Y, Z \subset X$, we can theoretically find cycle $Z' \sim Z$ s.t. $Y \cap Z'$ is pure of expected codim (dim'lly transverse). If Y & Z' are Cohen–Macaulay (technical! but lci $\Rightarrow$ CM, so this cond is very mild) at gen pts of $Y \cap Z'$, then

aff loc, codim k sch cut out by k eqns

$$[Y][Z] = \sum m_i [W_i], \text{ where}$$

$$(Y \cap Z')_{red} = \bigcup_i W_i \quad \& \quad m_i = \mathrm{mult}_{W_i}(Y \cap Z')$$

Example:



$$\Rightarrow [Y][Z] = 2[p] + [q] + [r]$$

$\Rightarrow$ Bezout's Thm works w/ multiplicity

Functoriality

$A_0(X)$ $\ni$

To define $\#(Y \cap Z) = \deg([Y] \cdot [Z])$ $\forall X$ sm qproj

Def: $f: Y \to X$ proper btwn any schs, then $f_*: A(Y) \to A(X)$ is defined by: $\forall A \subset Y$ subvar,

$$f_*[A] := \begin{cases} 0 & \text{if } \dim f(A) < \dim A \\ n[f(A)] & \text{if } \dim f(A) = \dim A \ \& \end{cases}$$

graded, i.e., $\downarrow$ $\deg f|_A = n$

$f_* : A_k(Y) \to A_k(X)$

<u>Def</u>: If X proper, then $\overset{\pi_*}{\overset{\shortparallel}{\deg}} : A(X) \to \mathbb{Z}$

$(\pi : X \to \operatorname{Spec} \Bbbk) \Rightarrow \#(Y \cap Z) := \deg([Y][Z])$

For pullback,

<u>Thm-Def</u>: $f : Y \to X$ mor btwn sm q-proj vars,

then $\underbrace{f^*}_{} : \bigoplus A^k(X) \longrightarrow \bigoplus A^k(Y)$

graded ring hom

$$\begin{array}{ccc} \bigoplus A^k(X) & \longrightarrow & \bigoplus A^k(Y) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [A] & \longmapsto & [f^{-1}(A)] \end{array}$$

where $A \subset X$ is CM or gen transverse to f

(sch $f^{-1}(A)$ gen red & $\operatorname{codim}_Y f^{-1}(A) = \operatorname{codim}_X A$)

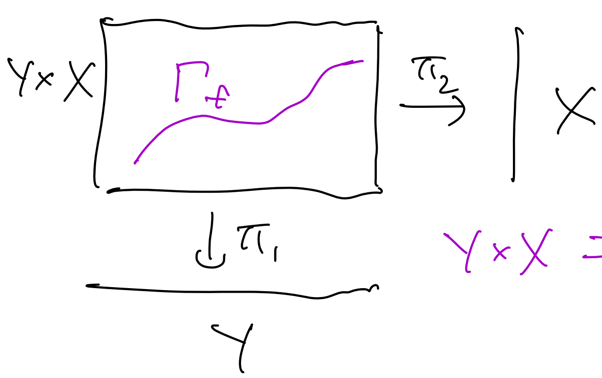
Alternatively, $f^*([A]) := \pi_{1*}([\Gamma_f] \cdot \pi_2^*[A])$,

where $\pi_2^* : A^\bullet(X) \longrightarrow A^\bullet(Y \times X)$

$[A] \longmapsto [Y \times A]$

no CM/(gen transv to f) cond

(b/c π_2 flat)



$Y \times X \supset \Gamma_f := \{(y, f(y)) : y \in Y\}$

graph of f

$\pi_1|_{\Gamma_f} : \Gamma_f \longrightarrow Y$ isom.

Then,
- gen transv cond $\Rightarrow$ $\Gamma_f \cap \overset{\pi_2^{-1}(A)}{\overset{\|}{(Y \times A)}}$ is gen transverse
- $A \subset X$ CM $\Rightarrow$ Γ_f & $Y \times A$ are CM along $\Gamma_f \cap (Y \times A)$

Rmk: I don't understand this detail so don't worry!

<u>Next Time</u>: Push-Pull Formula then application into examples by Kaleb.