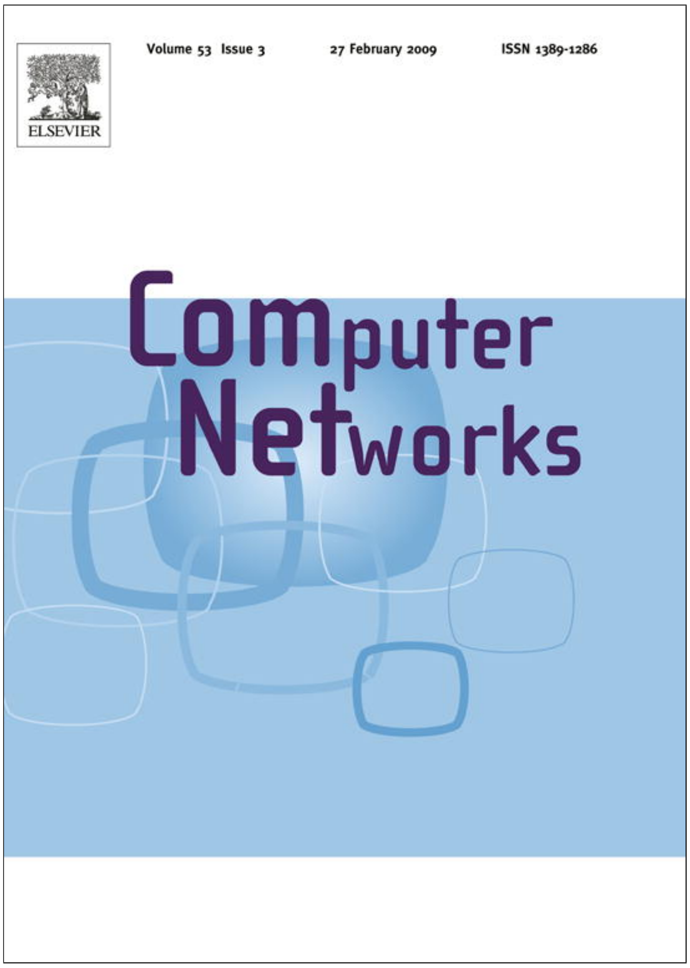
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Stability analysis of multiple-bottleneck networks

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ABSTRACT

A TCP/RED (Transmission Control Protocol/Random Early Detection) system with multiple-bottleneck links could be unstable even if its system parameters are set the same as those in a stable single-bottleneck system [D. Bauso, L. Giarre, G. Neglia, Active queue management stability in multiple bottleneck networks, IEEE ICC'04, vol. 4, June 2004, pp. 2267–2271]. In this paper, we study the stability of more general AIMD (Additive Increase and Multiplicative Decrease)/RED system with multiple bottlenecks that may incur non-negligible packet losses. We develop a general mathematical model to analyze network stability for both delay-free and delayed AIMD/RED systems. Sufficient conditions for the asymptotic stability of multiple-bottleneck systems with heterogeneous delays are derived by appealing to Lyapunov stability theory with Lyapunov–Razumikhin conditions, and these conditions can be easily assessed by using LMI (Linear Matrix Inequality) Toolbox. Numerical results with Matlab and simulation results with NS-2 are given to validate the analytical results.

1. Introduction

The Transmission Control Protocol (TCP) congestion control mechanism is a key technology for the world-wide information infrastructure, the Internet, and is deployed in the end-systems to maintain network stability and integrity in a distributed manner. The core of the Internet can be simple and scalable, and an IP router serves all incoming packets in a First-In First-Out (FIFO) fashion. In particular, in the TCP/RED (Random Early Detection) mechanism, the excessive arrival packets are buffered in a queue when the aggregated packet arrival rate exceeds the packet departure rate, and a RED-enabled router discards the incoming packets randomly when the average queue length exceeds a certain threshold ($\min_{th}$) and all incoming packets are discarded when the average queue length exceeds another higher threshold ($\max_{th}$). Since packet losses in the Internet are mainly due to network congestion, a TCP sender uses packet losses as network congestion indicators and adjusts its sending rate accordingly with a congestion window, which limits the maximum number of packets being sent without acknowledgments. TCP congestion control is implemented via the Additive Increase and Multiplicative Decrease (AIMD) mechanism: when there is no congestion indication (no packet loss), the TCP congestion window size is increased linearly by one packet per round-trip time; otherwise, the TCP congestion window size is reduced by half. To support heterogeneous traffic and multimedia applications, instead of increase-by-one and decrease-by-half, a generalized AIMD mechanism can use a pair of parameters (α, β) to set the increase rate and the decrease ratio [3–5], and the parameter pair can be flexibly chosen according to the TCP-friendly condition [5] and the applications quality of service (QoS) requirements. On the other hand, to distribute the network congestion indicators fairly to all on-going flows, active queue management (AQM) [6,7], e.g., the RED queue management scheme, has been promoted to be deployed in the intermediate

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nodes. With the RED schemes [8,9], the packet loss rate of each flow is roughly proportional to the flow sending rate. The AIMD congestion control, coupled with the RED queue in the core network, is essential to Internet stability and integrity [10].

Instead of proposing new control mechanisms, we focus on the stability and performance of the currently dominant AIMD congestion control mechanism over RED queue management, since the AIMD/RED in the core network has been acknowledged as one of the key factors to the overwhelming success of the Internet [10]. Needless to say, an in-depth understanding of AIMD/RED systems is of vital importance for healthy proliferation of the Internet in the future. Stability problems associated with TCP (e.g., AIMD) controlled flows over RED queues have been investigated extensively in the literature, for network models with a single bottleneck [2,6,13,14,17]. Notably, [6] proved the stability of TCP/RED systems without feedback delay, using a fluid model. The stability of TCP/RED systems with feedback delay has been studied in [15], which indicated that TCP/RED becomes unstable when the delay increases or, more strikingly, when link capacity increases. Our previous results provided some sufficient conditions to guarantee the stability of single-bottleneck AIMD/RED systems, considering feedback delays [26].

However, for the vast-scale Internet, the single-bottleneck topology may no longer be representative and a flow may traverse multiple links with non-negligible packet losses. In [1], it is shown that a multiple-bottleneck network may be unstable even if the same system parameters are used as those in a single-bottleneck, stable network. In fact, the congestion signals from multiple links sharing by different flows may lead to chaotic behaviors. Clearly, the results from single bottleneck networks can not be directly applied to multiple-bottleneck networks. In a nutshell, the stability property of multiple-bottleneck networks remains an important open issue beckoning for further investigation.

In this paper, after developing a general mathematical model of multiple-bottleneck AIMD/RED system, we study the stability properties, mainly asymptotic stability, of the system, considering possible heterogeneous feedback delays and propagation delays. The definitions of stability are listed below, which follow those in [24]

Definition 1. Consider dynamic systems with time delay of the following form:

$$\frac{dx}{dt} = f(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))),$$

where $x \in R^n, f : I \times R^n \times R^n \times \ldots \times R^n \to R^n$ is continuous. Let $\tau = \sup_{i=1,\ldots,m} \tau_i(t)$. The trivial solution of the system is said to be

- *stable* if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists some $\delta = \delta(t_0, \epsilon) > 0$ such that for any $\xi(t) \in C[[t_0 - \tau, t_0], R^n], \|\xi\| < \delta$ implies $\|x(t, t_0, \xi)\| < \epsilon$ for all $t \geqslant t_0$;
- *asymptotically stable* if the system is stable and for every $t_0 \in \mathbb{R}_+$, there exists some $\eta = \eta(t_0) > 0$ such that $\lim_{t\to\infty} \|x(t, t_0, \xi)\| = 0$ whenever $\|\xi\| < \eta$.

The main contributions of the paper are summarized as follows. First, the fluid model of a general multiple-bottleneck AIMD/RED system without feedback delay is proved to be *Globally asymptotically stable*, independent of the number of flows in each bottleneck, flow parameter pairs (α, β), and their round-trip delays, etc. Next, we consider the multi-bottleneck system with feedback delays where global stability is often difficult to attain, due to the highly nonlinear nature and the effect of delays. We present two sufficient conditions to guarantee *local asymptotic stability* of the system and note that these results are for general multiple-bottleneck scenarios. Numerical results with Matlab and simulation results with NS-2 [29] have validated the analytical results with an example of two-bottleneck topology. The theoretical findings can be used as a guideline for tuning the system parameters to maintain network stability and enhance system performance, and the analytical and simulation results provide important insight to understand the stability and performance of multi-bottleneck networks.

The remainder of the paper is organized as follows. Section 2 gives a brief review of related work. In Section 3, we provide background on the fluid model for stability analysis of the Internet, building on which we develop a general model for multi-bottleneck scenarios. We investigate in Section 4 the stability properties with delay-free marking, and prove the global asymptotic stability of the fluid model system by using Lyapunov stability theory and LaSalle's Invariance Principle. Section 5 studies the multi-bottleneck system considering feedback delays. Numerical results by MATLAB and simulation results by NS-2 are presented in Section 6, followed by concluding remarks in Section 7.

2. Related work

Internet stability analysis has recently received much attention. In particular, the stability of TCP systems has been studied from the point of window-based flow control [6,13–17,26,27] and rate control [19,21]. A linear model of a RED gateway with TCP connections was developed in [13,14] to characterize the stability region of the system and guideline for setting RED parameters was provided for the single-node analysis. A nonlinear discrete-time model was proposed in [18] to further study the dynamics of TCP/RED system over large parameter variations and [20] revealed a more comprehensive reason about the oscillation of TCP/RED system from the viewpoint of nonlinear control theory. New control mechanisms such as [23] were also proposed for the Internet, aiming to achieve quick convergence to efficiency, stability, fair bandwidth sharing, and low packet loss rate.

In practice, it is very likely that flows would experience heterogeneous round-trip delays and some flows may undergo multiple bottlenecks. To date, little work has been done on the stability and analysis of multiple bottleneck networks. It has been showed in [1] that RED configuration based on a single-bottleneck assumption may not prevent traffic instability when congestion occurs at the same time in two different locations of the network. Recent work [11] studied a class of TCP/RED multiple-bottleneck model and

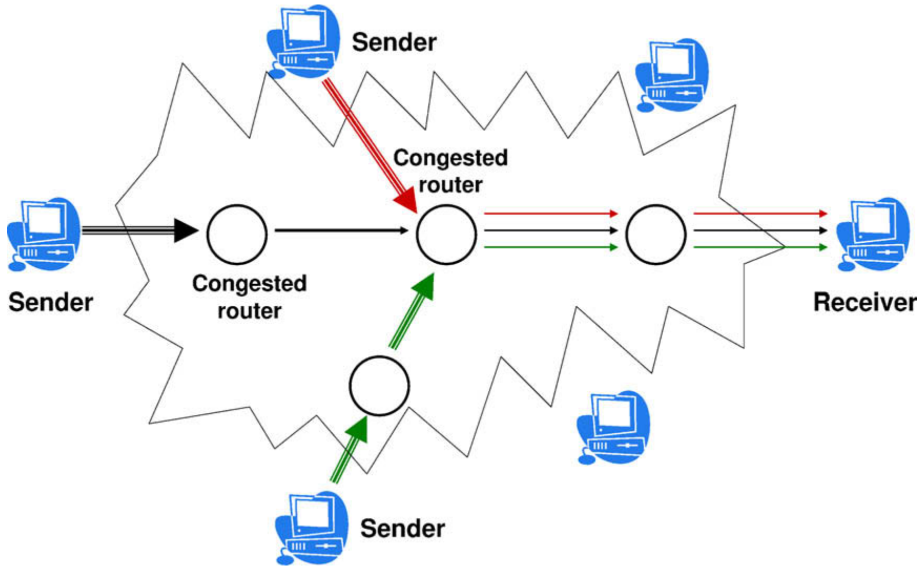


Fig. 1. General case of a multiple-bottleneck network.

tried to avoid network congestion by imposing some restrictions of AQM parameters. Middleton et al. [12] presented a matrix model and derived results that predict a degree of fairness in resource allocation between flows that compete directly with each other. In this paper, we study the general case of multiple-bottleneck AIMD/RED systems and obtain sufficient conditions for the asymptotic stability with and without feedback delays. It is illustrated that appropriate parameters for RED can be chosen to make the system asymptotically stable.

3. A class of fluid-flow models

3.1. Single-bottleneck network model

A stochastic model of TCP behavior was developed using fluid-flow analysis and stochastic differential equations [2]. Simulation results have demonstrated that this model accurately captures the dynamics of TCP. We extend the fluid-flow model for general AIMD(α,β) congestion control: the window size is increased by α packets per round-trip time (rtt) if no packet loss occurs; and otherwise, it is reduced to β times its current value [3–5].

We first consider that all AIMD-controlled flows have the same (α,β) parameter pair and round-trip delay. The AIMD/RED fluid model relates to the *ensemble averages* of key network variables and it is described by the following coupled, nonlinear differential equations:

$$\frac{dW(t)}{dt} = \frac{\alpha}{R(t)} - \frac{2(1-\beta)}{1+\beta} W(t) \frac{W(t-R(t))}{R(t-R(t))} p(t-R(t))$$
$$\frac{dq(t)}{dt} = \begin{cases} \frac{N(t)\cdot W(t)}{R(t)} - C, & q > 0 \\ \left\{\frac{N(t)\cdot W(t)}{R(t)} - C\right\}^+, & q = 0, \end{cases} \tag{1}$$

where $\{a\}^+ = \max\{a,0\}, \alpha > 0, \beta \in [0,1]$; W is the AIMD window size (in the unit of packets); q is the queue length; $R(t)$ is the round-trip time with $R(t) = \frac{q(t)}{C} + T_p$ (secs) where C is the link capacity (packets/sec) and T_p is the deterministic round-trip delay; and $N(t)$ is the number of AIMD flows and $p(t) \in [0,1]$ is the probability of a packet being marked or dropped. It should be noted that, in the fluid model, Queue length q and window size W are positive and bounded quantities; i.e. $W \in [0, W_{\max}]$ and $q \in [0, q_{\max}]$ where $W_{\max}$ and $q_{\max}$ denote buffer capacity and maximum window size, respectively. q and W approximate the ensemble averages of queue length and window size in real systems. Assuming ergodicity, the values of q and W in the fluid model can be used to approximate their time averages over a round.[1] Given the AIMD window size oscillating between $2\beta W/(1+\beta)$ and $2W/(1+\beta)$, the duration of a round equals $\frac{2(1-\beta)WR}{(1+\beta)\alpha}$.

With RED, the packet-marking (or dropping) probability, p, is proportional to the average queue length: $p = K_p(q - \min_{th})$ with $K_p > 0$ and $p \in [0,1]$, where $\min_{th}$ is the minimum threshold. When $q \leqslant \min_{th}$, the marking probability is zero, i.e., $\frac{dW(t)}{dt} = \frac{\alpha}{R}$, and the window size of AIMD flows would keep increasing till $q > \min_{th}$. In the following, we discuss the stability properties of Multiple-bottleneck model when $q > \min_{th}$. In addition, since the RED queue behaves the same as a Drop-Tail queue once the queue length exceeds the maximum threshold, $\max_{th}$, we choose $\max_{th}$ to be sufficiently large such that $p_{\max} = 1$. For convenience, we assume $\min_{th} = 0$, which does not impact the stability properties of AIMD/RED networks per se.

3.2. Multiple-bottleneck network model

A general scenario of a multiple-bottleneck AIMD/RED system is shown in Fig. 1. In the system, all AIMD flows pass through multiple links which causes more than one congested routers. The thick lines with arrow in the figure

[1] *A round* is defined as time interval between two instants at which the sender reduces its window size consecutively.

represent the size of traffic on each link and the traffic becomes smaller each time after passing through a congested router. Assume that a packet can only be marked at most once, following the idea of modeling in [27], a multiple-bottleneck AIMD/RED system that contains N groups of AIMD flows and M congested links can be mathematically modeled as follows:

$$\frac{dW_1(t)}{dt} = \frac{\alpha_1}{R_1(t)} - \frac{2(1-\beta_1)}{1+\beta_1} W_1(t) \frac{W_1(t-R_1(t))}{R_1(t-R_1(t))} \times \sum_{i\in r(1)} (K_{p_i} q_i(t-R_1(t))),$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\frac{dW_N(t)}{dt} = \frac{\alpha_N}{R_N(t)} - \frac{2(1-\beta_N)}{1+\beta_N} W_N(t) \frac{W_N(t-R_N(t))}{R_N(t-R_N(t))} \times \sum_{j\in r(N)} (K_{p_j} q_j(t-R_N(t))),$$

$$\frac{dq_1(t)}{dt} = \begin{cases} \sum_{n\in f(1)} \frac{N_n W_n(t)}{R_n(t)} - C_1, & q_1 > 0 \\ \left\{ \sum_{n\in f(1)} \frac{N_n W_n(t)}{R_n(t)} - C_1, \right\}^+, & q_1 = 0 \end{cases}$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\frac{dq_M(t)}{dt} = \begin{cases} \sum_{m\in f(M)} \frac{N_m W_m(t)}{R_m(t)} - C_M, & q_M > 0 \\ \left\{ \sum_{m\in f(M)} \frac{N_m W_m(t)}{R_m(t)} - C_M, \right\}^+, & q_M = 0, \end{cases} \tag{2}$$

where $r(i), i = 1, \ldots, N$, denotes the set of congested routers that flow i passes through, and $f(m), m = 1, \ldots, M$, denotes the set of flows that pass through the congested router m.

4. Stability analysis with delay-free marking

Stability properties of the single-bottleneck AIMD/RED system with delay-free marking, i.e., no time delay term is involved in the packet-marking function, have been presented in [6,26]. When studying the multiple-bottleneck system, we first take delay-free case into consideration for a comparison with the single-bottle system. Our analysis in the paper also show that stability properties with delay-free marking are different from those with delayed marking. These results provide theoretical support on how time delay will affect the stability of AIMD/RED systems with multiple bottlenecks.

In this section, we study the dynamics of the multi-bottleneck networks in the absence of feedback delays by using Lyapunov stability theory and LaSalle's Invariance Principle. Assume that the round-trip time R_i is time-invariant, i.e., $R_i(t) = R_i$ for $i = 1, 2, \ldots, N$. We shall show that the equilibrium point of this delay-free system is globally asymptotically stable for all positive gains.

For delay-free marking multiple-bottleneck AIMD/RED system, the equilibrium point $(W_1^*, \ldots, W_N^*, q_1^*, \ldots, q_M^*)$ can be obtained by

$$2(1-\beta_1) \cdot (W_1^*)^2 \left(\sum_{i\in r(1)} (K_{p_i} q_i^*) \right) = \alpha_1(1+\beta_1),$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$2(1-\beta_N) \cdot (W_N^*)^2 \left(\sum_{j\in r(N)} K_{p_j} q_j^* \right) = \alpha_N(1+\beta_N), \quad \sum_{n\in f(1)} N_n \cdot W_n^*/R_n = C_1, \tag{3}$$

$$\cdots\cdots \quad \cdots\cdots$$

$$\sum_{m\in f(M)} N_m \cdot W_m^*/R_m = C_M$$

One observation is that, if all flows have the same AIMD parameter pair, the flow that traverses more bottlenecks always suffers more packet losses than other flows, and its window size is always smaller than those of others.

Remark 1. The analysis throughout this paper is about the stability property of the equilibrium point of system (2). Since the equilibrium point is typically inside the desired operating region of the system, its stability property, i.e., the convergence of system trajectories to the desired operating area (especially, the equilibrium point), will guarantee network performance in terms of packet loss, delay, and jitter.

With the transformed variables $\widetilde{W}_i(t) = W_i(t) - W_i^*$, for $i = 1, \ldots, N; \tilde{q}_j(t) = q_j(t) - q_j^*$, for $j = 1, \ldots, M$; we can use the following Lyapunov function to establish the asymptotic stability of delay-free marking system:

$$
\begin{aligned}
&V(\widetilde{W}_1(t), \ldots, \widetilde{W}_N(t), \tilde{q}_1(t), \ldots, \tilde{q}_M(t)) \\
&\quad = \frac{1}{2}\sum_{i=1}^{N} \frac{(1+\beta_i)N_i}{(1-\beta_i)\widetilde{W}_i^{*2}} \widetilde{W}_i^2(t) + \frac{1}{2}\sum_{j=1}^{M} K_{p_j}\tilde{q}_j^2(t)
\end{aligned} \tag{4}
$$

The time-derivative of V along the solution of system (2) is non-positive, i.e., $\dot{V} \leqslant 0$. By applying LaSalle's Invariance Principle, all the trajectories converge to the unique equilibrium point of system (2). Thus, the global asymptotic stability of system (2) is obtained (see Appendix A. for details). The results can be summarized by the following theorem.

Theorem 1. *For any $K_{p_1} > 0, \ldots, K_{p_M} > 0$, the equilibrium point of the delay-free marking AIMD/RED system is globally asymptotically stable for any positive pairs $(\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)$ and any positive $R_1, \ldots, R_N$.*

In the above analysis, the AIMD parameter pairs for all the flows in group $i, i = 1, \ldots, N$, are the same. In reality, there are always different kinds of AIMD flows within one group. As an example, we consider the case when two types of AIMD flows are within the group I: N_{I1} AIMD $(\alpha_{I1}, \beta_{I1})$ flows denoted by W_{I1}, and N_{I2} AIMD $(\alpha_{I2}, \beta_{I2})$ flows denoted by W_{I2}, with round-trip time R_{I1} and R_{I2}, respectively. In this case, we can still obtain the globally asymptotic stability by choosing the following proper Lyapunov function and LaSalle's Invariance Principle (see Appendix B. for details).

Remark 2. Note that a similar analysis can be carried out for more general cases, i.e., when there are more than two kinds of AIMD flows in each group sharing the link capacities. For this, the corresponding mathematical models can be constructed along similar lines as above, by extending the model (2) to higher dimensions to include more terms, each representing another type of flow.

Remark 3. Time delay in the packet-marking can be ignored in the case with very high bandwidth and short distance communication links. The analysis with "delay-free marking" assumption offers some insight for the ultra high speed networks.

5. Stability analysis with feedback delays

5.1. Stability criteria for general multiple-bottleneck systems

In this section, we study the stability properties of the delayed system (2) in Section 3. With ever-increasing link capacity and appropriate congestion control mechanism, variation of queuing delays becomes relatively small to propagation delays. In fact, recent work [22] reveals that the variable nature of *rtt* due to queueing delay variation helps to stabilize the TCP/RED system. Therefore, we can ignore the effect of the delay jitter on the round-trip time and derive sufficient conditions for the asymptotic stability of multiple-bottleneck system assuming *rtt* to be constant. Clearly, these sufficient conditions will be still applicable if *rtt* is actually time-varying.

The equilibrium points $(W_1^*, \ldots, W_N^*, q_1^*, \ldots, q_M^*)$ of system (2) are defined by (3) with $R_i = R_i^*$ for $i = 1, \ldots, N$, where $R_i^* = T_{p_i} + \sum_{j \in r(i)} \frac{q_j^*}{C_j}$.

Due to the highly nonlinear nature and the effect of delays in the system, no suitable Lyapunov function could be constructed to prove global asymptotic stability of the equilibrium. We linearize system (2) about the equilibrium point (by doing so, we ignore the dependence of the time delay argument in $q(t - R_i)$ and fix the time-varying delay to its equilibrium value R_i^*, for $i = 1, \ldots, N$.) and write it in the following form:

$$
\dot{x}(t) = \overline{A}x(t) + \sum_{i=1}^{N} \overline{B}_i x(t - R_i^*), \tag{5}
$$

with $x = (\widetilde{W}_1(t), \ldots, \widetilde{W}_N(t), \tilde{q}_1(t), \ldots, \tilde{q}_M(t))^T$, $\overline{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$, $\overline{B} = \begin{bmatrix} B_{i11} & B_{i12} \\ 0 & 0 \end{bmatrix}$, where A_{ij}, B_{i11} and B_{i12} are known real constant matrices with appropriate dimensions with following forms:

$$
A_{11} = \begin{bmatrix} -\frac{\alpha_1}{R_1^* W_1^*} & 0 & \ldots & 0 \\ 0 & -\frac{\alpha_2}{R_2^* W_2^*} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -\frac{\alpha_N}{R_N^* W_N^*} \end{bmatrix}
$$

$$
(A_{21})_{ij} = \begin{cases} \frac{\lambda_j}{R_j^*}, & \text{if} \quad j \in r(i) \\ 0, & \text{otherwise} \end{cases}.
$$

$$
(A_{22})_{ij} = \begin{cases} -\frac{1}{C_j} \sum\limits_{l \in f(i)} \frac{\lambda_l W_l^*}{R_l^{*2}}, & \text{for} \quad i = j \\ -\frac{1}{C_j} \sum\limits_{l \in f(i) \cap f(j)} \frac{\lambda_l W_l^*}{R_l^{*2}}, & \text{otherwise} \end{cases}.
$$

$$
(B_{i11})_{jk} = \begin{cases} -\frac{\alpha_i}{R_i^* W_i^*}, & \text{for} \quad j = k = i \\ 0, & \text{otherwise} \end{cases}.
$$

$$
(B_{i12})_{jk} = \begin{cases} -\frac{2(1-\beta_i)}{1+\beta_i} \frac{W_i^{*2}}{R_i^*} K_{pk}, & \text{for} \quad j = i \quad \text{and} \quad k \in f(i) \\ 0 & \text{otherwise}. \end{cases}.
$$

It can be checked by the Routh Criterion that $\overline{A}$ is a Hurwitz matrix, which implies that for any positive definite matrix Q, there exists positive definite matrix P, such that $\overline{A}^T P + P\overline{A} = -Q$. We next give some sufficient conditions for the local asymptotic stability of system (2) by applying the direct method of Lyapunov. Let $M = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$, where $\lambda(P)$ denotes eigenvalues of matrix P, we can obtain a sufficient condition to guarantee the local asymptotic stability of the multiple-bottleneck system.

Theorem 2. *If there exists positive definite P, Q satisfying $\overline{A}^T P + P\overline{A} = -Q$ such that matrix $Q - 2M \cdot \left(\sum_{i=1}^{N} \|P\overline{B}_i\|\right) \cdot I$ is positive definite, then the equilibrium point of (2) is locally asymptotically stable.*

Proof: With (5), we choose Lyapunov function $V(x) = x^T P x$, then

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\
&= [\overline{A}x(t) + \sum_{i=1}^{N} \overline{B}_i x(t - R_i^*)]^T P x + x^T P[\overline{A}x(t) + \sum_{i=1}^{N} \overline{B}_i x(t - R_i^*)] \\
&= x^T(t)(\overline{A}^T P + P\overline{A})x(t) + 2\sum_{i=1}^{N} x^T(t - R_i^*)\overline{B}_i^T P x(t) \\
&= -x^T(t) Q x(t) + 2\sum_{i=1}^{N} x^T(t - R_i^*)\overline{B}_i^T P x(t).
\end{aligned}$$

Let $R^* = \max\{R_1^*, \ldots, R_N^*\}$. Applying the Lyapunov–Razumikhin condition, with $\mu > 1$ such that

$$V(\xi) \leqslant \mu^2 V(t) \quad \text{for} \quad t - R^* \leqslant \xi \leqslant t,$$

which implies that $\|x(\xi)\| \leqslant M \cdot \mu \cdot \|x(t)\|$. Thus,

$$\begin{aligned}
\dot{V} &\leqslant -x^T(t) Q x(t) + 2\|x(t - R^*)\| \left(\sum_{i=1}^{N} \|\overline{B}_i^T P\|\right) \|x(t)\| \\
&\leqslant -x^T(t)[Q - 2\mu M \left(\sum_{i=1}^{N} \|P\overline{B}_i\|\right) I] x(t),
\end{aligned}$$

thereby establishing the asymptotic stability of system (2). □

Observe that the Lyapunov–Razumikhin condition is used in Theorem (2) to deal with the delayed terms in $\dot{V}$. Lyapunov functional is another method that can be applied when studying the stability of delayed systems. Our next result gives another sufficient condition for the local asymptotic stability of system (2) in terms of linear matrix equality by applying the method of Lyapunov functional.

Theorem 3. *If there exist positive definite P, Q satisfying $\overline{A}^T P + P\overline{A} = -Q$ and positive definite G_i for $i = 1, \ldots, N$ such that the following matrix is positive definite:*

$$\begin{bmatrix}
Q - \sum_{i=1}^{N} G_i & -PB_1 & \ldots & -PB_N \\
-B_1^T P & G_1 & 0 & 0 \\
\vdots & 0 & \ddots & 0 \\
-B_N^T P & 0 & 0 & G_N
\end{bmatrix} > 0,$$

then the equilibrium point of (2) is locally asymptotically stable.

Proof. With (5), we choose Lyapunov functional

$$V(x) = x^T P x + \sum_{i=1}^{N} \int_{t-R_i^*}^{t} x^T(s) G_i x(s) ds,$$

then

$$\begin{aligned}
\dot{V} &= x^T(t)(\overline{A}^T P + P\overline{A})x(t) + 2\sum_{i=1}^{N} x^T(t - R_i^*)\overline{B}_i^T P x(t) \\
&\quad + x^T(t)\left(\sum_{i=1}^{N} G_i\right)x(t) - \sum_{i=1}^{N} x^T(t - R_i^*) G_i x(t - R_i^*) \\
&= -x^T(t)\left(Q - \sum_{i=1}^{N} G_i\right)x(t) + 2\sum_{i=1}^{N} x^T(t - R_i^*) B_i^T P x(t) \\
&\quad - \sum_{i=1}^{N} x^T(t - R_i^*) G_i x(t - R_i^*) \\
&= -(x^T(t), x^T(t - R_1^*), \ldots, x^T(t - R_N^*)) \\
&\quad \cdot D \cdot (x^T(t), x^T(t - R_1^*), \ldots, x^T(t - R_N^*))^T,
\end{aligned}$$

where D denotes the matrix $\begin{bmatrix}
Q - \sum_{i=1}^{3} G_i & -PB_1 & \ldots & -PB_N \\
-B_1^T P & G_1 & 0 & 0 \\
\vdots & 0 & \ddots & 0 \\
-B_N^T P & 0 & 0 & G_N
\end{bmatrix}$. Thus, system (2) is locally asymptotically stable if D is positive definite. □

It is worth pointing out that sufficient conditions derived in Theorems 2 and 3 are both given in terms of linear matrix inequalities (LMI). These conditions can be easily assessed by applying the LMI Control Toolbox with Matlab [28].

In general, Theorems 2 and 3 shed some light on how the network parameters impact the network stability. Specifically, we have the following intuitive interpretation of the conditions in these theorems. To guarantee the local asymptotic stability of system (2), $\dot{V}$ in Theorems 2 and 3 is required to be negative definite. It can be seen from the proof that the more negative $\overline{A}^T P + P\overline{A}$ and the smaller $\|P\overline{B}_i\|$, $i = 1, \ldots, N$, the more likely $\dot{V} < 0$. In other words, the term $\overline{A}^T P + P\overline{A}$ should be dominant in $\dot{V}$ and the absolute values of $\lambda(\overline{A})$ are expected to be sufficiently large. Notice that $\overline{A}$ has been checked to be a Hurwitz matrix and $W_i^*, i = I, \ldots, N$ has the form of RC/N. From the expression of $\overline{A}$ and $\overline{B}_i$, we know that the smaller the terms $R_i^*, i = 1, \ldots, N$, $C_j, j = 1, \ldots, M$, the larger the absolute values of $\lambda(\overline{A})$ and the smaller the $\|PB_i\|$, and hence the better the chance that the system is asymptotically stable. These observations are also consistent with those in [15]: TCP/RED will become unstable when delay increases, or when link capacity increases.

5.2. Case study: a class of two-bottleneck topology

In this section, we consider a basic multi-bottleneck scenario, as depicted in Fig. 2. Three groups of flows are sharing the links between four routers. AIMD flows in group I compete with flows in group II over link L_1, and also compete with 50 flows in group III over link L_2. We assume all routers are RED-enabled and there is no packet loss and delay jitter in the non-bottleneck links. All routers are RED-enabled. Links L_1 and L_2 are bottlenecks with the capacity of C_1 and C_2, respectively. The round-trip delays for the three groups of traffic are R_1, R_2, and R_3, respectively. The results with this topology are also applicable to the scenar-

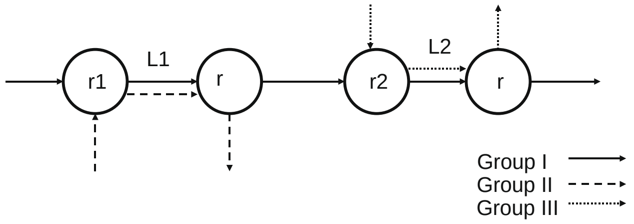


Fig. 2. Multiple-bottleneck topology.

ios when the three groups of flows traverse other non-bottleneck links before/after they enter/leave L_1 or L_2.

In this multi-bottleneck topology, let K_{p_1} and K_{p_2} denote the marking probability on L_1 and L_2, and (α_1, β_1), (α_2, β_2) and (α_3, β_3) be AIMD parameter pairs for the three groups of flows, respectively. For the first group of flows, the marking probabilities on L_1 and L_2 are $p_1(t-R_1) = K_{p_1}q_1(t-R_1)$ and $p_2(t-R_1) = K_{p_2}q_2(t-R_1)$, respectively. Since we assume that a packet can only be marked at most once, the probability of a flow I packet receiving a mark is $p_1(t-R_1) + p_2(t-R_1) - p_1(t-R_1)p_2(t-R_1)$. The marking probability can be approximated by $p_1(t-R_1) + p_2(t-R_1)$ given that p_1 and p_2 are very small. The closed-loop dynamics can be modeled as follows:

$$
\begin{aligned}
\frac{dW_I(t)}{dt} &= \frac{\alpha_1}{R_1(t)} - \frac{2(1-\beta_1)}{1+\beta_1} W_I(t) \frac{W_I(t-R_1)}{R_1(t-R_1)} (K_{p_0} q_0(t-R_1) + K_{p_2} q_2(t-R_1)) \\
\frac{dW_{II}(t)}{dt} &= \frac{\alpha_2}{R_2(t)} - \frac{2(1-\beta_2)}{1+\beta_2} W_{II}(t) \frac{W_{II}(t-R_2)}{R_2(t-R_2)} K_{p_0} q_0(t-R_2) \\
\frac{dW_{III}(t)}{dt} &= \frac{\alpha_3}{R_3(t)} - \frac{2(1-\beta_3)}{1+\beta_3} W_{III}(t) \frac{W_{III}(t-R_3)}{R_3(t-R_3)} K_{p_2} q_2(t-R_3) \\
\frac{dq_1(t)}{dt} &= \begin{cases} \frac{N_1 W_I(t)}{R_1(t)} + \frac{N_2 W_{II}(t)}{R_2(t)} - C_1, & q_0 > 0 \\ \{\frac{N_1 W_I(t)}{R_1(t)} + \frac{N_2 W_{II}(t)}{R_2(t)} - C_1\}^+, & q_0 = 0 \end{cases} \\
\frac{dq_2(t)}{dt} &= \begin{cases} \frac{N_1 W_I(t)}{R_1(t)} + \frac{N_3 W_{III}(t)}{R_3(t)} - C_2, & q_2 > 0 \\ \left\{\frac{N_1 W_I(t)}{R_1(t)} + \frac{N_3 W_{III}(t)}{R_3(t)} - C_2\right\}^+, & q_2 = 0. \end{cases}
\end{aligned}
\tag{6}
$$

Next, we give a numerical example to get a more concrete sense of the sufficient conditions in Theorem 2 on local asymptotic stability for the AIMD/RED system with heterogeneous delays. Let $N_1 = N_2 = N_3 = 5, C_1 = 3 \times 10^3$ packet/sec, $C_2 = 5 \times 10^3$ packet/sec with $K_{p1} = K_{p2} = 0.0005$. Choose $(\alpha_1, \beta_1) = (1,\ 0.5)$ with $T_{p1} = 0.020$ s, $(\alpha_2, \beta_2) = (0.2,\ 0.875)$ with $T_{p2} = 0.013$ s and $(\alpha_3, \beta_3) = (1,\ 0.5)$ with $T_{p3} = 0.007$ s, respectively. Solving the LMI in Theorem 2 with Matlab Control Toolbox, one feasible solution we obtain is as follow: positive definite matrix

$$
Q = 10^3 * \begin{bmatrix} 4.2596 & -1.2369 & 2.3752 & -1.8226 & -1.9184 \\ -1.2369 & 4.5479 & -3.1861 & -2.0736 & 0.8033 \\ 2.3752 & -3.1861 & 2.8241 & 0.5329 & -1.2195 \\ -1.8226 & -2.0736 & 0.5329 & 2.4057 & 0.6722 \\ -1.9184 & 0.8033 & -1.2195 & 0.6722 & 0.8817 \end{bmatrix}
$$

and

$$
P = \begin{bmatrix} 2.217 & -2.6696 & 2.4213 & 0.3497 & -1.091 \\ -2.669 & 4.9555 & -3.8606 & -1.3247 & 1.439 \\ 2.421 & -3.8606 & 3.3280 & 0.9250 & -1.279 \\ 0.349 & -1.3247 & 0.9250 & 0.6115 & -0.229 \\ -1.091 & 1.4386 & -1.2789 & -0.2295 & 0.559 \end{bmatrix}.
$$

We can also check that the eigenvalues of matrix $Q - 2M(\|PB_1\| + \|PB_2\| + \|PB_3\|)I$ are: 1.0e+003*[9.0769, 5.8269, 0.0088, 0.0044, 0.0001], which implies that $Q - 2M(\|PB_1\| + \|PB_2\| + \|PB_3\|)I$ is positive definite. Thus, the condition of Theorem 2 holds and the system is locally asymptotically stable. Simulation results using the same parameters will be given in Section 6.

Remark 4. Notice that Theorems 2 and 3 give two different sets of sufficient conditions for the asymptotic stability of system (6). These conditions can be easily checked by the LMI Toolbox, which allow us to use any of them at our convenience.

Remark 5. By the similar idea of this section, we can obtain the local stability of the network when it is shared by more than three groups of flows as well. Mathematical models can be established following the idea in Section 3.2 and the technique used in this section can be applied to obtain sufficient conditions, in terms of LMI, for asymptotic stability of any given scenarios.

Remark 6. We note that the results in this section are for local stability only, whereas the results obtained in Section 4 are for global stability. This is due to the difficulty in constructing a suitable Lyapunov-type function for the nonlinear multiple-bottleneck AIMD/RED system with heterogeneous delays. A plausible approach to resolve this issue is to develop a sequence of upper and lower bounds of system trajectories and use these bounds in Razumikhin's theorem to derive conditions for global stability in the presence of heterogeneous delays, and our study along this line is underway. Also, studying the stability properties of the general case of multiple-bottleneck AIMD/RED networks by directly using the model (2) is an important open issue for further investigation.

6. Numerical results and performance evaluation

With the two-bottleneck topology described in Section 5, we first obtain the system evolution trajectories by using *Matlab* to verify the asymptotic stability proved in Sections

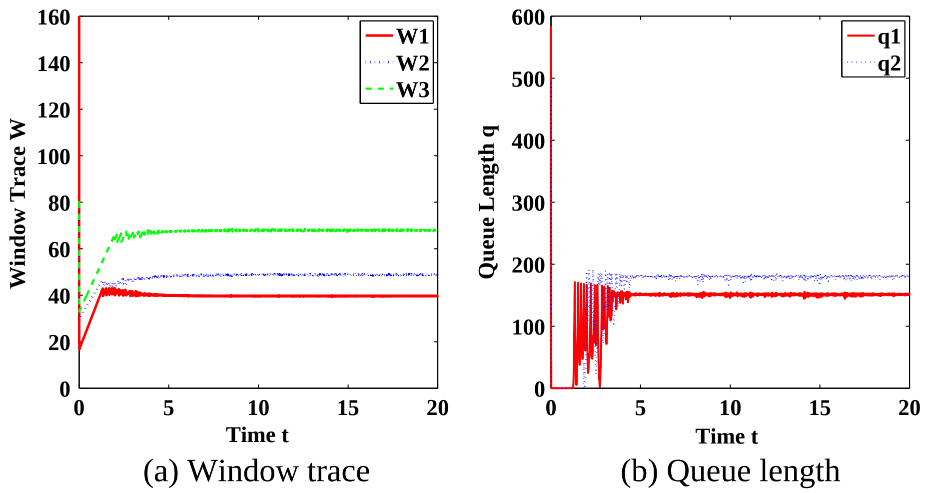


Fig. 3. Homogeneous TCP flows, delay-free.

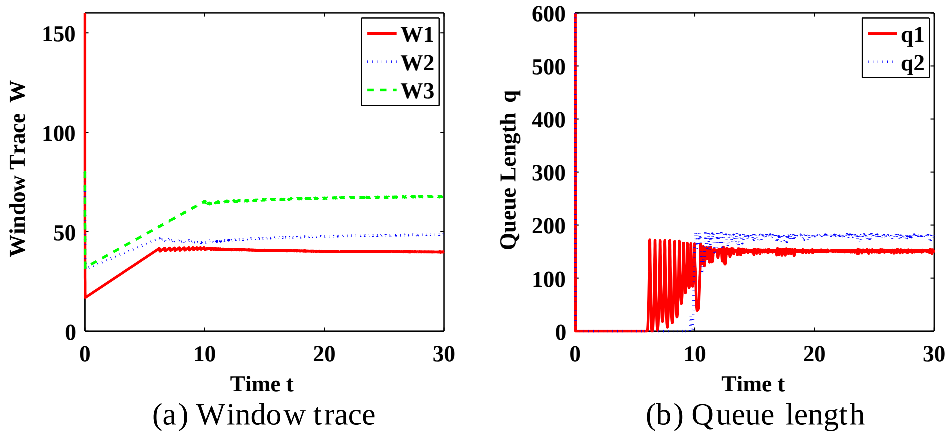


Fig. 4. Homogeneous AIMD(0.2, 0.875) flows, delay-free.

4 and 5. Network simulator, NS-2, is then used to further study the performance of the systems.

6.1. Numerical results

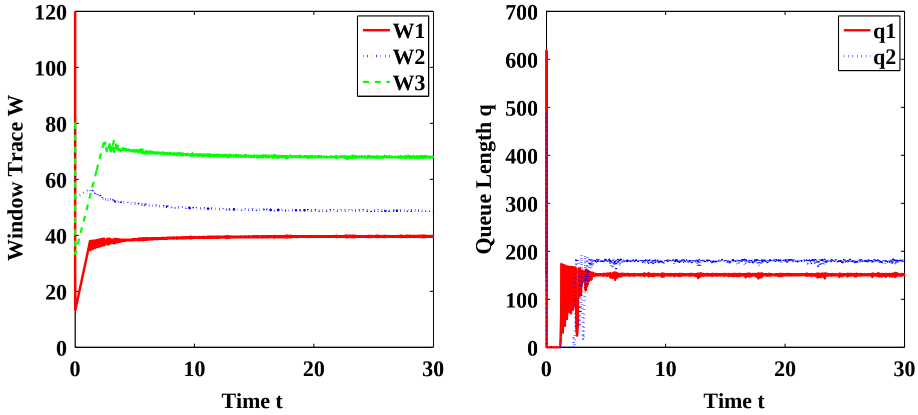
6.1.1. System without feedback delays

Figs. 3–5 show the traces of window size and queue length under the topology of Fig. 2, modeled by (6). The capacity of L_1 is $C_1 = 1 \times 10^5$ packet/sec, that of L_2 is $C_2 = 12 \times 10^4$ packet/sec. The number of flows in each groups are $N_1 = 80$, $N_2 = 60$ and $N_3 = 50$, respectively. The deterministic round-trip times of these groups are $T_{p_1} = 0.05$ s, $T_{p_2} = 0.08$ s and $T_{p_3} = 0.06$ s, respectively. We choose $K_{p_1} = 0.0006, K_{p_2} = 0.0008, Q_{min1} = 150$ packets and $Q_{min2} = 180$ packets.

In Fig. 3, all flows are TCP flows, i.e., $(\alpha, \beta) = (1, 0.5)$. In Fig. 4, all flows are AIMD flows with the same parameter pair, $(\alpha, \beta) = (0.2, 0.875)$. W_i in Figs. 3a and 4a represents the average window size of flows in the i-th group, and q_1 and q_2 in Figs. 3b and 4b represent the bottleneck queue lengths at r_1 and r_2, respectively. It can be seen both the average window sizes and queue lengths converge to constants in steady state. Although the convergence speed of homogeneous TCP flows is faster than that of homogeneous AIMD flows, their average windows and the average queue lengths in steady state are the same.

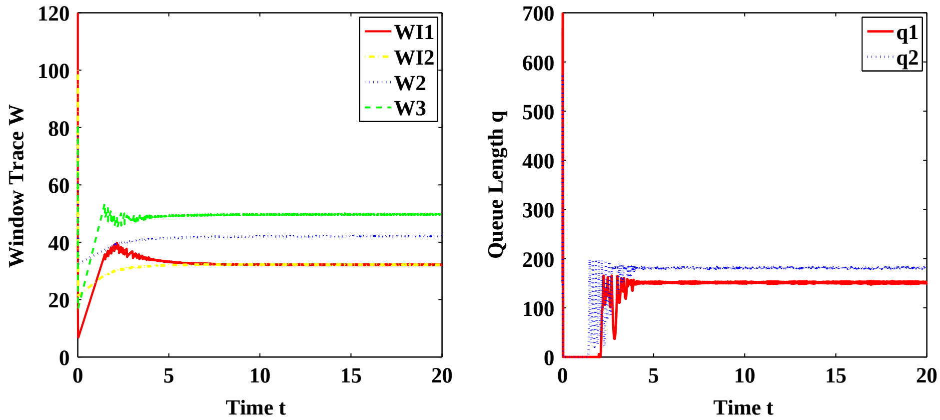
We further investigate the case that different groups of flows use different AIMD parameters. The flow parameters of the three groups in Fig. 5 are $(\alpha_1, \beta_1) = (1, 0.5)$, $(\alpha_2, \beta_2) = (0.2, 0.875)$ and $(\alpha_3, \beta_3) = (1, 0.5)$, respectively. The numerical results show that the average window sizes of the three groups of flows and queue lengths of the two-bottleneck routers converge to constants. Since all the trajectories are asymptotically stable, thereby validating the Theorem 1. In addition, the average window sizes of each groups in Figs. 3–5 are the same in steady state, which means AIMD $(0.2, 0.875)$ flows are TCP-friendly.[2] This property can be further illustrated in the following case.

[2] *TCP-friendliness* is defined as the average throughput of non-TCP-transported flows over a large time scale does not exceed that of any conformant TCP-transported ones under the same circumstance [10]. It has been shown that if an AIMD flow with the parameter pair satisfying the condition $\frac{\alpha(1+\beta)}{1-\beta} = 3$, the AIMD flow is TCP-friendly [5,26].



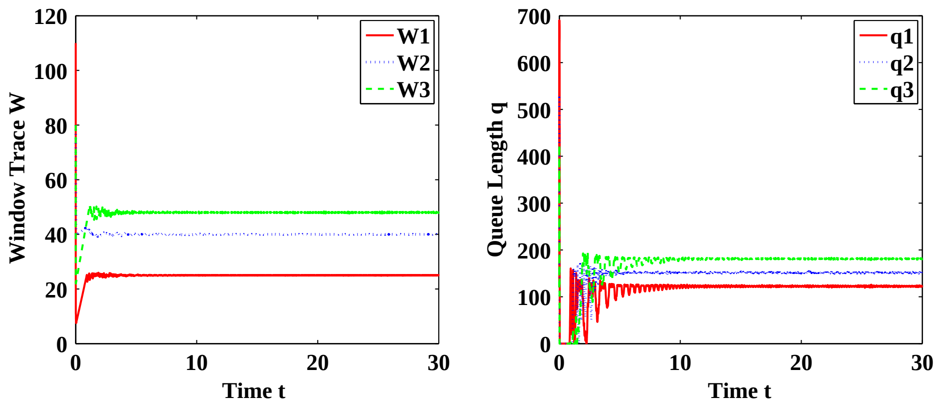
(a) Window trace (b) Queue length

Fig. 5. TCP and AIMD(0.2, 0.875) flows, delay-free.



(a) Window trace (b) Queue length

Fig. 6. TCP and AIMD(0.2,0.875) flows, delay-free, heterogeneous traffic in group I.



(a) Window trace (b) Queue length

Fig. 7. TCP and AIMD(0.2, 0.875) flows, delay-free, three bottleneck links.

The traces of window size and queue lengths when there are two different classes of flows in group I are shown in Fig. 6, which is modeled by (B.1). Here the number of flows within each group is chosen as $N_{11} = N_{12} = 40$, $N_2 = 60$ and $N_3 = 50$. Their deterministic *rtts* are $T_{p_{11}} = 0.05$ s, $T_{p_{12}} = 0.04$ s, $T_{p_2} = 0.06$ s and $T_{p_3} = 0.04$ s, respectively. Also, we have $C_1 = 1 \times 10^5$ packet/sec and $C_2 = 1.2 \times 10^5$ packet/sec as in Figs. 3–5 with $K_{p_1} = 0.0006$

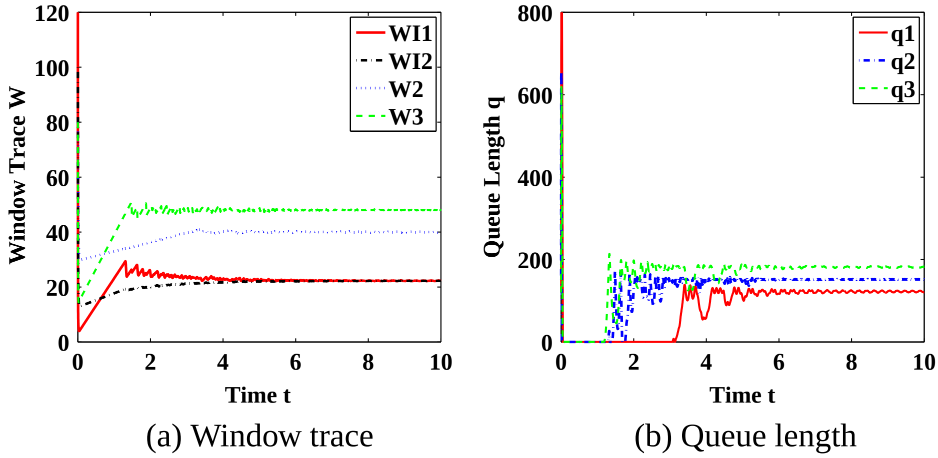


Fig. 8. TCP and AIMD(0.2,0.875) flows, delay-free, three bottleneck links.

and $K_{p_2} = 0.0008$. The AIMD parameter pairs in this case are $(\alpha_{11}, \beta_{11}) = (\alpha_3, \beta_3) = (1, 0.5)$ and $(\alpha_{12}, \beta_{12}) = (\alpha_2, \beta_2) = (0.2, 0.875)$, respectively. It can be seen that both the window size and queue length are asymptotically stable and are consistent with our analysis, and the AIMD$(0.2, 0.875)$ flows are truly TCP-friendly.

Figs. 7 and 8 show how the window size and queue length evolve when the link capacity of rr_2 for group I flows, C_3, is so small that the link rr_2 becomes the third bottleneck. Consequently, there are three bottlenecks in the network under the topology shown in Fig. 2. We choose $N_1 = 80, N_2 = 60$ and $N_3 = 50, C_1 = 8 \times 10^4$ packet/sec, $C_2 = 1 \times 10^5$ packet/sec and $C_3 = 4 \times 10^4$ packet/sec with $K_{p_1} = 0.0004, K_{p_2} = 0.0006$ and $K_{p3} = 0.0008$, respectively. The deterministic *rtts* are chosen as $T_{p_1} = 0.05$ s, $T_{p_2} = 0.06$ s and $T_{p_3} = 0.04$ s. In Fig. 7, $(\alpha_1, \beta_1) = (\alpha_3, \beta_3) = (1, 0.5)$, and $(\alpha_2, \beta_2) = (0.2, 0.875)$. In Fig. 8, there are two types of flows in group I, with $N_{11} = 40$, $N_{12} = 40$; and $T_{p_{11}} = 0.05$ s, $T_{p_{12}} = 0.04$ s. Other parameters are chosen as $(\alpha_{11}, \beta_{11}) = (\alpha_3, \beta_3) = (1, 0.5)$, $(\alpha_{12}, \beta_{12}) = (\alpha_2, \beta_2) = (0.2, 0.875)$. We can observe the property of the asymptotic stability of these systems from the numerical results.

6.1.2. System with feedback delays

Figs. 5–8 show the asymptotic stability of the multiple-bottleneck system without feedback delays, in which the property of stability is global. Figs. 9–11 illustrate the local asymptotic stability of the system with feedback delays. We choose $N_1 = N_2 = N_3 = 5, C_1 = 3 \times 10^3$ packet/sec, $C_2 = 5 \times 10^3$ packet/sec with $K_{p_1} = K_{p_2} = 0.0005$. The deterministic *rtts* for the flows are chosen as $T_{p_1} = 0.020$ s, $T_{p_2} = 0.013$ s and $T_{p_3} = 0.007$ s, respectively. The parameters used are the same as those in the numerical example of Theorem 2. In Fig. 9, $(\alpha_i, \beta_i) = (1, 0.5)$ for $i = 1, 2, 3$; in Fig. 10, $(\alpha_i, \beta_i) = (0.2, 0.875)$ for $i = 1, 2, 3$; and in Fig. 11, $(\alpha_1, \beta_1) = (\alpha_3, \beta_3) = (1, 0.5)$, $(\alpha_2, \beta_2) = (0.2, 0.875)$. As shown in the figures, all the trajectories are locally asymptotically stable, and the numerical results validate the theorems.

In the last part of this section, we give an example of an unstable multiple-bottleneck RED network. We choose $N_1 = N_3 = 4$, $N_2 = 8$, $C_1 = 1000$ packet/sec, $C_2 = 1000$ packet/sec with $K_{p_1} = K_{p_2} = 0.05$ and $(\alpha_i, \beta_i) = (1, 0.5)$ for $i = 1, 2, 3$ with $T_{p_1} = 0.03$ s, $T_{p_2} = 0.03$ s and $T_{p_3} = 0.04$ s. This case has been shown unstable in [1] and it is

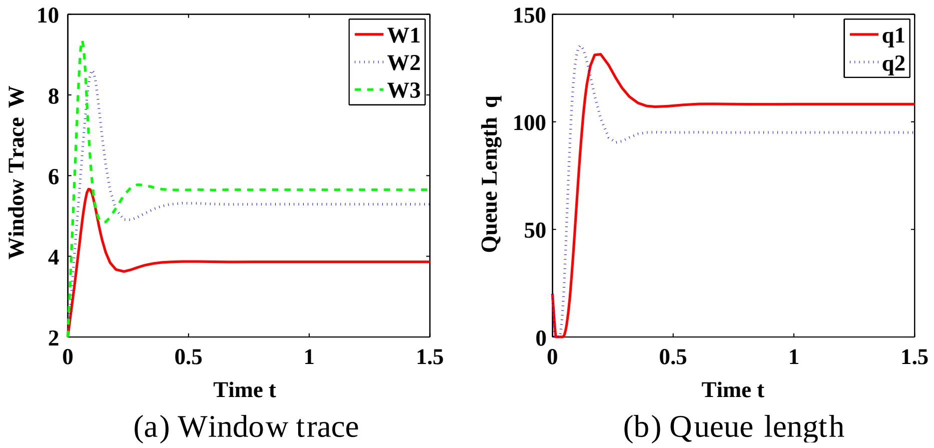


Fig. 9. Homogeneous TCP flows, with feedback delay.

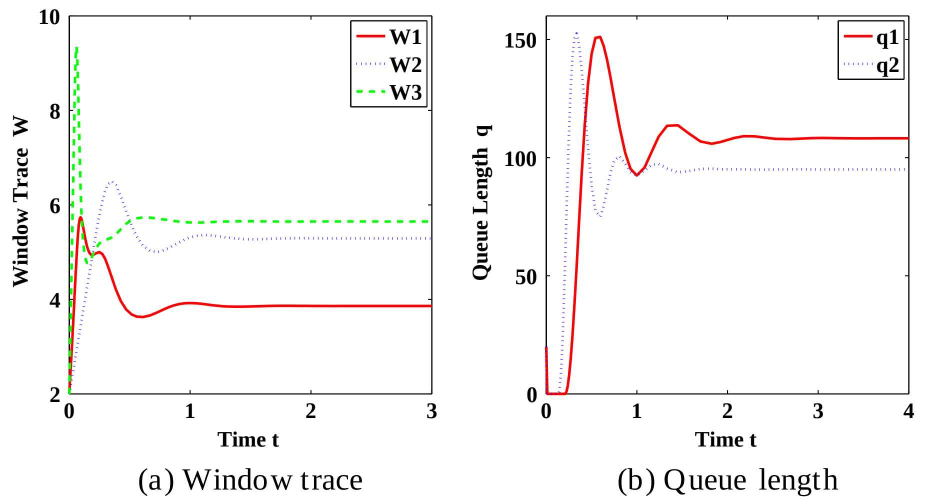


Fig. 10. Homogeneous AIMD flows, with feedback delay.

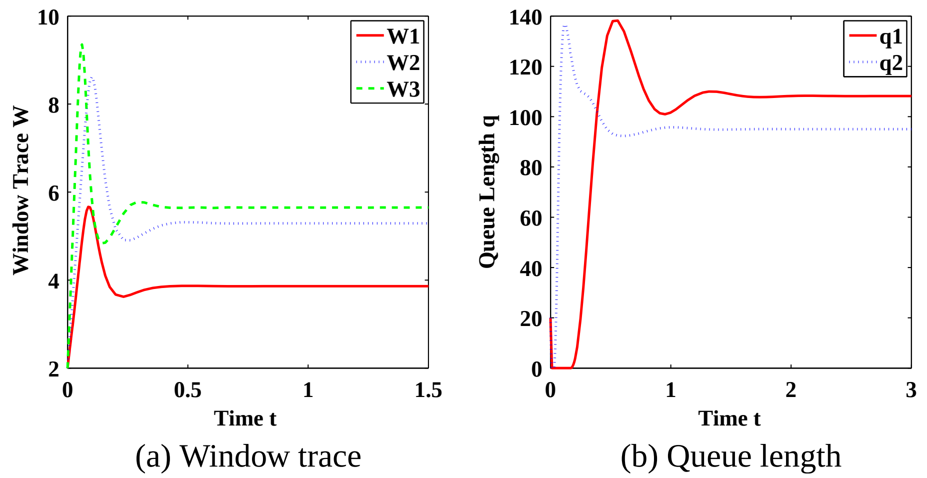


Fig. 11. TCP and AIMD flows, with feedback delay.

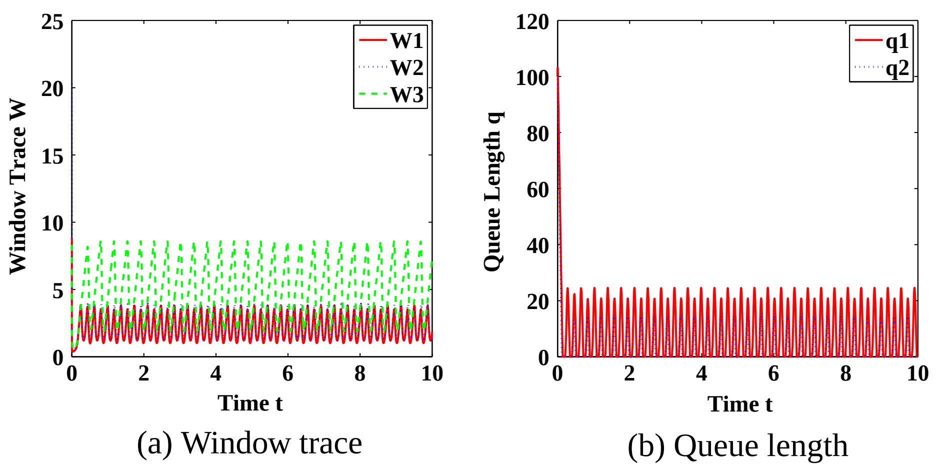


Fig. 12. Homogeneous TCP flows: unstable case.

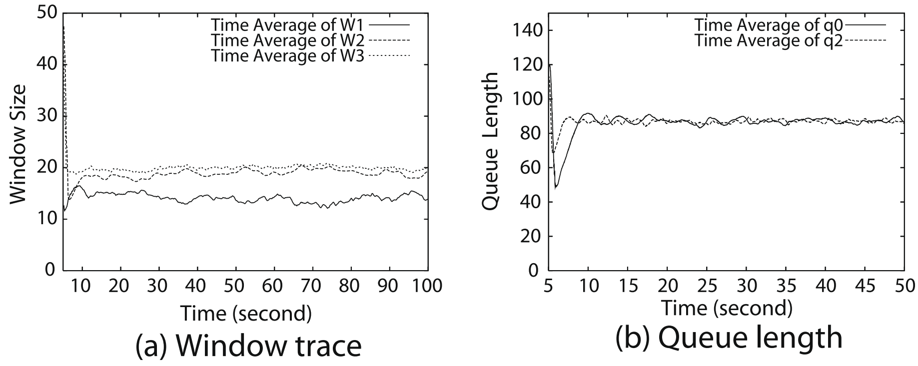


Fig. 13. Simulation results for a stable system.

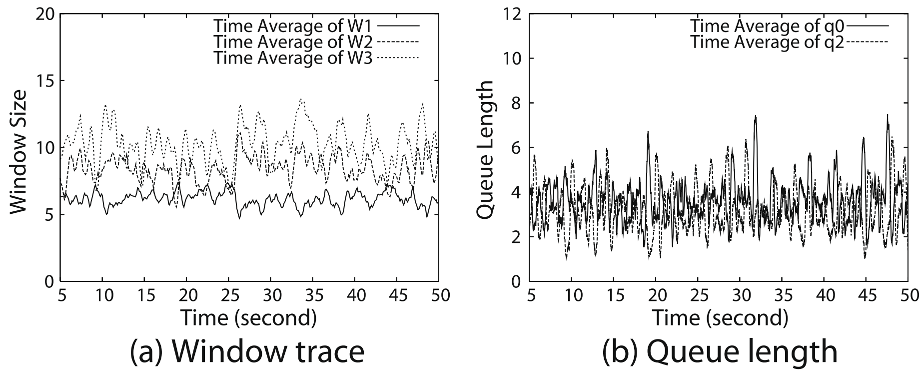


Fig. 14. Simulation results for an unstable system.

consistent with our results in Fig. 12. It is easy to check that this case does not satisfy the conditions of Theorems 2 and 3.

6.2. Simulation results

We use network simulator (NS-2) to further study the performance of the AIMD/RED system with realistic protocols and network topologies. The same multiple-bottleneck topology as in Fig. 2 is used in the simulations.

We first validate a theoretically stable setting. The parameters used are the same as those used for Fig. 11. It should be mentioned that, since the fluid model describes the *ensemble averages* of window size and queue length, the asymptotically stable property applies to the ensemble averages or time averages over a round. Here, a round is defined as the time interval between two instants at which the sender reduces its window size consecutively. Therefore, we focus on the time averages of the window size and queue length over a round. Fig. 13 shows that the time averages of the flow window sizes and queue lengths are converging to certain values, i.e., their time averages over a round are asymptotically stable. The average window sizes in the NS-2 simulation results are slightly larger than the numerical results. This is because the numerical simulations with Matlab ignore the queuing delay in *rtts*, which under-estimates the window size.

We also run the simulation for the unstable case with the same parameters as those used in Fig. 12, and the results are shown in Fig. 14. It can be seen that even averaging over a round, the window sizes and queue lengths are still highly oscillating. The oscillation of queue length with NS-2 is smaller than that with Matlab because larger window size in NS-2 causes smaller queue length (since both Eqs. (2) and (6) show that the product of window size square and queue length is a constant in equilibrium). In addition, the variation of queuing delay and the noises coherent in NS-2 also moderate the oscillations of queue length as mentioned in [22]. As shown in the figures, the queue length in equilibrium in Fig. 14b is also smaller than that in Fig. 12b. Since the simulation results show that the oscillation amplitude of the queue length w.r.t. that in equilibrium is very significant and the queue in Fig. 14b is still unstable. Thus, the simulation results validate the analytical ones.

7. Conclusions

In this paper, we have developed a class of general AIMD/RED models for multi-bottleneck systems and have studied stability properties for the models with delay-free marking and with heterogeneous delays. Global asymptotic stability is proven for the multiple-bottleneck AIMD/RED systems without feedback delay and sufficient conditions have been derived for the asymptotic (local) stability of multiple-bottleneck AIMD/RED systems with heterogeneous delays, by applying the methods of Lyapunov functional and Lyapunov function with the Razumikhin condition. These results are obtained for general multiple-bottleneck scenarios and provide important guidelines for setting system parameters that guarantee the efficient utilization of network resources in multi-bottleneck networks without excessive delay jitter. We are currently investigating sufficient conditions for establishing global stability in the presence of heterogeneous delays, by developing a sequence of upper and lower bounds of system trajectories and applying these bounds in Razumikhin's Theorem. The generalization of stability analysis for networks with mesh topologies should also be an interesting research direction.

Acknowledgement

This work has been supported in part by a research grant from the Natural Science and Engineering Council of Canada.

Appendix A. Proof of Theorem 1

With the transformed variables $\widetilde{W}_i(t) = W_i(t) - W_i^*$, for $i = 1, \dots, N$; $\tilde{q}_j(t) = q_j(t) - q_j^*$, for $j = 1, \dots, M$; the delay-free marking system becomes

$$\begin{aligned}
\dot{\widetilde{W}}_1(t) &= -\frac{2(1-\beta_1)}{1+\beta_1}\frac{(\widetilde{W}_I(t)+W_1^*)^2}{R_1}\left(\sum_{i\in r(1)} K_{p_i}\tilde{q}_i(t)\right)\\
&\quad -\frac{2(1-\beta_1)}{1+\beta_1}\frac{\widetilde{W}_1^2(t)+2W_1^*\widetilde{W}_1(t)}{R_1}\left(\sum_{i\in r(1)} K_{p_i}q_i^*\right)\\
\cdots &\quad \cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots\\
\dot{\widetilde{W}}_N(t) &= -\frac{2(1-\beta_N)}{1+\beta_N}\frac{(\widetilde{W}_N(t)+W_N^*)^2}{R_N}\left(\sum_{j\in r(N)} K_{p_j}\tilde{q}_j(t)\right)\\
&\quad -\frac{2(1-\beta_N)}{1+\beta_N}\frac{\widetilde{W}_N^2(t)+2W_N^*\widetilde{W}_N(t)}{R_N}\left(\sum_{j\in r(N)} K_{p_j}q_j^*\right)\\
\dot{\tilde{q}}_1(t) &= \sum_{n\in f(1)}\frac{N_n\cdot\widetilde{W}_n(t)}{R_n}\\
\cdots &\quad \cdots \quad \cdots\cdots\\
\dot{\tilde{q}}_M(t) &= \sum_{m\in f(M)}\frac{N_m\cdot\widetilde{W}_m(t)}{R_m}.
\end{aligned} \tag{A.1}$$

with the equilibrium point $(\widetilde{W}_1, \dots, \widetilde{W}_N, \tilde{q}_1, \dots \tilde{q}_M) = (0, 0, \dots, 0, 0)$. With system (A.1), we choose Lyapunov function with the following form:

$$\begin{aligned}
&V(\widetilde{W}_1(t), \dots, \widetilde{W}_N(t), \tilde{q}_1(t), \dots, \tilde{q}_M(t))\\
&= \frac{1}{2}\sum_{i=1}^{N}\frac{(1+\beta_i)N_i}{(1-\beta_i)W_i^{*2}}\widetilde{W}_i^2(t) + \frac{1}{2}\sum_{j=1}^{M}K_{p_j}\tilde{q}_j^2(t).
\end{aligned} \tag{A.2}$$

Computing the time-derivative of V along the solution of system (A.1) gives,

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N}\frac{(1+\beta_i)N_i}{(1-\beta_i)\widetilde{W}_i^{*2}}\widetilde{W}_i(t)\dot{\widetilde{W}}_i + \sum_{j=1}^{M}K_{p_j}\tilde{q}_j(t)\dot{\tilde{q}}_j\\
&= -\sum_{i=1}^{N}\frac{N_i\widetilde{W}_i(t)}{\widetilde{W}_i^{*2}R_i}\cdot\Bigg[(\widetilde{W}_i(t)+\widetilde{W}_i^*)^2\cdot\sum_{j\in r(i)}(K_{p_j}\tilde{q}_j(t))\\
&\qquad +(\widetilde{W}_i^2+2\widetilde{W}_i^*\widetilde{W}_i(t))\cdot\sum_{j\in r(i)}(K_{p_j}q_j^*)\Bigg]\\
&\qquad +\sum_{j=1}^{M}(K_{p_j}\tilde{q}_j(t))\cdot\sum_{m\in f(i)}\frac{N_m\widetilde{W}_m(t)}{R_m}\\
&= -\sum_{k=1}^{N}\frac{N_k}{W_k^{*2}R_k}\cdot[\widetilde{W}_k^2(t)\cdot(\widetilde{W}_k(t)+2\widetilde{W}_k^*)]\\
&\qquad \cdot\sum_{i\in r(k)}K_{p_i}(\tilde{q}_i(t)+q_i^*).
\end{aligned}$$

The last step above is derived because the term $\sum_{j=1}^{M}(K_{p_j}\tilde{q}_j(t))\cdot\sum_{m\in f(i)}\frac{N_m\widetilde{W}_m(t)}{R_m}$ can be canceled by part of $\sum_{i=1}^{N}\frac{(1+\beta_i)N_i}{(1-\beta_i)W_i^{*2}}\widetilde{W}_i(t)\dot{\widetilde{W}}_i$. Note that $\widetilde{W}_k(t) + W_k^* = W_k(t) \geqslant 0$ for $k = I, \dots, N$; and $\tilde{q}_i(t) + q_i^* = q_i(t) \geqslant 0$ for $i = 1, \dots, M$; which implies $\dot{V} \leqslant 0$. Thus, we prove that the equilibrium point of system (A.1) is stable. Next, we show the globally asymptotic stability of the system by applying LaSalle's Invariance Principle. Consider the set of states where $\dot{V} = 0$,

$$\begin{aligned}
\mathscr{M} &:= \{(\widetilde{W}_1, \dots, \widetilde{W}_N, \tilde{q}_1, \dots, \tilde{q}_M) : \dot{V} = 0\}\\
&= \{(\widetilde{W}_1, \dots, \widetilde{W}_N, \tilde{q}_1, \dots, \tilde{q}_M) :\\
&\qquad \widetilde{W}_1 = \dots = \widetilde{W}_N = 0;\\
&\qquad \text{or } \tilde{q}_1 = -q_1^*, \dots, \tilde{q}_M = -q_M^*.\}.
\end{aligned}$$

Applying LaSalle's Invariance Principle [24,25], trajectories of (A.1) converge to the largest invariant set contained in $\mathscr{M}$. We then prove that the only invariant set contained in $\mathscr{M}$ is the equilibria $(0, 0, \dots, 0, 0)$. If $(\widetilde{W}_1, \dots, \widetilde{W}_N, \tilde{q}_1, \dots \tilde{q}_M)$ is equal to $(0, \dots, 0, \tilde{q}_1, \dots, \tilde{q}_M)$ or $(\widetilde{W}_1, \dots, \widetilde{W}_N, -q_1^*, \dots - q_M^*)$, we can then conclude that $(\widetilde{W}_1(t^+), \dots, \widetilde{W}_N(t^+), \tilde{q}_1(t^+), \dots \tilde{q}_M(t^+))$ is not in $\mathscr{M}$ by applying (A.1), which implies that no trajectory can stay in $\mathscr{M}$, other than the equilibrium point $(0, 0, \dots, 0, 0)$. Therefore, the equilibrium point of system (A.1) is asymptotically stable. □

Appendix B. Two types of AIMD flows within the group *I*

Assume there are two types of AIMD flows within the group I: N_{I1} AIMD $(\alpha_{I1}, \beta_{I1})$ flows denoted by W_{I1}, and N_{I2} AIMD $(\alpha_{I2}, \beta_{I2})$ flows denoted by W_{I2}, with round-trip time R_{I1} and R_{I2}, respectively. Then these flows can be modeled as follows:

$$\begin{aligned}
\frac{dW_{I1}(t)}{dt} &= \frac{\alpha_{I1}}{R_{I1}} - \frac{2(1-\beta_{I1})}{1+\beta_{I1}}\frac{W_{I1}^2(t)}{R_{I1}}\left(\sum_{i\in r(I)}K_{p_i}q_i(t)\right)\\
\frac{dW_{I2}(t)}{dt} &= \frac{\alpha_{I2}}{R_{I2}} - \frac{2(1-\beta_{I2})}{1+\beta_{I2}}\frac{W_{I2}^2(t)}{R_{I2}}\left(\sum_{i\in r(I)}K_{p_i}q_i(t)\right)
\end{aligned} \tag{B.1}$$

We can then obtain the global asymptotic stability by choose the following Lyapunov function:

$$
\begin{aligned}
&V(\widetilde{W}_{I1}(t), \widetilde{W}_{I2}(t), \ldots, \widetilde{W}_N(t), \tilde{q}_1(t), \ldots, \tilde{q}_M(t)) \\
&\quad = \frac{1}{2}\frac{(1+\beta_{I1})N_{I1}}{(1-\beta_{I1})\widetilde{W}^{*2}_{I1}}\widetilde{W}^2_{I1}(t) + \frac{1}{2}\frac{(1+\beta_{I2})N_{I2}}{(1-\beta_{I2})\widetilde{W}^{*2}_{I2}}\widetilde{W}^2_{II2}(t) \\
&\qquad + \frac{1}{2}\sum_{i\neq I}\frac{(1+\beta_i)N_i}{(1-\beta_i)\widetilde{W}^{*2}_i}\widetilde{W}^2_i(t) + \frac{1}{2}\sum_{j=1}^{M}K_{p_j}\tilde{q}^2_j(t)
\end{aligned}
\tag{B.2}
$$

Similar to the analysis as in Appendix A, global asymptotic stability for this case can be proven. Same conclusion can be drawn for more general cases, i.e., when more than two kinds of AIMD flows in each group are sharing the link capacities. The corresponding mathematical models can be constructed along similar lines as above, by extending the model (6) to higher dimensions to include more terms, each representing another kind of flow.

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