Stability Criteria for Impulsive Systems with Time Delay and Unstable System Matrices

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***Abstract*— This paper studies stability problems of a class of impulsive systems with time delay whose linear parts contain unstable system matrices. By using the method of variation of parameters, Lyapunov functions and inequalities, several stability criteria are established for both linear and nonlinear impulsive systems with time delay. It is shown that the time delay systems can be stabilized by impulses even if the system matrices are unstable. Several numerical examples are given to illustrate the results.**

***Index Terms*— Impulsive delay differential system, time delay, stability, impulsive stabilization.**

I. Introduction

Impulsive dynamical systems have attracted considerable interest in science and engineering in recent years because they provide a natural framework for mathematical modelling of many real world problems where the reactions undergo abrupt changes [1]-[7]. However, most research results on impulsive systems do not consider time delay in their system models. This is mainly due to some theoretical difficulties in the study of impulsive delay systems which have been unsolved until recently. It is well known that time delay is inevitable in many practical problems. Hence it is important to study impulsive systems with time delay. Generally speaking, the study of impulsive delay systems is more difficult than that of impulsive systems without time delay. It is even more challenging when there are delayed impulses.

There are several research works appeared in the literature on impulsive delay differential equations. In [8], Ballinger and Liu have proved some existence and uniqueness results for general impulsive delay differential equations, and the results for some special classes of impulsive differential equations have been obtained in [9]-[11]. In [12], Liu and Shen have obtained two criteria on asymptotic behavior for a class of nonlinear impulsive neutral differential equations. Using fundamental matrices, exponential stability has been investigated for some linear impulsive delay differential equations by Berezansky and Idels [10] and Anokhin, Berezansky, and Braverman [11]. Impulsive integro-differential equations in a Banach space have been studied by Guo and Liu [13]. Lyapunov function method has been applied to the stability analysis of impulsive delay differential equations by Guan [14].

Stability is one of the most important issues in the study of impulsive delay differential equations [10]–[18]. However, most of the research results on the stability are based on the assumption that the system matrix is stable. Recently, using Lyapunov functional, Lyapunov function method and Razumikhin technique, the stability issue has been studied in [18] for some linear time-invariant impulsive control system with time delay where the system matrix is unstable. Several criteria on asymptotic stability are established, and those results show that a system can be stabilized by impulses even if it contains an unstable system matrix. In this paper, we investigate the stability problems of impulsive time-varying linear and nonlinear delay systems which contain unstable system matrix and/or there are time delays at impulsive moments by using the method of variation of parameters, Lyapunov functions and differential inequalities. For the linear impulsive delay differential system, our results are more applicable than those in [18] in the sense that we consider the equations with system matrix having eigenvalues with both positive or zero or negative real parts (see Example 3.1), while in [18] only system matrix having all eigenvalues with either positive real parts or negative real parts is studied.

The remainder of this paper is organized as follows. In Section 2, we introduce notation and definitions. We develop impulsive delay inequalities in Section 3 and then establish several stability criteria for linear and nonlinear impulsive systems with time delay. Numerical examples are given to illustrate our results. Finally, conclusions are given in Section 4.

II. Preliminaries

Let $\mathbb{N}^*$ be the set of positive integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the space of n-dimensional column vectors $x = col(x_1, \cdots, x_n)$ with the Euclidean norm $\|\cdot\|$. For any matrix $A \in \mathbb{R}^{n\times n}$, let $\lambda(A)$ denote the eigenvalue of A, A^T the transpose of A, $\lambda_{max}(A)$ (or $\lambda_{min}(A)$) the maximum (or minimum) eigenvalue of A, $\|A\| = \{\lambda_{max}(A^TA)\}^{\frac{1}{2}}$ the norm of A induced by the Euclidean vector norm.

For $a, b \in \mathbb{R}$ with $a < b$ and for $S \subseteq \mathbb{R}^n$, we define the

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following classes of functions:

$$PC[[a,b],S] = \big\{\psi : [a,b] \to S \big| \psi(t) = \psi(t^+), \forall t \in [a,b);$$
$$\psi(t^-) \text{ exists in } S, \forall t \in (a,b], \text{ and } \psi(t^-) = \psi(t) \text{ for all but}$$
$$\text{at most a finite number of points } t \in (a,b]\big\},$$
$$PC[[a,\infty),S] = \big\{\psi : [a,\infty) \to S \big| \forall b > a, \psi \in PC[[a,b],S]\big\}.$$

Let $J_a = [a,\infty)$, $C_r = PC[[-r,0],\mathbb{R}^n]$, and $\|\phi_t\| = \sup\limits_{t-r\le\theta\le t} \|\phi(\theta)\|$ denote the norm of function $\phi \in C_r$, where $r > 0$ is a constant.

Consider the impulsive delay system

$$\begin{cases} x'(t) &= A(t)x + f(t,x_t),\ t \ge t_0,\ t \ne t_k \\ x(t) &= I(t^-, x_{t^-}),\ \ t = t_k,\ k \in \mathbb{N}^* \\ x_{t_0} &= \phi, \end{cases} \tag{1}$$

where $t_0 \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $A \in C[J_{t_0}, \mathbb{R}^{n\times n}]$, $f, I \in PC[J_{t_0} \times C_r, \mathbb{R}^n]$, $\phi \in C_r$, and $f(t,0) = I(t,0) = 0$. Assume that $r \le t_{k+1} - t_k \le \overline{r} = constant$. Let $\Phi(t,s)$ be the fundamental matrix of the system $x' = A(t)x$, i.e., $\Phi(s,s) = E$ (identity matrix) and $\frac{\partial\Phi(t,s)}{\partial t} = A(t)\Phi(t,s)$. For conditions on the existence and uniqueness of the solution of system (1), see [8].

Denote with $x(t,t_0,x_{t_0})$ the solution of system (1) with initial function $x_{t_0} = \phi$. We define the stability of trivial solution of system (1) as:

Definition 2.1: The trivial solution $x = 0$ of system (1) is

i) stable, if for any given $t_0 \in \mathbb{R}_+$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_{t_0}\| < \delta$ implies that $\|x(t,t_0,x_{t_0})\| \le \epsilon$, $t \ge t_0$;

ii) asymptotically stable, if it is stable and there exists a $\sigma > 0$ such that $\|x_{t_0}\| < \sigma$ implies
$$\lim_{t\to\infty} x(t,t_0,x_{t_0}) = 0;$$

iii) exponentially stable, if there exist $\sigma > 0$, $\alpha > 0$ and $M > 0$ such that $\|x_{t_0}\| < \sigma$ implies
$$\|x(t,t_0,x_{t_0})\| \le Me^{-\alpha(t-t_0)},\ \ t \ge t_0.$$

For simplicity, we call the system stable (or exponentially stable) if its trivial solution $x = 0$ is stable (or exponentially stable).

III. Main Results

A. Linear impulsive delay systems

Consider the time-varying impulsive delay system

$$\begin{cases} x'(t) &= A(t)x(t) + B(t)x(t-r),\ \ t \ne t_k \\ x(t_k) &= D_k x(t_k^-) \\ x_{t_0} &= \phi, \end{cases} \tag{2}$$

where $A, B \in C[J_{t_0}, \mathbb{R}^{n\times n}]$, $D_k \in \mathbb{R}^{n\times n}$, and $\phi \in C_r$.

Theorem 3.1: Assume that

i) there exist constant $M > 0$ and function $\alpha \in C[J_{t_0}, \mathbb{R}]$ such that $\|\Phi(t,s)\| \le Me^{\,{}_s^t \alpha(\eta)d\eta}$, $M\|B(t+r)\| + \alpha(t) \ge 0$, $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^*$;

ii)
$$1 \le e^{\,{}_t^{t+r} \alpha(\eta)d\eta} < \infty,\ \ t \ge t_0.$$

Then

$$\|x(t)\| \quad \le \quad M[\|x(t_0)\| + \|x_{t_0}\| l_0] e^{\,{}_{t_0}^{t} \delta(s)ds} \prod_{j=1}^{k} M_j,$$
$$t \in [t_k, t_{k+1}),\ k \in \{0\} \cup \mathbb{N}^*,$$

where $x(t) \triangleq x(t,t_0,x_{t_0})$ is the solution of system (2), $\delta(t) = M\|B(t+r)\| + \alpha(t)$, $M_k = M[\|D_k\| + l_k \max\{\|D_k\|, 1\}]$, $l_k = \int_{t_k}^{t_k+r} \|B(s)\| e^{-\,{}_{t_k}^{s}\alpha(\eta)d\eta} ds$.

Proof. By the method of variation of parameters, the solution of (2) for $t \in [t_0, t_1)$ is given by

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,s)B(s)x(s-r)ds.$$

Then, we obtain that

$$\begin{aligned} \|x(t)\| \quad &\le \quad \|\Phi(t,t_0)\|\|x(t_0)\| + \int_{t_0}^{t} \|\Phi(t,s)\|\|B(s)\| \\ &\quad\quad \times\|x(s-r)\|ds \\ &\le \quad Me^{\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t_0)\| + \int_{t_0}^{t} Me^{\,{}_s^t\alpha(\eta)d\eta}\|B(s)\| \\ &\quad\quad \times\|x(s-r)\|ds \\ &\le \begin{cases} Me^{\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t_0)\| + \int_{t_0}^{t} Me^{\,{}_s^t\alpha(\eta)d\eta} \\ \times\|B(s)\|\|x(s-r)\|ds, \ \text{ if } \ t \in [t_0, t_0+r), \\ \\ Me^{\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t_0)\| + \int_{t_0}^{t_0+r} Me^{\,{}_s^t\alpha(\eta)d\eta} \\ \times\|B(s)\|\|x(s-r)\|ds + \int_{t_0+r}^{t} Me^{\,{}_s^t\alpha(\eta)d\eta} \\ \times\|B(s)\|\|x(s-r)\|ds, \ \text{ if } t \in [t_0+r, t_1). \end{cases} \end{aligned}$$

For $t \in [t_0, t_0+r)$,

$$\begin{aligned} \|x(t)\| \ &\le Me^{\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t_0)\| + \|x_{t_0}\|M \\ &\times \int_{t_0}^{t_0+r} e^{\,{}_s^t\alpha(\eta)d\eta}\|B(s)\|ds, \end{aligned}$$

thus,

$$e^{-\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t)\| \le M[\|x(t_0)\| + \|x_{t_0}\| l_0].$$

For $t \in [t_0+r, t_1)$,

$$\begin{aligned} \|x(t)\| &\le Me^{\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t_0)\| + \|x_{t_0}\|M\int_{t_0}^{t_0+r} e^{\,{}_s^t\alpha(\eta)d\eta} \\ &\quad \times\|B(s)\|ds + \int_{t_0+r}^{t} Me^{\,{}_s^t\alpha(\eta)d\eta}\|B(s)\|\|x(s-r)\|ds \\ &\le Me^{\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t_0)\| + \|x_{t_0}\|M\int_{t_0}^{t_0+r} e^{\,{}_s^t\alpha(\eta)d\eta} \\ &\quad \times\|B(s)\|ds + \int_{t_0}^{t-r} Me^{\,{}_{s+r}^{t}\alpha(\eta)d\eta}\|B(s+r)\|\|x(s)\|ds. \end{aligned}$$

Multiply $e^{-\,{}_{t_0}^{t}\alpha(\eta)d\eta}$ on both sides, we obtain that

$$\begin{aligned} &e^{-\,{}_{t_0}^{t}\alpha(\eta)d\eta}\|x(t)\| \le M\|x(t_0)\| + \|x_{t_0}\|M\int_{t_0}^{t_0+r}\|B(s)\| \\ &\times e^{\,{}_s^{t_0}\alpha(\eta)d\eta}ds + \int_{t_0}^{t-r} Me^{-\,{}_{t_0}^{s+r}\alpha(\eta)d\eta}\|B(s+r)\|\|x(s)\|ds. \end{aligned}$$

Let $y(t)=e^{-\int_{t_0}^{t}\alpha(\eta)d\eta}\|x(t)\|$, then

$$\begin{aligned} y(t) &\le M[\|x(t_0)\|+\|x_{t_0}\|l_0]+\int_{t_0}^{t-r}Me^{-\int_{s}^{s+r}\alpha(\eta)d\eta} \\ &\quad\times\|B(s+r)\|y(s)ds \\ &\le M[\|x(t_0)\|+\|x_{t_0}\|l_0]+\int_{t_0}^{t}M\|B(s+r)\|y(s)ds, \\ &\qquad t\in[t_0+r,t_1). \end{aligned}$$

It is obvious that the inequality holds over the interval $[t_0,t_1)$, i.e.,

$$\begin{aligned} y(t) &\le M[\|x(t_0)\|+\|x_{t_0}\|l_0]+\int_{t_0}^{t}M\|B(s+r)\|y(s)ds, \\ &\qquad t\in[t_0,t_1). \end{aligned}$$

Then, the Gronwall inequality implies

$$y(t)\le M[\|x(t_0)\|+\|x_{t_0}\|l_0]e^{\int_{t_0}^{t}M\|B(s+r)\|ds}.$$

Thus, for $t\in[t_0,t_1)$

$$\|x(t)\|\le M[\|x(t_0)\|+\|x_{t_0}\|l_0]e^{\int_{t_0}^{t}\delta(s)ds}.$$

Similarly, for $t\in[t_k,t_{k+1})$, we obtain that

$$\|x(t)\|\le M[\|x(t_k)\|+\|x_{t_k}\|l_k]e^{\int_{t_k}^{t}\delta(s)ds}.$$

Since

$$\begin{aligned} \|x_{t_k}\| &= \sup_{t_k-r\le s\le t_k}\|x(s)\| \\ &= \max\{\|x(t_k)\|,\sup_{t_k-r\le s<t_k}\|x(s)\|\} \\ &= \max\{\|D_k\|\|x(t_k^-)\|,\sup_{t_k-r\le s<t_k}\|x(s)\|\} \\ &\le \max\{\|D_k\|,1\}M[\|x(t_{k-1})\|+\|x_{t_{k-1}}\|l_{k-1}] \\ &\quad\times e^{\int_{t_{k-1}}^{t_k}\delta(s)ds}, \end{aligned}$$

for $t\in[t_k,t_{k+1})$,

$$\begin{aligned} \|x(t)\| &\le M[\|x(t_k)\|+\|x_{t_k}\|l_k]e^{\int_{t_k}^{t}\delta(s)ds} \\ &\le M[\|D_k\|\|x(t_k^-)\|+\|x_{t_k}\|l_k]e^{\int_{t_k}^{t}\delta(s)ds} \\ &\le M[\|D_k\|+l_k\max\{\|D_k\|,1\}]\|\widetilde{x}(t_k)\|e^{\int_{t_k}^{t}\delta(s)ds} \\ &\le M_k e^{\int_{t_k}^{t}\delta(s)ds}\|\widetilde{x}(t_k)\|, \end{aligned}$$

where $\|\widetilde{x}(t_k)\|=Me^{\int_{t_{k-1}}^{t_k}\delta(s)ds}[\|x(t_{k-1})\|+\|x_{t_{k-1}}\|l_{k-1}]$. By induction,

$$\begin{aligned} \|\widetilde{x}(t_k)\| &\le M_{k-1}e^{\int_{t_{k-1}}^{t_k}\delta(s)ds}\|\widetilde{x}(t_{k-1})\| \\ &\le\cdots \\ &\le\prod_{j=1}^{k-1}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}\|\widetilde{x}(t_1)\|, \end{aligned}$$

and hence

$$\begin{aligned} \|x(t)\| &\le M_ke^{\int_{t_k}^{t}\delta(s)ds}\prod_{j=1}^{k-1}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}\|\widetilde{x}(t_1)\| \\ &= M[\|x(t_0)\|+\|x_{t_0}\|l_0]e^{\int_{t_0}^{t}\delta(s)ds}\prod_{j=1}^{k}M_j, \\ &\qquad t\in[t_k,t_{k+1}). \end{aligned}$$

■

Corollary 3.1: Assume that the conditions of Theorem 3.1 hold. Then system (2) is

i) stable, if there exists an $\widetilde{M}>0$ such that $\prod_{j=1}^{k}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}\le\widetilde{M}<\infty$;

ii) asymptotically stable, if

$$\lim_{k\to\infty}\prod_{j=1}^{k}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}=0;$$

iii) exponentially stable, if there exist constants $\widetilde{M}>0$ and $\beta>0$ such that

$$\prod_{j=1}^{k}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}\le\widetilde{M}e^{-\beta(t_{k+1}-t_0)},$$

where $k\in\{0\}\cup\mathbb{N}^*$.

The proof is straightforward from Theorem 3.1, and thus omitted.

Corollary 3.2: Assume that the conditions of Theorem 3.1 hold. Furthermore, there exist η_k such that $|\eta_k|\le 1$ and $M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}\le 1+\eta_k$ for all $k\in\{0\}\cup\mathbb{N}^*$.

Then system (2) is

i) stable, if $\sum_{k=1}^{\infty}|\eta_k|<\infty$;

ii) asymptotically stable, if $\lim_{k\to\infty}\prod_{j=1}^{k}(1+\eta_j)=0$;

iii) exponentially stable, if there exist an $\eta>0$ and $N\in\mathbb{N}^*$ such that $1+\eta_k\le e^{-\eta(t_{k+1}-t_k)}$ for $k\ge N$.

Proof. For $t\in[t_k,t_{k+1})$, we have

$$\prod_{1\le j\le k}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds}\le\prod_{j=1}^{k}(1+|\eta_j|).$$

System (2) is stable since $\sum_{k=1}^{\infty}|\eta_k|<\infty$ implies $\prod_{k=1}^{\infty}(1+|\eta_k|)<\infty$. This proves $i)$.

$ii)$ follows from the fact that

$$\begin{aligned} 0 &\le \limsup_{k\to\infty}\prod_{1\le j\le k}M_je^{\int_{t_j}^{t_{j+1}}\delta(s)ds} \\ &\le \lim_{k\to\infty}\prod_{j=1}^{k}(1+\eta_j)=0; \end{aligned}$$

To prove $iii)$, let $N = 1$. Note that

$$\prod_{1\le j\le k} M_j e^{\int_{t_j}^{t_{j+1}} \delta(s)ds} \le \prod_{j=1}^{k}(1+\eta_j) \le \prod_{j=1}^{k} e^{-\eta(t_{j+1}-t_j)}$$
$$= e^{-\eta(t_{k+1}-t_1)} = e^{(t_1-t_0)}e^{-\eta(t_{k+1}-t_0)}$$
$$\le e^{(t_1-t_0)}e^{-\eta(t-t_0)}, \qquad t\in[t_k,t_{k+1}),$$

which implies that system (2) is exponentially stable. ■

For the special case $A(t) = A$ and $B(t) = B$, system (2) becomes

$$\begin{cases} x'(t) &= Ax + Bx(t-r), \quad t \neq t_k \\ x(t_k) &= D_k x(t_k^-) \\ x_{t_0} &= \phi, \end{cases} \tag{3}$$

and $\Phi(t,s) = e^{A(t-s)}$. Let σ denote the maximum of the real part of the eigenvalues of the matrix A. Then for $\alpha > \sigma$, there exists $M > 0$ such that $\|\Phi(t,s)\| \le Me^{\alpha(t-s)}$. It is also noticed that $\|B\| = \sqrt{\lambda_{max}(B^TB)}$ and $\|D_k\| = \sqrt{\lambda_{max}(D_k^TD_k)}$. By Theorem 3.1 we have the following corollary.

Corollary 3.3: Assume that there exist constants $M > 0$ and $\alpha \ge 0$ such that $\|\Phi(t,s)\| \le Me^{\alpha(t-s)}$, $M_k = M[\|D_k\| + r\|B\|\max\{\|D_k\|, 1\}]$, and $\lambda = M\|B\| + \alpha$. Then

$$\|x(t)\| \le e^{\lambda(t-t_0)}M[\,\|x(t_0)\| + r\|B\|\,\|x_{t_0}\|\,]\prod_{j=1}^{k} M_j,$$
$$t\in[t_k,t_{k+1}),\ k\in\{0\}\cup\mathbb{N}^*,$$

where $x(t)$ is any solution of system (3).

Remark 3.1. To achieve the stability, Corollary 3.3 provides some hints for the choice of parameters r, $\|B\|$, and $\|D_k\|$. For instance,

(i) for given $\|B\|$, delay $r \ll 1$ (small delay system) implies that the stability of system (3) is determined by $\|D_k\|$ and impulsive interval $t_{k+1} - t_k$;

(ii) for given delay r, $\|B\| \ll 1$ (the term $Bx(t-r)$ is a small perturbation of system (3)) implies that the stability of system (3) is determined by $\|D_k\|$ and the length of impulsive interval.

Hence, when $r\|B\| \ll 1$, we can achieve stability, asymptotic stability and so on by adjusting the value of $\|D_k\|$. This implies that the impulsive matrices D_k are essential in system stabilization and impulsive control.

By Corollary 3.3, we obtain the following results:

Corollary 3.4: Assume that conditions of Corollary 3.3 hold. Then system (3) is

i) stable, if $\prod_{j=1}^{k} M_j e^{\lambda(t_{j+1}-t_j)} \le \widetilde{M} < \infty, \quad k\in\mathbb{N}^*$;

ii) asymptotically stable, if
$$\lim_{k\to\infty}\prod_{j=1}^{k} M_j e^{\lambda(t_{j+1}-t_j)} = 0, \quad k\in\mathbb{N}^*;$$

iii) exponentially stable, if there exist constants β, $\overline{M} > 0$ such that $\prod_{j=1}^{k} M_j e^{\lambda(t_{j+1}-t_j)} \le \overline{M}e^{-\beta(t_{k+1}-t_0)}$, $k\in\mathbb{N}^*$.

Corollary 3.5: Assume that the conditions of Corollary 3.3 hold and there exist constants ξ, $\overline{M} > 0$ such that $t_k - t_{k-1} = \xi \ge r$ and $M_k \le \overline{M}$ for $k\in\{0\}\cup\mathbb{N}^*$.

Then

i) $\overline{M} = e^{-(M\|B\|+\alpha)\xi}$ implies that system (3) is stable;

ii) $\overline{M} < e^{-(M\|B\|+\alpha)\xi}$ implies that system (3) is exponentially stable.

Proof. $i)$ is straightforward from Corollary 3.3 and hence we only prove $ii)$.

Since $M_k \le \overline{M} < e^{-(M\|B\|+\alpha)\xi}$, there exists a $\delta > 1$ such that $\overline{M} = e^{-\delta(M\|B\|+\alpha)\xi}$.

Thus
$$M_k \le \overline{M} = e^{-\delta(M\|B\|+\alpha)\xi}, \quad k\in\mathbb{N}^*.$$

Then by Corollary 3.3, for $t\in[t_k,t_{k+1})$ and $k\in\{0\}\cup\mathbb{N}^*$,

$$\begin{aligned}\|x(t)\| &\le e^{\lambda(t_{k+1}-t_0)}M[\,\|x(t_0)\| + r\|B\|\,\|x_{t_0}\|\,]\prod_{j=1}^{k}M_j \\ &\le e^{\lambda(k+1)\xi}M^*e^{-\delta\lambda(k+1)\xi} = M^*e^{-(\delta-1)\lambda(k+1)\xi} \\ &\le M^*e^{-\beta(t_{k+1}-t_0)} \le M^*e^{-\beta(t-t_0)},\end{aligned}$$

where $\lambda = M\|B\| + \alpha$, $\beta = (\delta-1)\lambda > 0$, and $M^* = M[\,\|x(t_0)\| + r\|B\|\,\|x_{t_0}\|\,]$. Therefore, $ii)$ is true. ■

Corollary 3.6: Assume that the conditions of Corollary 3.3 hold and there exist constants $\overline{M}$, $p > 0$ such that $M_k \le \overline{M}$ for all $k\in\{0\}\cup\mathbb{N}^*$ and

$$\lim_{T\to\infty}\frac{\eta(t,t+T)}{T} = p,$$

where $\eta(t,t+T)$ denotes the number of impulses in time interval $[t,t+T)$.

Then

i) $\overline{M} = e^{-\frac{\lambda}{p}}$ implies that system (3) is stable;

ii) $\overline{M} < e^{-\frac{\lambda}{p}}$ implies that system (3) is exponentially stable.

Proof. Since $\lim_{T\to\infty}\frac{\eta(t,t+T)}{T} = p$, it follows that for any $\epsilon > 0$, there exist $T > 0$ and $\widetilde{M} = \widetilde{M}(t_0) > 0$, such that $t \ge t_0 + T$ implies $\frac{\eta(t_0,t)}{t-t_0} \le p + \epsilon$, and

$$\prod_{t_0\le t_j\le t} M_j = e^{\sum_{t_0\le t_j\le t}\ln M_j} \le e^{\frac{\eta(t_0,t)}{t-t_0}(t-t_0)\ln\overline{M}}$$
$$\le \widetilde{M}e^{(p+\epsilon)\ln\overline{M}\,(t-t_0)}.$$

Hence,

$$\prod_{t_0\le t_j\le t} M_j e^{\lambda(t_j-t_{j-1})} \le \widetilde{M}e^{[(p+\epsilon)\ln\overline{M}+\lambda](t-t_0)}.$$

Since ϵ can be chosen arbitrarily small, the last inequality implies the results. ■

Next we use an example to illustrate our result.

Example 3.1: Consider the following linear differential system with delayed feedback control and impulsive control

$$\begin{cases} x'(t) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} \frac{1}{30} & -\frac{1}{500} \\ \frac{1}{20} & \frac{1}{27} \end{bmatrix} u(t-1),\ t \neq t_k \\ x(t_k) &= D_k x(t_k^-), \quad k\in\mathbb{N}^* \\ x_{t_0} &= \phi, \end{cases} \tag{4}$$

where $x,\ u\in\mathbb{R}^2$, $t_{k+1} - t_k = 1$, $\phi\in C_1$, and $D_k\in\mathbb{R}^{2\times 2}$.

If we choose the feedback control as

$$u(t) = \begin{bmatrix} \frac{1069}{3450} & -\frac{18}{1081} \\ -\frac{9}{23} & \frac{481}{2162} \end{bmatrix} x(t),$$

we can rewrite the original system as

$$\begin{cases} x'(t) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} \frac{1}{90} & -\frac{1}{1000} \\ \frac{1}{1000} & \frac{1}{135} \end{bmatrix} x(t-1), \\ & \qquad t \neq t_k \\ x(t_k) &= D_k x(t_k^-), \quad k \in \mathbb{N}^* \\ x_{t_0} &= \phi, \end{cases} \tag{5}$$

Let

$$\|D_k\| \le \begin{cases} \frac{1}{4e^2}, & k = 10i + \mu, \quad i \in \mathbb{N}^*, \quad \mu = 1, 2, \cdots, 9 \\ e^2, & k = 10i, \qquad i \in \mathbb{N}^*. \end{cases} \tag{6}$$

Using the notations in Corollary 3.3, we obtain that $\alpha = 1$, $M = 1$, $r = 1$, and $\|B\| \le \frac{1}{4e^2}$.

$$\begin{aligned} M_k &= M[\|D_k\| + \|B\| \max\{\|D_k\|, 1\}] \\ &\le \begin{cases} \frac{1}{2e^2}, & k = 10i + \mu, \quad i \in \mathbb{N}^*, \ \mu = 1, 2, \cdots, 9 \\ e^2 + \frac{1}{4}, & k = 10i, \qquad i \in \mathbb{N}^*, \end{cases} \end{aligned}$$

and

$$\lambda = M\|B\| + \alpha = \|B\| + 1 \le \frac{1}{4e^2} + 1.$$

Then, for $k = 10i + \mu$, where $i \in \mathbb{N}^*$ and $\mu = 1, 2, \cdots, 9$,

$$\begin{aligned} e^{\lambda(t_{k+1} - t_1)} \prod_{j=1}^{k} M_j &= \prod_{j=1}^{10i+\mu} [M_j e^{\lambda(t_{j+1} - t_j)}] \\ &\le \prod_{j=1}^{10i} [M_j e^{\lambda(t_{j+1} - t_j)}] \prod_{j=10i+1}^{10i+\mu} [M_j e^{\lambda(t_{j+1} - t_j)}] \\ &\le \{[\tfrac{1}{2e^2} e^{1 + \frac{1}{4e^2}}]^9 [(e^2 + \tfrac{1}{4}) e^{1 + \frac{1}{4e^2}}]\}^i [\tfrac{1}{2e^2} e^{1 + \frac{1}{4e^2}}]^\mu \\ &\le e^{-\frac{1}{2}(t_{k+1} - t_1)}. \end{aligned}$$

For $t \in [t_k, t_{k+1})$, we obtain that

$$\begin{aligned} \|x(t)\| &\le e^{\lambda(t - t_0)} M[\|x(t_0)\| + r\|B\| \|x_{t_0}\|] \prod_{j=1}^{k} M_j \\ &\le e^2 [\|x_{t_0}\| + \tfrac{1}{4e^2} \|x_{t_0}\|] e^{-\frac{1}{2}(t - t_0)}. \end{aligned}$$

The last inequality implies that system (5) is exponentially stable.

Remark 3.2. In the example, the corresponding system without impulses is unstable since A has a positive eigenvalue and $\|B\|$ is small, the numerical result of this delay differential equation with initial functions

$$\begin{aligned} \phi_1(t) &= \begin{cases} 0, & t \in [-1, 0) \\ -2.1, & t = 0, \end{cases} \\ \phi_2(t) &= \begin{cases} 0, & t \in [-1, 0) \\ 2.1, & t = 0, \end{cases} \end{aligned}$$

is shown in Figure 1.

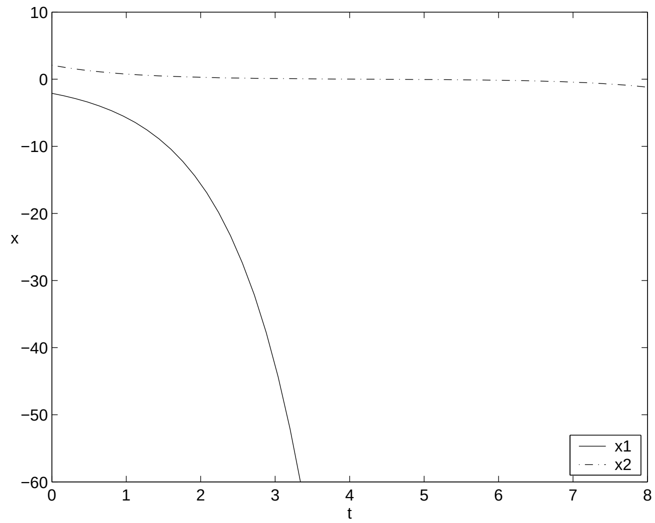

Figure 1. System without impulses.

However, if we choose D_k such that (6) holds, for instance $D_k = \frac{1}{36} I$ for $k \in \mathbb{N}^*$, where I is the identity matrix, by Corollary 3.3, the unstable delay differential equation can be exponentially stabilized by impulses, as shown in Figure 2.

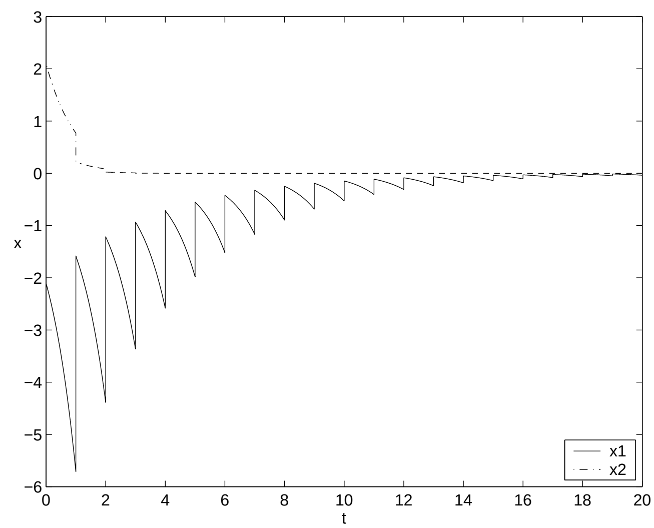

Figure 2. Impulsive system.

Notice that $\|D_k\| \le 1$ is not required for all $k \in \mathbb{N}^*$ to achieve stability, i.e., the trajectories of system (2) are not required to be decreasing at all impulse moments even if the system matrix is unstable. In addition, the results in [18] are not applicable here since the system matrix A in Example 3.1 has both negative and positive eigenvalues.

B. Nonlinear impulsive delay system

Using similar approach, we can study the nonlinear impulsive delay system

$$\begin{cases} x'(t) &= A(t)x + f(t, x(t), x(t - r(t))), \quad t \neq t_k \\ x(t_k) &= D_k x(t_k^-) \\ x_{t_0} &= \phi, \end{cases} \tag{7}$$

where $A \in C[J_{t_0}, \mathbb{R}^{n\times n}]$, $f \in C[\mathbb{R} \times \mathbb{R}^n \times C_r, \mathbb{R}^n]$, $D_k \in \mathbb{R}^{n\times n}$, $k \in \mathbb{N}^*$, $0 \le r(t) \le r$, and $\phi \in C_r$.

Theorem 3.2: Assume that there exists some positive definite and symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

i) there exist functions $\alpha \in C[\mathbb{R}, \mathbb{R}]$ and $a_1, a_2 \in C[\mathbb{R}, \mathbb{R}_+]$ such that $\lambda(A^T(t)P + PA(t)) \le \alpha(t)$ and $2|x^T Pf(t, x, y)| \le a_1(t)\|x\|^2 + a_2(t)\|y\|^2$;

ii)

$$\begin{aligned} l_k^2 &= \int_{t_k}^{t_k+r} e^{\int_s^{t_k} q_0(\eta)d\eta} \frac{a_2(s)}{\lambda_{min}(P)} ds, \\ M_k &= \|D_k\| + l_k \max\{\|D_k\|, 1\}, \\ \beta_k &= \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} e^{\frac{1}{2}\int_{t_k}^{t_{k+1}} q(s)ds}; \end{aligned}$$

where $q_0(\eta) = \frac{\alpha(\eta)+a_1(\eta)}{\lambda_{min}(P)}$ and

$$q(s) = \frac{1}{\lambda_{min}(P)}\big(a_2(s+r) + \alpha(s) + a_1(s)\big) e^{\int_{s+r}^{s} q_0(\eta)d\eta}.$$

iii) $\int_{t_k}^{t} q(s)ds \ge 0$, for $t \in [t_k, t_{k+1})$.

Then

i) if there exists an $\overline{M} > 0$ such that $\prod_{j=1}^{k} M_j\beta_j \le \overline{M} < \infty$, $k \in \mathbb{N}^*$, system (7) is stable;

ii) if $\lim\limits_{k\to\infty} \prod_{j=1}^{k} M_j\beta_j = 0$, system (7) is asymptotically stable;

iii) if there exist $\overline{M}$ and $\beta > 0$ such that $\prod_{j=1}^{k} M_j\beta_j \le \overline{M} e^{-\beta(t_{k+1}-t_0)}$, system (7) is exponentially stable.

Proof. Let $V(t) = x^T(t)Px(t)$, where $x(t)$ is the solution of system (7). Then for $t \ne t_k$, the derivative of V along system (7) is

$$\begin{aligned} V'(t) &= x^T(A^T(t)P + PA(t))x \\ &+ 2x^T Pf(t, x(t), x(t-r)) \\ &\le \alpha(t)x^T x + a_1(t)\|x\|^2 + a_2(t)\|x(t-r)\|^2 \\ &= q_0(t)V + \frac{a_2(t)}{\lambda_{min}(P)} V(t-r). \end{aligned}$$

Thus, for $t \in [t_0, t_1)$,

$$\begin{aligned} V(t) &\le V(t_0)e^{\int_{t_0}^{t} q_0(s)ds} + \int_{t_0}^{t} e^{\int_s^t q_0(\eta)d\eta} \\ &\quad \times \frac{a_2(s)}{\lambda_{min}(P)} V(s-r)ds. \end{aligned}$$

For $t \in [t_0, t_0 + r)$,

$$\begin{aligned} V(t) &\le V(t_0)e^{\int_{t_0}^{t} q_0(s)ds} + \|V_{t_0}\| \\ &\quad \times \int_{t_0}^{t} e^{\int_s^t q_0(\eta)d\eta} \frac{a_2(s)}{\lambda_{min}(P)} ds \\ &\le V(t_0)e^{\int_{t_0}^{t} q_0(s)ds} + \|V_{t_0}\| \\ &\quad \times \int_{t_0}^{t_0+r} e^{\int_s^t q_0(\eta)d\eta} \frac{a_2(s)}{\lambda_{min}(P)} ds, \end{aligned}$$

and for $t \in [t_0 + r, t_1)$,

$$\begin{aligned} V(t) &\le V(t_0)e^{\int_{t_0}^{t} q_0(s)ds} + \|V_{t_0}\| \\ &\quad \times \int_{t_0}^{t_0+r} e^{\int_s^t q_0(\eta)d\eta} \frac{a_2(s)}{\lambda_{min}(P)} ds \\ &\quad + \int_{t_0+r}^{t} e^{\int_s^t q_0(\eta)d\eta} \frac{a_2(s)}{\lambda_{min}(P)} V(s-r)ds \\ &\le V(t_0)e^{\int_{t_0}^{t} q_0(s)ds} + \|V_{t_0}\| \\ &\quad \times \int_{t_0}^{t_0+r} e^{\int_s^t q_0(\eta)d\eta} \frac{a_2(s)}{\lambda_{min}(P)} ds \\ &\quad + \int_{t_0}^{t-r} e^{\int_{s+r}^t q_0(\eta)d\eta} \frac{a_2(s+r)}{\lambda_{min}(P)} V(s)ds. \end{aligned}$$

Multiplying $e^{-\int_{t_0}^{t} q_0(s)ds}$ on both sides of the last inequality gives

$$\begin{aligned} V(t)e^{-\int_{t_0}^{t} q_0(s)ds} &\le V(t_0) + \|V_{t_0}\| \int_{t_0}^{t_0+r} \frac{a_2(s)}{\lambda_{min}(P)} \\ &\quad \times e^{\int_s^{t_0} q_0(\eta)d\eta} ds + \int_{t_0}^{t-r} e^{\int_{s+r}^{t_0} q_0(\eta)d\eta} \frac{a_2(s+r)}{\lambda_{min}(P)} V(s)ds \\ &\le V(t_0) + \|V_{t_0}\| l_0^2 + \int_{t_0}^{t-r} e^{\int_{s+r}^{s} q_0(\eta)d\eta} \frac{a_2(s+r)}{\lambda_{min}(P)} \\ &\quad \times e^{\int_s^{t_0} q_0(\eta)d\eta} V(s)ds. \end{aligned}$$

Let $y(t) = V(t)e^{-\int_{t_0}^{t} q_0(s)ds}$, then for $t \in [t_0 + r, t_1)$

$$y(t) \le V(t_0) + \|V_{t_0}\| l_0^2 + \int_{t_0}^{t} e^{\int_{s+r}^{s} q_0(\eta)d\eta} \frac{a_2(s+r)}{\lambda_{min}(P)} y(s)ds.$$

It can be seen that the last inequality holds for all $t \in [t_0, t_1)$, i.e.,

$$y(t) \le V(t_0) + \|V_{t_0}\| l_0^2 + \int_{t_0}^{t} e^{\int_{s+r}^{s} q_0(\eta)d\eta} \frac{a_2(s+r)}{\lambda_{min}(P)} y(s)ds.$$

Then, the Gronwall-Bellman inequality implies that

$$\begin{aligned} y(t) &\le [V(t_0) + \|V_{t_0}\| l_0^2] e^{\int_{t_0}^{t} e^{\int_{s+r}^{s} q_0(\eta)d\eta} \frac{a_2(s+r)}{\lambda_{min}(P)} ds}, \\ &\qquad t \in [t_0, t_1). \end{aligned}$$

Therefore,

$$\begin{aligned} \|x(t)\| &\le e^{\frac{1}{2}\int_{t_0}^{t} q(s)ds} \sqrt{\frac{V(t_0)+\|V_{t_0}\| l_0^2}{\lambda_{min}(P)}} \\ &\le e^{\frac{1}{2}\int_{t_0}^{t} q(s)ds} \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} [\|x(t_0)\| + \|x_{t_0}\| l_0], \\ &\qquad t \in [t_0, t_1). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|x(t)\| &\le e^{\frac{1}{2}\int_{t_k}^{t} q(s)ds} \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \\ &\quad \times [\|x(t_k)\| + \|x_{t_k}\| l_k], \quad t \in [t_k, t_{k+1}). \end{aligned}$$

Furthermore,

$$\|x(t_k)\| \le \|D_k\| \|\widetilde{x}(t_k)\|,$$

and

$$\begin{aligned}\|x_{t_k}\| &= \sup_{t_k-r\le t\le t_k}\|x(t)\| \\ &= \max\{\|x(t_k)\|, \sup_{t_k-r\le t<t_k}\|x(t)\|\} \\ &\le \max\{\|D_k\|\|x(t_k^-)\|, \sup_{t_k-r\le t<t_k}\|x(t)\|\} \\ &\le \max\{\|D_k\|,1\}\|\widetilde{x}(t_k)\|,\end{aligned}$$

where

$$\|\widetilde{x}(t_k)\| = [\|x(t_{k-1})\| + \|x_{t_{k-1}}\|l_{k-1}]\beta_{k-1}.$$

Then

$$\begin{aligned}&\|\widetilde{x}(t_k)\| = [\|x(t_{k-1})\| + \|x_{t_{k-1}}\|l_{k-1}]\beta_{k-1} \\ &\le [\|D_{k-1}\| + l_{k-1}\max\{\|D_{k-1}\|,1\}]\|\widetilde{x}(t_{k-1})\|\beta_{k-1} \\ &= M_{k-1}\|\widetilde{x}(t_{k-1})\|\beta_{k-1} \\ &\le\ M_{k-1}\beta_{k-1}M_{k-2}\beta_{k-2}\|\widetilde{x}(t_{k-2})\| \\ &\le \cdots\ \le\ \prod_{j=1}^{k-1}M_j\beta_j\|\widetilde{x}(t_1)\|.\end{aligned}$$

Thus, for $t\in[t_k,t_{k+1})$,

$$\begin{aligned}\|x(t)\| &\le \sqrt{\tfrac{\lambda_{max}(P)}{\lambda_{min}(P)}}[\|x(t_k)\| + \|x_{t_k}\|l_k]e^{\frac{1}{2}\int_{t_k}^{t}q(s)ds} \\ &\le \sqrt{\tfrac{\lambda_{max}(P)}{\lambda_{min}(P)}}M_k\prod_{j=1}^{k-1}M_j\beta_j\|\widetilde{x}(t_1)\|e^{\frac{1}{2}\int_{t_k}^{t}q(s)ds} \\ &\le \|\widetilde{x}(t_{k+1})\|\ \le\ \prod_{j=1}^{k}M_j\beta_j\|\widetilde{x}(t_1)\|.\end{aligned}$$

This inequality implies the results of Theorem 3.2. ■

Remark 3.3. In Theorems 3.1 and 3.2, $\alpha(t)\ge 0$ is not required, which means it may be negative for some t. And for some special case such as $t_k - t_{k-1} = \eta$ with $k\in\mathbb{N}^*$, additional useful and simple results similar to Corollaries 3.5 and 3.6 can be obtained.

C. System with delayed impulses

Consider the following linear impulsive system with delayed impulses

$$\begin{cases} x'(t) = A(t)x + B(t)x(t-r), & t\ne t_k \\ x(t_k) = D_kx(t_k^-) + E_kx(t_k^- - r_1) \\ x_{t_0} = \phi, \end{cases} \tag{8}$$

where $A,B\in C[J_{t_0},\mathbb{R}^{n\times n}]$, $D_k,E_k\in\mathbb{R}^{n\times n}$, $k\in\mathbb{N}^*$, $\phi\in C_{r^*}$ with $r^*=\max\{r,r_1\}$, and $0\le r,r_1<t_k-t_{k-1}\le \sup\{t_k-t_{k-1}\} = constant < \infty$. Denote with $\Phi(t,t_0)$ the fundamental matrix of system $x' = A(t)x$.

Theorem 3.3: Assume that there exist constant $M>0$ and function $\alpha\in C[J_{t_0},\mathbb{R}]$ such that $\|\Phi(t,s)\|\le Me^{\int_s^t\alpha(\eta)d\eta}$, $\delta(t)\ge 0$, $t\in[t_{k-1},t_k)$, and $1\le e^{\int_t^{t+r}\alpha(\eta)d\eta}<\infty$ for $t\ge t_0$.

Then system (8) is

i) stable, if there exists an $\widetilde{M}<\infty$ such that

$$\prod_{j=1}^{k}M_je^{\int_{t_0}^{t}\delta(s)ds}\le\widetilde{M},\quad k\in\mathbb{N}^*;$$

ii) asymptotically stable, if

$$\lim_{k\to\infty}\prod_{j=1}^{k}M_je^{\int_{t_0}^{t}\delta(s)ds} = 0,\qquad k\in\mathbb{N}^*;$$

iii) exponentially stable, if there exist M^*, $\beta>0$ such that

$$\prod_{j=1}^{k}M_je^{\int_{t_0}^{t}\delta(s)ds}\le M^*e^{-\beta(t_{k+1}-t_0)},\quad k\in\mathbb{N}^*,$$

where $l_k = \int_{t_k}^{t_k+r}\|B(s)\|e^{-\int_{t_k}^{s}\alpha(\eta)d\eta}ds$ and

$$\begin{aligned}M_k\ =\ & M[\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds} + \\ & l_k\max\{\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds},\ 1\}].\end{aligned}$$

Proof. Based on the proof of Theorem 3.1, we have

$$\begin{aligned}\|x(t)\|\ &\le\ M[\|x(t_k)\| + \|x_{t_k}\|l_k]e^{\int_{t_k}^{t}\delta(s)ds}, \\ &\qquad t\in[t_k,t_{k+1}),\ k\in\{0\}\cup\mathbb{N}^*.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|x(t_k)\|\ &\le\ \|D_k\|\|x(t_k^-)\| + \|E_k\|\|x(t_k^- - r_1)\| \\ &\le\ \|D_k\|\|\widetilde{x}(t_k)\| + \|E_k\|\|\widetilde{x}(t_k)\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds} \\ &=\ [\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds}]\|\widetilde{x}(t_k)\|,\end{aligned}$$

where

$$\|\widetilde{x}(t_k)\| = M[\|x(t_{k-1})\| + \|x_{t_{k-1}}\|l_{k-1}]e^{\int_{t_{k-1}}^{t_k}\delta(s)ds},$$

and

$$\begin{aligned}\|x_{t_k}\|\ &=\ \sup_{t_k-r\le t\le t_k}\|x(t)\| \\ &=\ \max\{\|x(t_k)\|, \sup_{t_k-r\le t<t_k}\|x(t)\|\} \\ &\le\ \max\{[\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds}]\|\widetilde{x}(t_k)\|, \\ &\qquad \sup_{t_k-r\le t<t_k}\|x(t)\|\} \\ &\le\ \max\{\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds},\ 1\}\|\widetilde{x}(t_k)\|.\end{aligned}$$

Thus,

$$\begin{aligned}&\|x(t)\|\le M[\|x(t_k)\| + l_k\|x_{t_k}\|]e^{\int_{t_k}^{t}\delta(s)ds} \\ &\le M[\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds} + l_k\|\widetilde{x}(t_k)\|e^{\int_{t_k}^{t}\delta(s)ds} \\ &\times\max\{\|D_k\| + \|E_k\|e^{\int_{t_k}^{t_k-r_1}\delta(s)ds},\ 1\}] \\ &= M_k\|\widetilde{x}(t_k)\|e^{\int_{t_k}^{t}\delta(s)ds},\ \ t\in[t_k,t_{k+1}),\ k\in\{0\}\cup\mathbb{N}^*.\end{aligned}$$

Since

$$\|\widetilde{x}(t_k)\| = M[\|x(t_{k-1})\| + \|x_{t_{k-1}}\|l_{k-1}]e^{\int_{t_{k-1}}^{t_k}\delta(s)ds}$$

$$\leq M[[\|D_{k-1}\| + \|E_{k-1}\|e^{\int_{t_{k-1}}^{t_{k-1}-r_1}\delta(s)ds}]\|\widetilde{x}(t_{k-1})\|$$

$$+l_{k-1}\max\{\|D_{k-1}\| + \|E_{k-1}\|e^{\int_{t_{k-1}}^{t_{k-1}-r_1}\delta(s)ds},\ 1\}$$

$$\times\|\widetilde{x}(t_{k-1})]\|e^{\int_{t_{k-1}}^{t_k}\delta(s)ds}$$

$$= M[\|D_{k-1}\| + \|E_{k-1}\|e^{\int_{t_{k-1}}^{t_{k-1}-r_1}\delta(s)ds}$$

$$+l_{k-1}\max\{\|D_{k-1}\| + \|E_{k-1}\|e^{\int_{t_{k-1}}^{t_{k-1}-r_1}\delta(s)ds},\ 1\}]$$

$$\times\|\widetilde{x}(t_{k-1})\|e^{\int_{t_{k-1}}^{t_k}\delta(s)ds}$$

$$= M_{k-1}\|\widetilde{x}(t_{k-1})\|e^{\int_{t_{k-1}}^{t_k}\delta(s)ds},$$

we have

$$\begin{aligned}\|x(t)\| &= M_k\|\widetilde{x}(t_k)\|e^{\int_{t_k}^{t}\delta(s)ds}\\ &\leq M_kM_{k-1}\|\widetilde{x}(t_{k-1})\|e^{\int_{t_{k-1}}^{t}\delta(s)ds}\\ &\leq \cdots \leq \prod_{j=1}^{k}M_j\|\widetilde{x}(t_1)\|e^{\int_{t_1}^{t}\delta(s)ds}\\ &\leq M[\|x(t_0)\| + \|x_{t_0}\|l_0]e^{\int_{t_0}^{t}\delta(s)ds}\prod_{j=1}^{k}M_j,\\ &\qquad t\in[t_k,t_{k+1}),\ k\in\{0\}\cup\mathbb{N}^*.\end{aligned}$$

This inequality implies the results of Theorem 3.3. ■

Remark 3.4.

- Theorem 3.3 (also Theorems 3.1 or 3.2) indicates that even if the system matrix A may be unstable, providing $\|B\|$ is not large, we can choose appropriate impulse matrices D_k, E_k to stabilize the original system.
- The result of Theorem 3.3 is a little stricter than that of Theorem 3.1. This is because there exists time delay at impulsive moments in system (8). Notice that time delay sometimes plays a very important role in system analysis. For example, if a solution of an ODE 'jumps' to 0 at time t_1, then the solution will be 0 after t_1 providing the system has only one solution passing through every point. But for delay system, jumping to 0 at one point does not effect the solution as much as ODE does since the delay determines the change of the system at a neighborhood of that point. Therefore, the investigation of delay system is much more difficult than that of the system without delay. Similarly, if time delay is included in the impulsive moment, it may dominate the value of the solution at the point. That is why the result of Theorem 3.3 is a little more restrictive than that of Theorem 3.1.

For impulsive system with nonlinear impulses with time delay, the system becomes

$$\begin{cases} x'(t) &= A(t)x + f(t,x,x_t),\quad t\neq t_k\\ x(t_k) &= D_kx(t_k^-) + g(x(t_k^-),x_{t_k^-})\\ x_{t_0} &= \phi,\end{cases}\tag{9}$$

where $x_t\in C_r$, $f\in C[\mathbb{R}\times\mathbb{R}^n\times C_r,\mathbb{R}^n]$, $g\in C[\mathbb{R}^n\times C_{r_1},\mathbb{R}^n]$, $\phi\in C_{r^*}$ with $r^*=\max\{r,r_1\}$, and $0\leq r,r_1<t_k-t_{k-1}\leq \sup\{t_k-t_{k-1}\}<\infty$ for $k\in\mathbb{N}^*$.

Theorem 3.4: Assume that

i) there exist functions $\alpha\in C[J_{t_0},\mathbb{R}]$, $a_1,a_2\in C[\mathbb{R},\mathbb{R}_+]$, and constants $b_{k1},b_{k2}\in\mathbb{R}_+$ such that

$$\lambda(A^T(t)+A(t))\leq\alpha(t),$$

$$2|x^Tf(t,x,y)|\leq a_1(t)\|x\|^2 + a_2(t)\|y\|^2,$$

$$\|g(x,y)\|\leq b_{k1}\|x\| + b_{k2}\|y\|;$$

ii)

$$\int_{t_k}^{t}q_1(s)ds\geq 0,\quad t\in[t_k,t_{k+1}),$$

$$\beta_k = e^{\frac{1}{2}\int_{t_k}^{t_{k+1}}q_1(s)ds},$$

$$\int_{t_k}^{t_k+r}e^{\int_s^{t_k}(\alpha(\eta)+a_1(\eta))d\eta}a_2(s)ds = l_k^2,$$

$$M_k = \|D_k\| + b_{k1} + b_{k2} + l_k\max\{\|D_k\| + b_{k1} + b_{k2},\ 1\},$$

where

$$q_1(s) = \big(a_2(s+r) + \alpha(s) + a_1(s)\big)e^{\int_{s+r}^{s}(\alpha(\eta)+a_1(\eta))d\eta}.$$

Then

$$\|x(t)\|\leq\ M_k\prod_{j=1}^{k-1}M_j\beta_j\|\widetilde{x}(t_1)\|e^{\frac{1}{2}\int_{t_k}^{t}q_1(s)ds},$$

where $x(t)$ is the solution of system (9) and $\|\widetilde{x}(t_1)\| = [\|x(t_0)\| + \|x_{t_0}\|l_0]\beta_0$.

Proof. Based on the proof of Theorem 3.2 with $P=I$, where I is the identity matrix, we have

$$\begin{aligned}\|x(t)\| &\leq [\|x(t_k)\| + \|x_{t_k}\|l_k]e^{\frac{1}{2}\int_{t_k}^{t}q_1(s)ds},\\ &\qquad t\in[t_k,t_{k+1}).\end{aligned}$$

Furthermore,

$$\begin{aligned}\|x(t_k)\| &\leq \|D_k\|\|x(t_k^-)\| + b_{k1}\|x(t_k^-)\| + b_{k2}\|x_{t_k^-}\|\\ &= (\|D_k\| + b_{k1} + b_{k2})\|\widetilde{x}(t_k)\|,\end{aligned}$$

where

$$\|\widetilde{x}(t_k)\| = [\|x(t_{k-1})\| + \|x_{t_{k-1}}\|l_{k-1}]\beta_{k-1},$$

and

$$\|x_{t_k}\| = \sup_{t_k-r\leq t\leq t_k}\|x(t)\|$$

$$= \max\{\|x(t_k)\|,\ \sup_{t_k-r\leq t<t_k}\|x(t)\|\}$$

$$\leq\max\{(\|D_k\| + b_{k1} + b_{k2})\|\widetilde{x}(t_k)\|,\ \sup_{t_k-r\leq t<t_k}\|x(t)\|\}$$

$$\leq\max\{\|D_k\| + b_{k1} + b_{k2},\ 1\}\|\widetilde{x}(t_k)\|.$$

Then,

$$\|\widetilde{x}(t_k)\| = [\|x(t_{k-1})\| + \|x_{t_{k-1}}\|l_{k-1}]\beta_{k-1}$$

$$\leq [[\|D_{k-1}\| + b_{(k-1)1} + b_{(k-1)2} + l_{k-1}$$

$$\times \max\{\|D_{k-1}\| + b_{(k-1)1} + b_{(k-1)2}, 1\}]\|\widetilde{x}(t_{k-1})\|\beta_{k-1}$$

$$= M_{k-1}\|\widetilde{x}(t_{k-1})\|\beta_{k-1}$$

$$\leq M_{k-1}\beta_{k-1}M_{k-2}\beta_{k-2}\|\widetilde{x}(t_{k-2})\|$$

$$\leq \cdots \leq \prod_{j=1}^{k-1} M_j\beta_j\|\widetilde{x}(t_1)\|.$$

Thus, for $t \in [t_k, t_{k+1})$,

$$\|x(t)\| \leq [\|x(t_k)\| + \|x_{t_k}\|l_k]e^{\frac{1}{2}\int_{t_k}^{t} q_1(s)ds}$$

$$\leq M_k\|\widetilde{x}(t_k)\|e^{\frac{1}{2}\int_{t_k}^{t} q_1(s)ds} \leq \cdots$$

$$\leq M_k\prod_{j=1}^{k-1} M_j\beta_j\|\widetilde{x}(t_1)\|\, e^{\frac{1}{2}\int_{t_k}^{t} q_1(s)ds}.$$

■

Corollary 3.7: Assume that the conditions of Theorem 3.4 hold. Then,

i) if there exists an $\overline{M} > 0$ such that $\prod_{j=1}^{k} M_j\beta_j \leq \overline{M} < \infty$, $k \in \mathbb{N}^*$, system (9) is stable;

ii) $\lim_{k\to\infty}\prod_{j=1}^{k} M_j\beta_j = 0$, $k \in \mathbb{N}^*$ implies that system (9) is asymptotically stable;

iii) if there exist constants $\overline{M}$, $\beta > 0$ such that $\prod_{j=1}^{k} M_j\beta_j \leq \overline{M}e^{-\beta(t_{k+1}-t_0)}$, $k \in \mathbb{N}^*$, system (9) is exponentially stable.

Similar to Corollary 3.2, we obtain the following result:

Corollary 3.8: Assume that the conditions of Theorem 3.4 hold, $t_{k+1} - t_k = \eta < \infty$ and there exist η_k with $|\eta_k| \leq 1$ such that $M_k\beta_k \leq 1 + \eta_k \ \ for \ \ all \ \ k \in \{0\} \cup \mathbb{N}^*$. Then,

i) $\sum_{k=1}^{\infty}|\eta_k| < \infty$ implies system (9) is stable;

ii) $\prod_{k=1}^{\infty}(1 + \eta_k) = 0$ implies system (9) is asymptotically stable;

iii) if there exist $\gamma > 0$ and $N > 0$ such that when $k \geq N$, $1+\eta_k \leq e^{-\gamma(t_{k+1}-t_k)}$, system (9) is exponentially stable.

Example 3.2: Consider the nonlinear impulsive delay system with time delay at impulsive moments

$$\begin{cases} x'(t) &= \begin{bmatrix} \frac{1-|\sin t|}{2} & -0.1t \\ 0.1t & \frac{1-|\sin t|}{2} \end{bmatrix} x + f(t, x, x_t), \quad t \neq t_k \\ x(t_k) &= g(x(t_k^-), x_{t_k^-}), \ k \in \mathbb{N}^* \\ x_{t_0} &= \phi, \end{cases} \tag{10}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$, x_t, $\phi \in C_r$, $f \in C[\mathbb{R} \times \mathbb{R}^2 \times C_r, \mathbb{R}^2]$, $g \in C[\mathbb{R}^2 \times C_r, \mathbb{R}^2]$, $r = 0.2$, $t_k - t_{k-1} = 0.5$, and $D_k = 0.1I$, where I is the identity matrix. Assume that $f(t, x, x_t)$ and $g(x(t_k^-), x_{t_k^-})$ satisfy

$$2|x^T(t)f(t, x, x_t)| \leq (\frac{1}{2} + |\sin t|)\|x(t)\|^2 + \frac{1}{2}\|x(t-r)\|^2,$$

$$\|g(x(t_k^-), x_{t_k^-})\| \leq b_{k1}\|x(t_k^-)\| + b_{k2}\|x(t_k^- - r)\|,$$

where $b_{k1} + b_{k2} \leq 0.7$.
Using the notations of Theorem 3.4, we obtain that

$$\begin{array}{rcl} \alpha(t) &=& 1 - |\sin t|, \ a_1(t) = \frac{1}{2} + |\sin t|, \ a_2(t) = \frac{1}{2}, \\ \beta_k &=& e^{\frac{1}{2}\int_{t_k}^{t_{k+1}} q_1(s)ds} = 1.5962, \\ l_k^2 &=& \int_{t_k}^{t_k+r} e^{\int_s^{t_k}(\alpha(\eta)+a_1(\eta))d\eta}a_2(s)ds, \\ &=& \int_{t_k}^{t_k+r} e^{t_k-s}e^r ds = 0.0864, \\ M_k &=& \|D_k\| + b_{k1} + b_{k2} + l_k\max\{\|D_k\| + b_{k1} + b_{k2}, 1\} \\ &=& 0.1 + b_{k1} + b_{k2} + 0.2939. \end{array}$$

Thus $M_k\beta_k = (b_{k1} + b_{k2} + 0.3939)1.5962$.

By Corollary 3.8, we make the following conclusions

- $b_{k1} + b_{k2} \leq 0.2326$ implies that system (10) is stable;
- $b_{k1} + b_{k2} \leq 0.2326 - \frac{1}{k}$ implies that system (10) is asymptotically stable;
- $b_{k1} + b_{k2} \leq \tau < 0.2326$ implies that system (10) is exponentially stable.

To illustrate our conclusions numerically, we choose the functions as

$$f(t, x, x_t) = \begin{bmatrix} 0.5(sin(t)x_1(t) + x_2(t-0.2)) \\ 0.5(sin(t)x_2(t) + x_1(t-0.2)) \end{bmatrix},$$

$$g(x(t_k^-), x(t_{k^-})) = \begin{bmatrix} 0.05\sqrt{|x_1(t_k^-)x_2(t_k^- - 0.2)|} \\ 0.05\sqrt{|x_2(t_k^-)x_1(t_k^- - 0.2)|} \end{bmatrix},$$

so $b_{k_1} = b_{k_2} = 0.05$, and the initial functions are given by

$$\begin{array}{rcl} \phi_1(t) &=& \begin{cases} 0, & t \in [-0.2, 0) \\ -0.05, & t = 0, \end{cases} \\ \phi_2(t) &=& \begin{cases} 0, & t \in [-0.2, 0) \\ 0.1, & t = 0, \end{cases} \end{array}$$

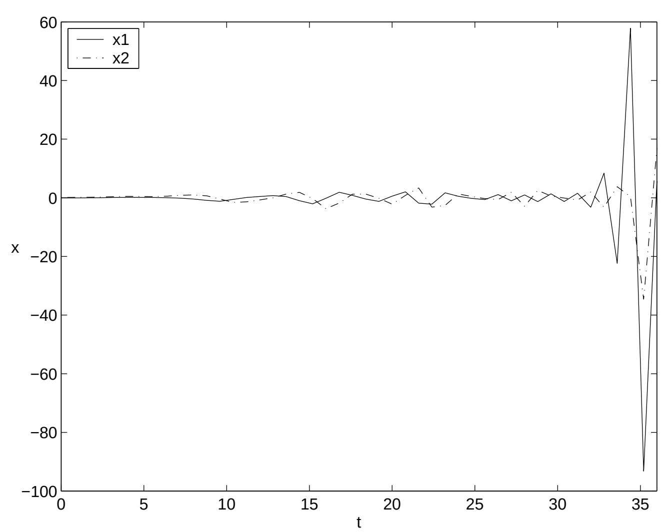

Figure 3. System without impulses.

Figure 3 shows that the corresponding system without impulses is unstable, but it can be exponentially stabilized by impulses, as shown in Figure 4.

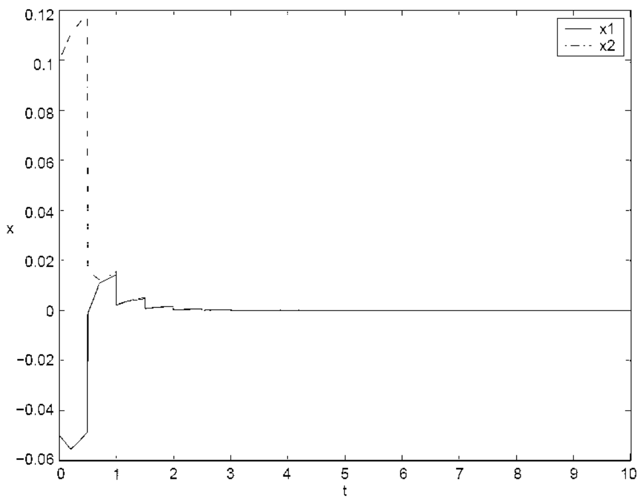

Figure 4. Impulsive system.

IV. Conclusions

We have investigated the stability issues of both linear and nonlinear impulsive delay systems which include unstable system matrix and/or time delay at impulsive moments. Some criteria on stability and exponential stability have been obtained. Our results show that the impulsive delay system with unstable matrix can be stabilized by adjusting impulsive value under certain conditions. Although only single delay has been considered in the paper, the study can be extended to the case with multiple delays. It should be mentioned that the results presented in this paper are not applicable to systems without impulses since the corresponding equations without impulses discussed in this paper are unstable.

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