Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis: Hybrid Systems 1 (2007) 44–58
Nonlinear Analysis: Hybrid Systems
www.elsevier.com/locate/nahs

Stability of switched systems with time delay

Yi Zhang[a], Xinzhi Liu[b,*], Xuemin Shen[c]

[a] Department of Mathematics and Physics, China University of Petroleum, Beijing, China
[b] Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
[c] Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Received 8 March 2006; accepted 24 March 2006

Abstract

In this paper, we study the qualitative properties of linear and nonlinear delay switched systems which have stable and unstable subsystems. First, we prove some inequalities which lead to the switching laws that guarantee: (a) the global exponential stability to linear switched delay systems with stable and unstable subsystems; (b) the local exponential stability of nonlinear switched delay systems with stable and unstable subsystems. In addition, these switching laws indicate that if the total activation time ratio among the stable subsystems, unstable subsystems and time delay is larger than a certain number, the switched systems are exponentially stable for any switching signals under these laws. Some examples are given to illustrate the main results.

1. Introduction

Switched systems are systems that consist of several subsystems and controlled by switching laws. Such systems are often encountered in biochemical systems, control systems, etc., (see [5,4]). For switched systems, one of the most important and challenging problems is to find the switching laws, i.e., what switching laws can guarantee the switched systems stable.

Recently, there has been increasing interest in the stability analysis of switched systems, and switching control design of such systems (see [8,9,11–13] and the references therein). Using common Lyapunov functions, [18] studies stability for linear switched systems and shows that a common Lyapunov function exists when the stability matrices A_i commute pairwise, i.e. $A_i A_j = A_j A_i, i, j = 1, 2, \ldots, N$. Exponential stability is studied for some special linear time-invariant switched system in [23,11]. Similar results are presented in [7]. Lie algebra is used to prove the existence of common Lyapunov function and then the stability of switched systems is derived in [16,15,1]. The stability for nonlinear-switched systems is proposed in [4,2,3] and the linearization method is developed in [2]. Multiple Lyapunov function techniques are used in [10–20] to investigate the stability of the switched systems. The stability of some slow-switched control systems is studied in [12–25]. Lagrange stability for switched systems is considered in [4,3,24]. A review of stability results covering the majority of research works can be found in [6]. Stability analysis of switched systems with stable and unstable subsystems is driven in [26,27].

* Corresponding author.
E-mail address: xzliu@math.uwaterloo.ca (X. Liu).

1751-570X/$ - see front matter
doi:10.1016/j.nahs.2006.03.001

However, the studies for the stabilities of switched systems have mainly focused on the systems without time delay due to its difficulty. In this paper, we study stability properties of linear and nonlinear switched delay systems with stable and unstable subsystems. Firstly, we prove some useful inequalities in Section 2. Then switching laws are proposed in Section 3, which ensures that the switched delay system is stable or exponentially stable for any switching signals under the laws. Linear switched delay systems are studied in Section 4 and some switching laws are given, which guarantee the switched system to be stable or globally exponentially stable. In Section 5, using delay inequalities instead of linearization, which may lose some nonlinear properties, we study a nonlinear switched system and propose a switching law that guarantees the stability or local exponential stability of the systems. Some examples are given to illustrate the main results. Section 6 concludes the paper.

2. Preliminary

In this section, some useful notations are given, and necessary inequalities are derived.

Let R be the set of real numbers, R_+ the set of nonnegative numbers, R^n the n-dimensional space, $R^{n\times m}$ the $n \times m$ matrix space, respectively. If $A = (a_{ij}) \in R^{n\times m}$, denote $\|A\| = \max_{1\le j\le m} \sum_{i=1}^{n} |a_{ij}|$ the norm of A. If $A = (a_{ij}) \in R^{n\times n}$, denote $\lambda(A)$ the eigenvalue of A, $\mathrm{Re}\lambda(A)$ the real part of $\lambda(A)$.

Let $C_\tau = \{\phi : \phi \in C[[-\tau, 0], R^n]\}$, where $\tau > 0$ is a constant. If $x \in C_\tau$, define $x_t = x(t+\theta), \theta \in [-\tau, 0]$ and $\|x_t\| = \sup_{t-\tau\le s\le t} \|x(s)\|$.

Denote

$$\Gamma(\lambda, t-s) = \begin{cases} 1, & t \le s, \\ \mathrm{e}^{\lambda(t-s)}, & t \ge s; \end{cases}$$

$$K_1 = \{\phi | \phi \in C[R_+, R_+], \phi(0) = 0, \phi \text{ is nondecreasing}\}.$$

$$\aleph = \{1, 2, \ldots\},$$

$$\aleph_i = \{1, 2, \ldots, N_i\}.$$

In this paper, we study the stability of the following switched delay system

$$x'(t) = f_{\sigma_i}(t, x(t), x_t), \quad \sigma_i \in \aleph_i, \tag{2.1}$$

where $x \in R^n$, $f_{\sigma_i} \in C[R \times R^n \times C_\tau, R^n]$, $f_{\sigma_i}(t, 0, 0) = 0$ for any $\sigma_i \in \aleph_i$, $i = 1, 2$.

In the rest of the paper, let $i_k \in \aleph_1 \cup \aleph_2$ denote the i_kth subsystem, t_k the switching points, $[t_k, t_{k+1})$ the time period over which the i_kth subsystem is activated. Let $x_{\sigma_1}(t)$ and $x_{\sigma_2}(t)$ denote the solution of systems

$$x'(t) = f_{\sigma_1}(t, x(t), x_t), \quad \sigma_1 \in \aleph_1, \tag{2.2}$$

and

$$x'(t) = f_{\sigma_2}(t, x(t), x_t), \quad \sigma_2 \in \aleph_2, \tag{2.3}$$

respectively, i.e.,

$$x'_{\sigma_1}(t) = f_{\sigma_1}(t, x_{\sigma_1}(t), x_{\sigma_1 t}), \quad \sigma_1 \in \aleph_1,$$
$$x'_{\sigma_2}(t) = f_{\sigma_2}(t, x_{\sigma_2}(t), x_{\sigma_2 t}), \quad \sigma_2 \in \aleph_2.$$

Let $k_{\sigma_i}(t)$ be the number of times that the σ_ith subsystem of (2.1) is switched on during $[t_0, t)$, $\pi_{\sigma_i}(t)$ be the total activation time of σ_ith subsystem of (2.1) during $[t_0, t)$.

Assume that

(B1) $t_{k+1} - t_k \ge \tau$, $\lim_{k\to\infty} t_k = \infty$;
(B2) The switching sequence is minimal, i.e., $i_k \ne i_{k+1}$;
(B3) Each $f_{\sigma_i}(t, x, y)$ is locally Lipschitzian in x and y.

Our aim is to find switching laws under which switched delay system (2.1) is stable or exponentially stable. Assumption (B1) is introduced to prevent the problem from being trivial. In fact, if after $t \ge t_k$ the system (2.1) is not switched any more, then the switched system becomes

$$x'_{i_k}(t) = f_{i_k}(t, x_{i_k}(t), x_{i_k t}), \quad t \ge t_k$$

and the convergent property of (2.1) is trivially dependent on the i_kth subsystem. Assumption (B2) means that for any consecutive interval $[t_{k-1}, t_k)$, $[t_k, t_{k+1})$, the active subsystems are different. The assumption (B3) guarantees that all subsystems have a unique solution for any initial condition.

Definition 2.1. For given switching signal, the equilibrium point of system (2.1) is said to be

(i) stable, if for any given t_0 and $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_{t_0}\| < \delta$ implies that $\|x(t; t_0, x_{t_0})\| \le \epsilon$ for $t \ge t_0$. Otherwise, the system is said to be unstable;

(ii) locally exponentially stable, if there exist $M > 0, \gamma > 0$ and $\epsilon > 0$ such that if $\|x_{t_0}\| < \gamma$, the solution $x(t; t_0, x_{t_0})$ of (2.1) satisfies

$$\|x(t; t_0, x_{t_0})\| \le M \mathrm{e}^{-\epsilon(t-t_0)}, \quad t \ge t_0;$$

(iii) globally exponentially stable, if there exist $M > 0$ and $\epsilon > 0$ such that the solution $x(t; t_0, x_{t_0})$ of (2.1) satisfies

$$\|x(t; t_0, x_{t_0})\| \le M \|x_{t_0}\| \mathrm{e}^{-\epsilon(t-t_0)}, \quad t \ge t_0.$$

Theorem 2.1. *Assume that $\alpha > 0$, $V \in C^1[R, R_+]$, $g \in K_1$, $h(s, \cdot) \in K_1$ for any fixed $s \in R$, and*

$$V'(t) \le -\alpha V(t) + g(\|V_t\|)\|V_t\| + \int_{t_0}^{t} h(t-s, V(s))V(s)\mathrm{d}s.$$

If there exist constants $r > 0$ and $\lambda > 0$ such that

$$g(r) + \int_0^\infty h(s, r)\mathrm{e}^{\lambda s}\mathrm{d}s < \alpha,$$

then there exists an $\epsilon > 0$ such that $\|V_{t_0}\| < r$ implies that

$$V(t) \le \|V_{t_0}\| \mathrm{e}^{-\epsilon(t-t_0)}, \quad t \ge t_0,$$

where ϵ satisfies

$$\epsilon + g(r)\mathrm{e}^{\epsilon\tau} + \int_0^\infty h(s, r)\mathrm{e}^{\epsilon s}\mathrm{d}s < \alpha.$$

Proof. Since $g(r) + \int_0^\infty h(s, r)\mathrm{e}^{\lambda s}\mathrm{d}s < \alpha$, there exists an ϵ, $0 < \epsilon \le \lambda$, such that

$$g(r)\mathrm{e}^{\epsilon\tau} + \int_0^\infty h(s, r)\mathrm{e}^{\epsilon s}\mathrm{d}s < \alpha - \epsilon.$$

Let

$$P(t) = V(t)\Gamma(\epsilon, t - t_0).$$

Then $P(t) \ge V(t)$, $\|P_t\| \ge \|V_t\|$ and for $t > t_0$,

$$\begin{aligned}
P'(t) &= V'(t)\Gamma(\epsilon, t-t_0) + \epsilon V(t)\Gamma(\epsilon, t-t_0) \\
&\le \Gamma(\epsilon, t-t_0)\left[-\alpha V(t) + g(\|V_t\|)\|V_t\| + \int_{t_0}^{t} h(t-s, V(s))V(s)\mathrm{d}s\right] + \epsilon V(t)\Gamma(\epsilon, t-t_0) \\
&\le -\alpha P(t) + g(\|V_t\|)\|P_t\|\mathrm{e}^{\epsilon\tau} + \int_{t_0}^{t} h(t-s, V(s))P(s)\mathrm{e}^{\epsilon(t-s)}\mathrm{d}s + \epsilon P(t) \\
&\le (-\alpha + \epsilon)P(t) + g(\|V_t\|)\|P_t\|\mathrm{e}^{\epsilon\tau} + \int_{t_0}^{t} h(t-s, V(s))P(s)\mathrm{e}^{\epsilon(t-s)}\mathrm{d}s.
\end{aligned} \tag{2.4}$$

We claim that $\|P_{t_0}\| < r$ implies $P(t) \le \|P_{t_0}\|$, which can be justified as follows.
For any $d \in (1, r/\|P_{t_0}\|)$, $P(t) \le d\|P_{t_0}\| = d\|V_{t_0}\| =: M$.
If it is not true, then there must exist a $t^* > t_0$, such that

$$P(t^*) = M, \qquad P(t) < M, \quad t < t^*.$$

Thus $P'(t^*) \geq 0$. On the other hand, it follows from (2.4) that

$$\begin{aligned} P'(t^*) &\leq (-\alpha+\epsilon)P(t^*) + g(\|V_{t^*}\|)\|P_{t^*}\|e^{\epsilon\tau} + \int_{t_0}^{t^*} h(t^*-s, V(s))P(s)e^{\epsilon(t^*-s)}ds \\ &\leq (-\alpha+\epsilon)M + g(\|P_{t^*}\|)e^{\epsilon\tau}M + \int_{t_0}^{t^*} h(t^*-s, P(s))Me^{\epsilon(t^*-s)}ds \\ &\leq M\left[(-\alpha+\epsilon) + g(r)e^{\epsilon\tau} + \int_0^{\infty} h(s,r)e^{\epsilon s}ds\right] < 0, \end{aligned}$$

this contradiction implies $P(t) \leq d\|P_{t_0}\|$. Let $d \to 1$, we obtain $P(t) \leq \|P_{t_0}\|$ and thus

$$V(t) \leq \|V_{t_0}\|e^{-\epsilon(t-t_0)}.$$

The proof is complete. □

3. Switching laws

In this section, we first prove an inequality for switched delay system (2.1). Based on the inequality, switching laws are derived which guarantee stability of switched delay system (2.1).

Theorem 3.1. *Assume that there exist $M_{\sigma_i}, \epsilon_{\sigma_i} > 0$ such that*

$$\|x_{\sigma_1}(t)\| \leq M_{\sigma_1}\|x_{\sigma_1 t_0}\|e^{-\epsilon_{\sigma_1}(t-t_0)}, \quad t \geq t_0,\ \sigma_1 \in \aleph_1, \tag{3.1}$$

and

$$\|x_{\sigma_2}(t)\| \leq M_{\sigma_2}\|x_{\sigma_2 t_0}\|e^{\epsilon_{\sigma_2}(t-t_0)}, \quad t \geq t_0,\ \sigma_2 \in \aleph_2. \tag{3.2}$$

Then for any function $x_{t_0} \in C_\tau$, the solution $x(t)$ of switched delay system (2.1) satisfies

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \aleph_i} M_{\sigma_i}^{k_{\sigma_i}} \exp\left\{\sum_{\sigma_1 \in \aleph_{\sigma_1}} \epsilon_{\sigma_1}[k_{\sigma_1}(t)\tau - \pi_{\sigma_1}(t)] + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)\right\}. \tag{3.3}$$

Proof. We replace $M_{\sigma_1}, M_{\sigma_2}, \epsilon_{\sigma_1}, \epsilon_{\sigma_2}$ with $M_{\sigma_1,1}, M_{\sigma_2,2}, \epsilon_{\sigma_1,1}, \epsilon_{\sigma_2,2}$. Let t_{j-1}, t_j and t_{j+1} be consecutive switching points.

(1) Suppose that the lth subsystem of (2.2) is active on the interval $[t_j, t_{j+1})$,
- if the ith subsystem of (2.2) is active on the interval $[t_{j-1}, t_j)$, then

$$\begin{aligned} \|x(t)\| &\leq M_{l,1}\|x_{t_j}\|e^{-\epsilon_{l,1}(t-t_j)} \\ &\leq M_{i,1}M_{l,1}\|x_{t_{j-1}}\|e^{-\epsilon_{i,1}(t_j-t_{j_1}-\tau)}e^{-\epsilon_{l,1}(t-t_j)}, \quad t \in [t_j, t_{j+1}); \end{aligned}$$

- if the ith subsystem of (2.3) is active on the interval $[t_{j-1}, t_j)$, then

$$\begin{aligned} \|x(t)\| &\leq M_{l,1}\|x_{t_j}\|e^{-\epsilon_{l,1}(t-t_j)} \\ &\leq M_{l,1}M_{i,2}\|x_{t_{j-1}}\|e^{-\epsilon_{l,1}(t-t_j)}e^{\epsilon_{i,2}(t_{j-1}-t_j)}, \quad t \in [t_j, t_{j+1}); \end{aligned}$$

(2) Suppose that the lth subsystem of (2.3) is active on the interval $[t_j, t_{j+1})$,
- if the ith subsystem of (2.2) is active on the interval $[t_{j-1}, t_j)$, then

$$\begin{aligned} \|x(t)\| &\leq M_{l,2}\|x_{t_j}\|e^{\epsilon_{l,2}(t-t_j)} \\ &\leq M_{i,1}M_{l,2}\|x_{t_{j-1}}\|e^{-\epsilon_{i,1}(t_{j-1}-t_j-\tau)}e^{\epsilon_{l,2}(t-t_j)}, \quad t \in [t_j, t_{j+1}); \end{aligned}$$

- if the ith subsystem of (2.3) is active on the interval $[t_{j-1}, t_j)$, then

$$\begin{aligned} \|x(t)\| &\leq M_{l,2}\|x_{t_j}\|e^{\epsilon_{l,2}(t-t_j)} \\ &\leq M_{i,2}M_{l,2}\|x_{t_{j-1}}\|e^{\epsilon_{i,2}(t_j-t_{j-1})}e^{\epsilon_{l,2}(t-t_j)}, \quad t \in [t_j, t_{j+1}). \end{aligned}$$

Using iterative method, we can obtain

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \aleph_i} M_{\sigma_i}^{k_{\sigma_i}} \exp\left\{ \sum_{\sigma_1 \in \aleph_{\sigma_1}} \epsilon_{\sigma_1}[k_{\sigma_1}(t)\tau - \pi_{\sigma_1}(t)] + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t) \right\}.$$

The proof is complete. □

Specially, if $\aleph_1 = \aleph_2 = \{1\}$, then (2.1) becomes

$$x'(t) = f_\sigma(t, x(t), x_t) \tag{3.4}$$

where $\sigma = 1, 2$. Let $x_1(t)$ and $x_2(t)$ be the solution of system, then

$$x_1'(t) = f_1(t, x_1(t), x_{1t}) \tag{3.5}$$

$$x_2'(t) = f_2(t, x_2(t), x_{2t}) \tag{3.6}$$

respectively, $k(t)$ be the times of subsystem (3.5) being switched on during the interval $[t_0, t)$, $\pi_1(t)$ and $\pi_2(t)$ are the total time of subsystems (3.5), (3.6) used respectively on $[t_0, t)$, then we have

Corollary 3.1. *Assume that there exist $M_1, M_2 > 0$ and $\epsilon_1, \epsilon_2 > 0$, such that*

$$\|x_1(t)\| \leq M_1\|x_{1t_0}\|\mathrm{e}^{-\epsilon_1(t-t_0)}, \quad t \geq t_0,$$
$$\|x_2(t)\| \leq M_2\|x_{2t_0}\|\mathrm{e}^{\epsilon_2(t-t_0)}, \quad t \geq t_0.$$

Then the solution of switched delay system (3.4) satisfies

$$\|x(t)\| \leq \|x_{t_0}\|(M_1M_2)^{k(t)} \exp\{\epsilon_1 k(t)\tau - \epsilon_1\pi_1(t) + \epsilon_2\pi_2(t)\}. \tag{3.7}$$

Define switching laws as follows:

(S_1) $\frac{\pi_2(t)}{\pi_1(t)} + \frac{k(t)\ln M_1M_2}{\epsilon_2\pi_1(t)} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)} \leq \frac{\epsilon_1}{\epsilon_2}$, $t \geq T$ for some T;
(S_2) $\limsup_{t\to\infty}[\frac{\pi_2(t)}{\pi_1(t)} + \frac{k(t)\ln M_1M_2}{\epsilon_2\pi_1(t)} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)}] = \frac{\epsilon_1}{\epsilon_2}$;
(S_3) $\limsup_{t\to\infty}[\frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\ln M_1M_2}{\epsilon_2\pi_1(t)} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)}] < 0$.

We can prove the following theorems.

Theorem 3.2. *Assume that the conditions of Corollary 3.1 hold. Then*

(1) *switching laws* (S_1) *and* (S_2) *imply that switched delay system* (3.4) *is stable;*
(2) *switching laws* (S_3) *implies that switched delay system* (3.4) *is exponentially stable.*

Proof. Without loss of generality, suppose that switched delay system (3.4) is

$$\begin{cases} x'(t) = f_1(t, x, x_t), & t \in [t_{2k}, t_{2k+1}), \\ x'(t) = f_2(t, x, x_t), & t \in [t_{2k-1}, t_{2k}), \end{cases}$$

where t_j are switching points, $j \in \aleph$. Then, when $t \in [t_{2k}, t_{2k+1})$

$$\begin{aligned} \|x(t)\| &\leq M_1\|x_{t_{2k}}\|\mathrm{e}^{-\epsilon_1(t-t_{2k})} \\ &\leq M_1M_2\|x_{t_{2k-1}}\|\mathrm{e}^{\epsilon_2(t_{2k}-t_{2k-1})}\mathrm{e}^{-\epsilon_1(t-t_{2k})} \\ &\leq M_1M_2M_1\|x_{t_{2k-2}}\|\mathrm{e}^{\epsilon_2(t_{2k}-t_{2k-1})}\mathrm{e}^{-\epsilon_1[(t-t_{2k})+(t_{2k-1}-t_{2k-2}-\tau)]} \\ &\leq (M_1M_2)^2\|x_{t_{2k-3}}\|\mathrm{e}^{\epsilon_2[(t_{2k}-t_{2k-1})+(t_{2k-2}-t_{2k-3})]}\mathrm{e}^{-\epsilon_1[(t-t_{2k})+(t_{2k-1}-t_{2k-2}-\tau)]} \\ &\leq \cdots\cdots \\ &\leq (M_1M_2)^k(t)\|x_{t_0}\|\mathrm{e}^{\epsilon_2\pi_2(t)}\mathrm{e}^{-\epsilon_1\pi_1(t)+k(t)\epsilon_1\tau} \\ &= \|x_{t_0}\|\mathrm{e}^{\epsilon_2\pi_2(t)-\epsilon_1\pi_1(t)+k(t)\epsilon_1\tau+k(t)\ln M_1M_2}. \end{aligned}$$

So

$$\|x(t)\| \le \|x_{t_0}\| \mathrm{e}^{\epsilon_2 \pi_1(t)\left[\frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\epsilon_1 \tau}{\epsilon_2 \pi_1(t)} + \frac{k(t)\ln M_1 M_2}{\epsilon_2 \pi_1(t)}\right]}.$$

It is easy to see that the result (1) of Theorem 3.2 is true from about inequality. Next, we prove the result (2). Let

$$\limsup_{t\to\infty} \left[\frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\ln M_1 M_2}{\epsilon_2 \pi_1(t)} + \frac{k(t)\epsilon_1 \tau}{\epsilon_2 \pi_1(t)}\right] = -2\epsilon^*.$$

Without loss of generality, suppose that

$$\frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\ln M_1 M_2}{\epsilon_2 \pi_1(t)} + \frac{k(t)\epsilon_1 \tau}{\epsilon_2 \pi_1(t)} < -\epsilon^*, \quad t \ge t_0.$$

Then $\pi_1(t) \ge \frac{\epsilon_2}{\epsilon_1}\pi_2(t)$, and

$$\begin{aligned}
\epsilon_2 \pi_1(t)\left[\frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\epsilon_1 \tau}{\epsilon_2 \pi_1(t)} + \frac{k(t)\ln M_1 M_2}{\epsilon_2 \pi_1(t)}\right] &\le -\epsilon^* \epsilon_2 \pi_1(t) \\
&\le -\epsilon^* \epsilon_2 \pi_1(t)/2 - \epsilon^* \epsilon_2 \pi_1(t)/2 \\
&\le -\frac{\epsilon^* \epsilon_2}{2}\pi_1(t) - \frac{\epsilon^* \epsilon_2^2}{2\epsilon_1}\pi_2(t) \\
&\le -\epsilon[\pi_1(t) + \pi_2(t)] = -\epsilon(t - t_0),
\end{aligned}$$

where $\epsilon = \min\{\frac{\epsilon^* \epsilon_2}{2}, \frac{\epsilon^* \epsilon_2^2}{2\epsilon_1}\}$. Thus

$$\|x(t)\| \le \|x_{t_0}\| \mathrm{e}^{-\epsilon(t - t_0)}.$$

The proof is complete. □

For switched delay system (2.1), define switching law as following:

(S4) $\sum_{\sigma_i \in \aleph_i} [k_{\sigma_i}(t)\ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau + \epsilon_{\sigma_2}\pi_{\sigma_2}(t)] \le \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t), \quad t \ge T$ for some T;

(S5) $\limsup_{t\to\infty} \frac{1}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i}(t)\ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau + \epsilon_{\sigma_2}\pi_{\sigma_2}(t)] = 1$;

(S6) $\limsup_{t\to\infty} \frac{1}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i}(t)\ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau + \epsilon_{\sigma_2}\pi_{\sigma_2}(t)] < 1$.

Theorem 3.3. *Assume that the conditions of Theorem 3.1 hold. Then the switched systems (2.1) is*

(1) *switching laws* (S4) *and* (S5) *imply that switched delay system (2.1) is stable;*
(2) *switching laws* (S6) *implies that switched delay system (2.1) is exponentially stable.*

The proof is similar to that of Theorem 3.2.

4. Linear switched delay systems

In this section, we study the linear switched delay systems

$$x'(t) = A_{\sigma_i} x(t) + B_{\sigma_i} x(t - \tau(t)) + \int_{t_0}^{t} h_{\sigma_i}(t - s)x(s)\mathrm{d}s, \tag{4.1}$$

where $x \in R^n$, $A_{\sigma_i}, B_{\sigma_i} \in R^{n\times n}$, $h_{\sigma_i} \in C[R_+, R^n]$, $\sigma_i \in \aleph_{\sigma_i}$, $i = 1, 2$, $0 \le \tau(t) \le \tau$.

Without loss of generality, we make the following assumptions.

(A1) $\Phi_{\sigma_i} \in R^{n\times n}$, satisfy

$$\begin{cases} \dfrac{\partial \Phi_{\sigma_i}(t, s)}{\partial t} = A_{\sigma_i} \Phi_{\sigma_i}(t, s), \\ \Phi_{\sigma_i}(s, s) = I, \end{cases}$$

where I is an identity matrix, and $\sigma_i \in \aleph_i$, $i = 1, 2$.

(A2) there exist $\lambda_{\sigma_i} > 0$ and $M_{\sigma_i} \geq 1$, such that

$$\|\Phi_{\sigma_1}(t, t_0)\| \leq M_{\sigma_1} \mathrm{e}^{-\lambda_{\sigma_1}(t-t_0)},$$

and

$$\|\Phi_{\sigma_2}(t, t_0)\| \leq M_{\sigma_2} \mathrm{e}^{\lambda_{\sigma_2}(t-t_0)}.$$

Let $x_{\sigma_1(t)}$ and $x_{\sigma_2(t)}$ be the solutions of the subsystems

$$x'_{\sigma_1}(t) = A_{\sigma_1} x_{\sigma_1} + B_{\sigma_1} x_{\sigma_1}(t - \tau(t)) + \int_{t_0}^{t} h_{\sigma_1}(t - s) x_{\sigma_1}(s) \mathrm{d}s, \tag{4.2}$$

$$x'_{\sigma_2}(t) = A_{\sigma_2} x_{\sigma_2} + B_{\sigma_2} x_{\sigma_2}(t - \tau(t)) + \int_{t_0}^{t} h_{\sigma_2}(t - s) x_{\sigma_2}(s) \mathrm{d}s, \tag{4.3}$$

then we have the following theorems.

Theorem 4.1. *Assume that* (A1), (A2) *hold and* $\exists \epsilon_{\sigma_i} > 0$, $i = 1, 2$, *such that*

$$-\lambda_{\sigma_1} + M_{\sigma_1} \|B_{\sigma_1}\| \mathrm{e}^{\epsilon_{\sigma_1} \tau} + \int_0^\infty M_{\sigma_1} \|h_{\sigma_1}(s)\| \mathrm{e}^{\epsilon_{\sigma_1} s} \mathrm{d}s < -\epsilon_{\sigma_1},$$

$$\lambda_{\sigma_2} + M_{\sigma_2} \|B_{\sigma_2}\| + \int_0^\infty M_{\sigma_2} \|h_{\sigma_2}(s)\| \mathrm{e}^{-\epsilon_{\sigma_2} s} \mathrm{d}s < \epsilon_{\sigma_2}.$$

Then switched delay systems (4.1) *is*

(1) *stable, if* $\exists T > 0$, *such that*

$$\frac{1}{\sum\limits_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i} \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] \leq 1, \quad t \geq T \tag{4.4}$$

and

$$\limsup_{t \to \infty} \frac{1}{\sum\limits_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i} \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] = 1, \tag{4.5}$$

(2) *globally exponentially stable, if*

$$\limsup_{t \to \infty} \frac{1}{\sum\limits_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i} \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] < 1. \tag{4.6}$$

Proof. By the variation of parameters, the solution of subsystem (4.2) is

$$x_{\sigma_1}(t) = \Phi_{\sigma_1}(t, t_0) x_{\sigma_1}(t_0) + \int_{t_0}^{t} \Phi_{\sigma_1}(t, s) B_{\sigma_1} x_{\sigma_1}(s - \tau(s)) \mathrm{d}s + \int_{t_0}^{t} \Phi_{\sigma_1}(t, s) \int_{t_0}^{s} h_{\sigma_1}(s - \theta) x_{\sigma_1}(\theta) \mathrm{d}\theta \mathrm{d}s.$$

Taking the norm on both sides, we have

$$\begin{aligned}
\|x_{\sigma_1}(t)\| \leq\ & \|\Phi_{\sigma_1}(t, t_0)\| \|x_{\sigma_1}(t_0)\| + \int_{t_0}^{t} \|\Phi_{\sigma_1}(t, s)\| \|B_{\sigma_1}\| \|x_{\sigma_1}(s - \tau(s))\| \mathrm{d}s \\
& + \int_{t_0}^{t} \int_{t_0}^{s} \|\Phi_{\sigma_1}(t, s)\| \|h_{\sigma_1}(s - \theta)\| \|x_{\sigma_1}(\theta)\| \mathrm{d}\theta \mathrm{d}s \\
\leq\ & M_{\sigma_1} \|x_{\sigma_1 t_0}\| \mathrm{e}^{-\lambda_{\sigma_1}(t-t_0)} + \int_{t_0}^{t} M_{\sigma_1} \mathrm{e}^{-\lambda_{\sigma_1}(t-s)} \|B_{\sigma_1}\| \|x_{\sigma_1 s}\| \mathrm{d}s \\
& + \int_{t_0}^{t} \int_{t_0}^{s} M_{\sigma_1} \mathrm{e}^{-\lambda_{\sigma_1}(t-s)} \|h_{\sigma_1}(s - \theta)\| \|x_{\sigma_1}(\theta)\| \mathrm{d}\theta \mathrm{d}s \\
=:\ & P^*_{\sigma_1}(t), \quad t \geq t_0.
\end{aligned}$$

Let

$$P_{\sigma_1}(t) = \begin{cases} P^*_{\sigma_1}(t), & t \geq t_0, \\ M_{\sigma_1}\|x_{\sigma_1 t_0}\|, & t_0 - \tau \leq t \leq t_0. \end{cases}$$

Then $P_{\sigma_1}(t) \geq \|x_{\sigma_1}(t)\|$, $t \geq t_0 - \tau$ and

$$\begin{aligned} P'_{\sigma_1}(t) &= -\lambda_{\sigma_1} P_{\sigma_1}(t) + M_{\sigma_1}\|B_{\sigma_1}\|\|x_{\sigma_1 t}\| + \int_{t_0}^{t} M_{\sigma_1}\|h_{\sigma_1}(t-s)\|\|x_{\sigma_1}(s)\|\mathrm{d}s \\ &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t) + M_{\sigma_1}\|B_{\sigma_1}\|\|P_{\sigma_1 t}\| + \int_{t_0}^{t} M_{\sigma_1}\|h_{\sigma_1}(t-s)\| P_{\sigma_1}(s)\mathrm{d}s. \end{aligned}$$

For any $K > 1$, we claim that

$$P_{\sigma_1}(t) < K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}. \tag{4.7}$$

In fact, if (4.7) is not true, then $\exists t^* > t_0$, such that

$$\begin{aligned} &P_{\sigma_1}(t^*) = K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t^*-t_0)}, \quad \text{and} \\ &P_{\sigma_1}(t) \leq K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}, \quad t < t^*. \end{aligned}$$

Thus $P'_{\sigma_1}(t^*) \geq -\epsilon_{\sigma_1} K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t^*-t_0)} = -\epsilon_{\sigma_1} P_{\sigma_1}(t^*)$. On the other hand

$$\begin{aligned} P'_{\sigma_1}(t^*) &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t^*) + M_{\sigma_1}\|B_{\sigma_1}\|\|P_{\sigma_1 t^*}\| + \int_{t_0}^{t^*} M_{\sigma_1}\|h_{\sigma_1}(t^*-s)\| P_{\sigma_1}(s)\mathrm{d}s \\ &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t^*) + M_{\sigma_1}\|B_{\sigma_1}\| P_{\sigma_1}(t^*)\mathrm{e}^{\epsilon_{\sigma_1}\tau} + \int_{t_0}^{t^*} K M_{\sigma_1}\|h_{\sigma_1}(t^*-s)\|\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(s-t_0)}\mathrm{d}s \\ &= P_{\sigma_1}(t^*)\left[-\lambda_{\sigma_1} + M_{\sigma_1}\|B_{\sigma_1}\|\mathrm{e}^{\epsilon_{\sigma_1}\tau} + \int_{t_0}^{t^*} M_{\sigma_1}\|h_{\sigma_1}(t^*-s)\|\mathrm{e}^{\epsilon_{\sigma_1}(t^*-s)}\mathrm{d}s\right] < -\epsilon_{\sigma_1} P_{\sigma_1}(t^*). \end{aligned}$$

This contradiction implies that $P_{\sigma_1}(t) < K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}$. Let $K \to 1$, we obtain $P_{\sigma_1}(t) \leq \|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}$ and hence

$$\|x_{\sigma_1}(t)\| \leq M_{\sigma_1}\|x_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}. \tag{4.8}$$

Similarly, we can obtain

$$\|x_{\sigma_2}(t)\| \leq M_{\sigma_2}\|x_{\sigma_2 t_0}\|\mathrm{e}^{\epsilon_{\sigma_2}(t-t_0)}. \tag{4.9}$$

Combining (4.8) and (4.9) and Theorem 3.1, we have

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \aleph_i} M_{\sigma_i}^{k_{\sigma_i}(t)} \exp\left\{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau - \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t) + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)\right\}.$$

The results of (1) and (2) of Theorem 4.1 can be obtained by Theorem 3.2. The proof is complete. □

Remark 1. It should be pointed out that, in Theorem 4.1, (4.4)–(4.6) are actually switching laws which guarantee the switched delay system (4.1) is stable or locally exponentially stable.

Corollary 4.1. *Assume that* $\exists \lambda_{\sigma_i} > 0$, $\epsilon_{\sigma_2} > 0$ *such that*

(i) $\lambda(A^T_{\sigma_1} + A_{\sigma_1}) \leq -2\lambda_{\sigma_1}$, $\lambda(A^T_{\sigma_2} + A_{\sigma_2}) \leq 2\lambda_{\sigma_2}$;
(ii) $-\lambda_{\sigma_1} + \|B_{\sigma_1}\|\mathrm{e}^{\epsilon_{\sigma_1}\tau} + \int_0^\infty \|h_{\sigma_1}(s)\|\mathrm{e}^{\epsilon_{\sigma_1}s}\mathrm{d}s < -\epsilon_{\sigma_1}$;
(iii) $\lambda_{\sigma_2} + \|B_{\sigma_2}\| + \int_0^\infty \|h_{\sigma_2}(s)\|\mathrm{e}^{-\epsilon_{\sigma_2}s}\mathrm{d}s < \epsilon_{\sigma_2}$.

Then the switched delay systems (4.1) is

(1) *stable, if* $\exists T > 0$, *such that*

$$\frac{1}{\sum\limits_{\sigma_1\in\aleph_1}\epsilon_{\sigma_1}\pi_{\sigma_1}(t)}\sum_{\sigma_i\in\aleph_i}[\epsilon_{\sigma_1}k_{\sigma_1}\tau+\epsilon_{\sigma_2}\pi_{\sigma_2}(t)]\le 1,\qquad t\ge T$$

and

$$\limsup_{t\to\infty}\frac{1}{\sum\limits_{\sigma_1\in\aleph_1}\epsilon_{\sigma_1}\pi_{\sigma_1}(t)}\sum_{\sigma_i\in\aleph_i}[\epsilon_{\sigma_1}k_{\sigma_1}\tau+\epsilon_{\sigma_2}\pi_{\sigma_2}(t)]=1,$$

(2) *globally exponentially stable, if*

$$\limsup_{t\to\infty}\frac{1}{\sum\limits_{\sigma_1\in\aleph_1}\epsilon_{\sigma_1}\pi_{\sigma_1}(t)}\sum_{\sigma_i\in\aleph_i}[\epsilon_{\sigma_1}k_{\sigma_1}\tau+\epsilon_{\sigma_2}\pi_{\sigma_2}(t)]<1.$$

Proof. Let $V_{\sigma_i}=x_{\sigma_i}^Tx_{\sigma_i}$, then the derivative of V_{σ_i} along the system $x'_{\sigma_i}=A_{\sigma_i}x_{\sigma_i}$ is

$$V'_{\sigma_i}=x_{\sigma_i}^T(A_{\sigma_i}^T+A_{\sigma_i})x_{\sigma_i}.$$

Thus

- $$V'_{\sigma_1}(t)\le -2\lambda_{\sigma_1}V_{\sigma_1}(t)\Rightarrow V_{\sigma_1}(t)\le V_{\sigma_1}(t_0)\mathrm{e}^{-2\lambda_{\sigma_1}(t-t_0)},\quad\text{and}\quad \|\Phi_{\sigma_1}(t,t_0)\|\le \mathrm{e}^{-\lambda_{\sigma_1}(t-t_0)};\tag{4.10}$$
- $$V'_{\sigma_2}(t)\le 2\lambda_{\sigma_2}V_{\sigma_2}(t)\Rightarrow V_{\sigma_2}(t)\le V_{\sigma_2}(t_0)\mathrm{e}^{2\lambda_{\sigma_2}(t-t_0)},\quad\text{and}\quad \|\Phi_{\sigma_2}(t,t_0)\|\le \mathrm{e}^{\lambda_{\sigma_2}(t-t_0)}.\tag{4.11}$$

All the results of this theorem can be obtained by (4.10) and (4.11) and Theorem 4.1. The proof is complete. □

We illustrate the results of Corollary 4.1 with a simple linear switched delay system.

Example 4.1.

$$A_1=\begin{bmatrix}-3 & 10\\ -10 & -3.5\end{bmatrix},\qquad B_1=\begin{bmatrix}\frac{1}{2\mathrm{e}} & 0\\ \frac{1}{\mathrm{e}} & 0\end{bmatrix},\qquad h_1=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$

$$A_2=\begin{bmatrix}-1 & -3\\ 3+\sqrt{12} & 1\end{bmatrix},\qquad B_2=\begin{bmatrix}\frac{1}{3\mathrm{e}} & \frac{1}{4\mathrm{e}}\\ 0 & 0\end{bmatrix},\qquad h_2=\begin{bmatrix}\mathrm{e}^t\cos^2 t & 0\\ \mathrm{e}^t\sin^2 t & 0\end{bmatrix}.$$

Hence $\lambda(A_1^{\mathrm{T}}+A_1)\le -6$, $\lambda(A_2^{\mathrm{T}}+A_2)\le 4$; $\|B_1\|=\frac{3}{2\mathrm{e}}$, $\|B_2\|=\frac{1}{3\mathrm{e}}$; $\|h_1\|=0$, $\|h_2\|=\mathrm{e}^t$. Take $\tau=1,\lambda_1=3,\lambda_2=2,\epsilon_1=1,\epsilon_2=4$.

Then,

$$-\lambda_1+\|B_1\|\mathrm{e}^{\epsilon_1\tau}+\int_0^\infty\|h_1(s)\|\mathrm{e}^{\epsilon_1 s}\mathrm{d}s=-3+3/2<-1=-\epsilon_1;$$
$$\lambda_2+1+\|B_2\|\mathrm{e}^{\tau}+\int_0^\infty\|h_2(s)\|\mathrm{e}^{(-\epsilon_2+1)s}\mathrm{d}s<2+1+1/3+1/2\le 4=\epsilon_2.$$

With the switching laws from Corollary 4.1, we have

(S_1) $\pi_1(t)>k_1(t)+4\pi_{\sigma_2}(t)$ and $\limsup_{t\to\infty}\frac{k_1(t)+4\pi_{\sigma_2}(t)}{\pi_1(t)}=1$

(S_2) $\limsup_{t\to\infty}\frac{k_1(t)+4\pi_{\sigma_2}(t)}{\pi_1(t)}<1$.

By Corollary 4.1, we know that

- if we choose the switching law (S_1), then the switched delay system

$$x'(t) = A_i x(t) + B_i x(t - \tau(t)) + \int_{t_0}^{t} h_i(t-s)x(s)\mathrm{d}s, \tag{4.12}$$

  where $i = 1, 2, 0 \le \tau(t) \le 1$, is stable;
- if we choose the switching law (S_2), then switched delay system (4.12) is globally exponentially stable.

5. Nonlinear switched delay systems

In this section, we study the nonlinear switched delay systems

$$x'(t) = A_{\sigma_i} x + f_{\sigma_i}(t, x_t) + \int_{t_0}^{t} h_{\sigma_i}(t-s, x(s))\mathrm{d}s, \tag{5.1}$$

where $x \in R^n$, $A_{\sigma_i} \in R^{n\times n}$, $f_{\sigma_i} \in C[R \times C_\tau, R^n]$, $h_{\sigma_i} \in C[R \times R^n, R^n]$, $\sigma_i \in \aleph_i$, $i = 1, 2$.

Denote $\widehat{f}_{\sigma_2}(t, y_t) = \mathrm{e}^{-\epsilon_{\sigma_2}(t-t_0)} f_{\sigma_2}(t, x_t)$, $\widehat{h}_{\sigma_2}(t-s, y(s)) = \mathrm{e}^{-\epsilon_{\sigma_2}(t-t_0)} h_{\sigma_2}(t-s, x(s))$, where $y(s) = x(s)\mathrm{e}^{-\epsilon_{\sigma_2}(s-t_0)}$, then we can obtain the following results about local exponential stability.

Theorem 5.1. *Assume that* (A1), (A2) *hold and*

(i) *there exist* $\widetilde{f}_{\sigma_i} \in K_1$ *and* $\widetilde{h}_{\sigma_i} \in C[R^2, R_+]$ *such that* $\widetilde{h}_{\sigma_i}(s, \cdot) \in K_1$ *for any* $s \in R_+$ *and*

$$\|f_{\sigma_1}(t, x_t)\| \le \widetilde{f}_{\sigma_1}(\|x_t\|)\|x_t\|, \qquad \|\widehat{f}_{\sigma_2}(t, x_t)\| \le \widetilde{f}_{\sigma_2}(\|x_t\|)\|x_t\|,$$
$$\|h_{\sigma_1}(t-s, x(s))\| \le \widetilde{h}_{\sigma_1}(t-s, \|x(s)\|)\|x(s)\|, \qquad \|\widehat{h}_{\sigma_2}(t-s, x(s))\| \le \widetilde{h}_{\sigma_2}(t-s, \|x(s)\|)\|x(s)\|;$$

(ii) $\exists \epsilon_{\sigma_1}, \epsilon_{\sigma_2}, r > 0$ *such that*

$$\epsilon_{\sigma_1} + M_{\sigma_1}\widetilde{f}_{\sigma_1}(r)\mathrm{e}^{\epsilon_{\sigma_1}\tau} + \int_0^{\infty} M_{\sigma_1}\widetilde{h}_{\sigma_1}(s, r)\mathrm{e}^{\epsilon_{\sigma_1}s}\mathrm{d}s < \lambda_{\sigma_1},$$
$$\lambda_{\sigma_2} + M_{\sigma_2}\widetilde{f}_{\sigma_2}(r)\mathrm{e}^{\epsilon_{\delta_2}\tau} + M_{\sigma_2}\int_0^{\infty} \widetilde{h}_{\sigma_2}(s, r)\mathrm{e}^{\epsilon_{\sigma_2}s}\mathrm{d}s < \epsilon_{\sigma_2}.$$

Then the switched delay systems (5.1) *is*

(1) *stable, if* $\exists T > 0$, *such that*

$$\frac{\sum\limits_{\sigma_i\in\aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}k_{\sigma_1}(t)\tau + \sum\limits_{\sigma_2\in\aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} \le 1, \quad t \ge T \tag{5.2}$$

and

$$\limsup_{t\to\infty} \frac{\sum\limits_{\sigma_i\in\aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}k_{\sigma_1}(t)\tau + \sum\limits_{\sigma_2\in\aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} = 1; \tag{5.3}$$

(2) *locally exponentially stable, if*

$$\limsup_{t\to\infty} \frac{\sum\limits_{\sigma_i\in\aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}k_{\sigma_1}(t)\tau + \sum\limits_{\sigma_2\in\aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} < 1. \tag{5.4}$$

Proof. Let $x_{\sigma_1}(t)$ be the solution of the σ_1th subsystem, $\sigma_1 \in \aleph_1$, i.e.

$$x'_{\sigma_1}(t) = A_{\sigma_1}x_{\sigma_1} + f_{\sigma_1}(t, x_{\sigma_1 t}) + \int_{t_0}^{t} h_{\sigma_1}(t-s, x_{\sigma_1}(s))\mathrm{d}s, \tag{5.5}$$

By the variation of parameters, we have

$$x_{\sigma_1}(t) = \Phi_{\sigma_1}(t, t_0)x_{\sigma_1}(t_0) + \int_{t_0}^{t} \Phi_{\sigma_1}(t, s) f_{\sigma_1}(t, x_{\sigma_1 s})\mathrm{d}s + \int_{t_0}^{t} \Phi_{\sigma_1}(t, s) \int_{t_0}^{s} h_{\sigma_1}(s-\theta, x_{\sigma_1}(\theta))\mathrm{d}\theta\mathrm{d}s.$$

Take norm on both sides, we have

$$\begin{aligned}\|x_{\sigma_1}(t)\| \le{}& \|\Phi_{\sigma_1}(t, t_0)\|\|x_{\sigma_1}(t_0)\| + \int_{t_0}^{t} \|\Phi_{\sigma_1}(t, s)\|\|f_{\sigma_1}(s, x_{\sigma_1 s})\|\mathrm{d}s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s} \|\Phi_{\sigma_1}(t, s)\|\|h_{\sigma_1}(s-\theta, x_{\sigma_1}(\theta))\|\mathrm{d}\theta\mathrm{d}s \\ \le{}& M_{\sigma_1}\|x_{\sigma_1 t_0}\|\mathrm{e}^{-\lambda_{\sigma_1}(t-t_0)} + \int_{t_0}^{t} M_{\sigma_1}\mathrm{e}^{-\lambda_{\sigma_1}(t-s)} \widetilde{f}_{\sigma_1}(\|x_{\sigma_1 s}\|)\|x_{\sigma_1 s}\|\mathrm{d}s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s} M_{\sigma_1}\mathrm{e}^{-\lambda_{\sigma_1}(t-s)}\widetilde{h}_{\sigma_1}(s-\theta, \|x_{\sigma_1}(\theta)\|)\|x_{\sigma_1}(\theta)\|\mathrm{d}\theta\mathrm{d}s \\ =:{}& P^*_{\sigma_1}(t), \quad t \ge t_0.\end{aligned}$$

Let

$$P_{\sigma_1}(t) = \begin{cases} P^*_{\sigma_1}(t), & t \ge t_0, \\ M_{\sigma_1}\|x_{\sigma_1 t_0}\|, & t_0 - \tau \le t \le t_0. \end{cases}$$

Then $P_{\sigma_1}(t) \ge \|x_{\sigma_1}(t)\|$, $t \ge t_0 - \tau$ and for $t \ge t_0$

$$\begin{aligned}P'_{\sigma_1}(t) &\le -\lambda_{\sigma_1}P_{\sigma_1}(t) + M_{\sigma_1}\widetilde{f}_{\sigma_1}(\|x_{\sigma_1 t}\|)\|x_{\sigma_1 t}\| + \int_{t_0}^{t} M_{\sigma_1}\widetilde{h}_{\sigma_1}(t-s, \|x_{\sigma_1}(s)\|)\|x_{\sigma_1}(s)\|\mathrm{d}s \\ &\le -\lambda_{\sigma_1}P_{\sigma_1}(t) + M_{\sigma_1}\widetilde{f}_{\sigma_1}(\|P_{\sigma_1 t}\|)\|P_{\sigma_1 t}\| + \int_{t_0}^{t} M_{\sigma_1}\widetilde{h}_{\sigma_1}(t-s, P_{\sigma_1}(s))P_{\sigma_1}(s)\mathrm{d}s.\end{aligned}$$

We claim that for any $0 < \|P_{\sigma_1 t_0}\| < r$, $K \in (1, \frac{r}{\|P_{\sigma_1 t_0}\|})$,

$$P_{\sigma_1}(t) < K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}. \tag{5.6}$$

For contradiction, suppose that $\exists t^* > t_0$, such that

$$\begin{aligned}&P_{\sigma_1}(t^*) = K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t^*-t_0)}, \quad \text{and} \\ &P_{\sigma_1}(t) \le K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}, \quad t < t^*.\end{aligned}$$

Thus $P'_{\sigma_1}(t^*) \ge -\epsilon_{\sigma_1}K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t^*-t_0)} = -\epsilon_{\sigma_1}P_{\sigma_1}(t^*)$. On the other hand

$$\begin{aligned}P'_{\sigma_1}(t^*) &= -\lambda_{\sigma_1}P_{\sigma_1}(t^*) + M_{\sigma_1}\widetilde{f}_{\sigma_1}(\|P_{\sigma_1 t^*}\|)\|P_{\sigma_1 t^*}\| + \int_{t_0}^{t^*} M_{\sigma_1}\widetilde{h}_{\sigma_1}(t^*-s, P_{\sigma_1}(s))P_{\sigma_1}(s)\mathrm{d}s \\ &\le -\lambda_{\sigma_1}P_{\sigma_1}(t^*) + M_{\sigma_1}\widetilde{f}_{\sigma_1}(r)P_{\sigma_1}(t^*)\mathrm{e}^{\epsilon_{\sigma_1}\tau} + \int_{t_0}^{t^*} M_{\sigma_1}\widetilde{h}_{\sigma_1}(t^*-s, r)P_{\sigma_1}(t^*)\mathrm{e}^{\epsilon_{\sigma_1}(t^*-s)}\mathrm{d}s \\ &\le P_{\sigma_1}(t^*)\left[-\lambda_{\sigma_1} + M_{\sigma_1}\widetilde{f}_{\sigma_1}(r)\mathrm{e}^{\epsilon_{\sigma_1}\tau} + \int_{0}^{\infty} M_{\sigma_1}\widetilde{h}_{\sigma_1}(s, r)\mathrm{e}^{\epsilon_{\sigma_1}s}\mathrm{d}s\right] \\ &< -\epsilon_{\sigma_1}P_{\sigma_1}(t^*).\end{aligned}$$

This contradiction implies that $P_{\sigma_1}(t) < K\|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}$. Let $K \to 1$, we obtain $P_{\sigma_1}(t) \le \|P_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}$ and hence

$$\|x_{\sigma_1}(t)\| \le M_{\sigma_1}\|x_{\sigma_1 t_0}\|\mathrm{e}^{-\epsilon_{\sigma_1}(t-t_0)}. \tag{5.7}$$

For subsystems

$$x'(t) = A_{\sigma_2}x + f_{\sigma_2}(t, x_t) + \int_{t_0}^{t} h_{\sigma_2}(t-s, x(s))\mathrm{d}s, \tag{5.8}$$

let $x_{\sigma_2}(t)$ be the solution (5.8) and let $y_{\sigma_2}(t) = x_{\sigma_2}(t)\Gamma(-\epsilon_{\sigma_2}, t-t_0)$, then

$$x'_{\sigma_2}(t) = A_{\sigma_2}x_{\sigma_2} + f_{\sigma_2}(t, x_{\sigma_2 t}) + \int_{t_0}^{t} h_{\sigma_2}(t-s, x_{\sigma_2}(s))\mathrm{d}s, \tag{5.9}$$

and for $t \geq t_0$

$$\begin{aligned} y'_{\sigma_2}(t) &= (A_{\sigma_2} - \epsilon_{\sigma_2} I)y_{\sigma_2}(t) + f_{\sigma_2}(t, x_t)\mathrm{e}^{-\epsilon_{\sigma_2}(t-t_0)} + \int_{t_0}^{t} h_{\sigma_2}(t-s, x(s))\mathrm{e}^{-\epsilon_{\sigma_2}(t-t_0)}\mathrm{d}s \\ &= (A_{\sigma_2} - \epsilon_{\sigma_2} I)y_{\sigma_2}(t) + \widehat{f}_{\sigma_2}(t, y_t) + \int_{t_0}^{t} \widehat{h}_{\sigma_2}(t-s, y(s))\mathrm{d}s. \end{aligned} \tag{5.10}$$

Let $\Phi^*_{\sigma_2}(t, t_0)$ be a matrix and satisfy

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi^*_{\sigma_2}(t,s) = (A_{\sigma_2} - \epsilon_{\sigma_2} I)\Phi^*_{\sigma_2}(t,s) \\ \Phi^*_{\sigma_2}(s,s) = I. \end{cases}$$

Then $\|\Phi^*_{\sigma_2}(t,s)\| \leq M_{\sigma_2}\mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-t_0)}$, and the solution $y_{\sigma_2}(t)$ of (5.10) is

$$y_{\sigma_2}(t) = \Phi^*(t, t_0)y_{\sigma_2}(t_0) + \int_{t_0}^{t} \Phi^*(t,s)\widehat{f}_{\sigma_2}(s, y_{\sigma_2 s})\mathrm{d}s + \int_{t_0}^{t}\int_{t_0}^{s} \Phi^*(t,s)\widehat{h}_{\sigma_2}(s-\theta, y_{\sigma_2}(\theta))\mathrm{d}\theta\mathrm{d}s.$$

Thus

$$\begin{aligned} \|y_{\sigma_2}(t)\| \leq\ & M_{\sigma_2}\|y_{\sigma_2 t_0}\|\mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-t_0)} + \int_{t_0}^{t} M_{\sigma_2}\mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\|\widehat{f}_{\sigma_2}(s, y_{\sigma_2 s})\|\mathrm{d}s \\ & + \int_{t_0}^{t}\int_{t_0}^{s} M_{\sigma_2}\mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\|\widehat{h}_{\sigma_2}(s-\theta, y_{\sigma_2}(\theta))\|\mathrm{d}\theta\mathrm{d}s \\ \leq\ & M_{\sigma_2}\left[\|y_{\sigma_2 t_0}\|\mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\widetilde{f}_{\sigma_2}(\|y_{\sigma_2 s}\|)\|y_{\sigma_2 s}\|\mathrm{d}s \right. \\ & \left. + \int_{t_0}^{t}\int_{t_0}^{s} \mathrm{e}^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\widetilde{h}_{\sigma_2}(s-\theta, \|y_{\sigma_2}(\theta)\|)\|y_{\sigma_2}(\theta)\|\mathrm{d}\theta\mathrm{d}s\right] \\ =:\ & P^*_{\sigma_2}(t), \quad t \geq t_0. \end{aligned}$$

Let

$$P_{\sigma_2}(t) = \begin{cases} P^*_{\sigma_2}(t), & t \geq t_0, \\ M_{\sigma_2}\|y_{\sigma_2 t_0}\|, & t_0 - \tau \leq t \leq t_0. \end{cases}$$

Then $P_{\sigma_2}(t) \geq \|y_{\sigma_2}(t)\|$, $t \geq t_0 - \tau$ and

$$P'_{\sigma_2}(t) \leq (\lambda_{\sigma_2} - \epsilon_{\sigma_2})P_{\sigma_2}(t) + M_{\sigma_2}\left[\widetilde{f}_{\sigma_2}(\|P_{\sigma_2 t}\|)\|P_{\sigma_2 t}\| + \int_{t_0}^{t}\widetilde{h}_{\sigma_2}(t-\theta, P_{\sigma_2}(\theta))P_{\sigma_2}(\theta)\mathrm{d}\theta\right].$$

Using a similar approach, we can obtain $P_{\sigma_2}(t) \leq \|P_{\sigma_2 t_0}\|$ and thus

$$\|y_{\sigma_2}(t)\| \leq P_{\sigma_2}(t) \leq M_{\sigma_2}\|y_{\sigma_2 t_0}\|.$$

Then

$$\|x_{\sigma_2}(t)\| \leq M_{\sigma_2}\|x_{\sigma_2 t_0}\|\mathrm{e}^{\epsilon_{\sigma_2}(t-t_0)}. \tag{5.11}$$

Combining (5.7) and (5.11) and Theorem 3.1, we have

$$\|x(t)\| \leq \|x_{t_0}\|\prod_{\sigma_i \in \aleph_i} M_{\sigma_1}^{k_{\sigma_1}(t)} \exp\left\{\sum_{\sigma_1 \in \aleph_1}\epsilon_{\sigma_1}k_{\sigma_1}(t)\tau - \sum_{\sigma_1 \in \aleph_1}\epsilon_{\sigma_1}\pi_{\sigma_1}(t) + \sum_{\sigma_2 \in \aleph_2}\epsilon_{\sigma_2}\pi_{\sigma_2}(t)\right\}.$$

By Theorem 3.2, we can obtain the results of this theorem. The proof is complete. □

Theorem 5.2. *Assume that* (A1), (A2) *hold and*

(i) *there exist* $\widetilde{f}_{\sigma_1} \in K_1$ *and* $\widetilde{h}_{\sigma_1} \in C[R^2, R_+]$, $b_{\sigma_2} \geq 0$ *and* $\widetilde{h}_{\sigma_2} \in C[R_+, R_+]$ *such that* $\widetilde{h}_{\sigma_1}(s, \cdot) \in K_1$ *for any* $s \in R_+$ *and*

$$\|f_{\sigma_1}(t, x_t)\| \leq \widetilde{f}_{\sigma_1}(\|x_t\|)\|x_t\|, \qquad \|f_{\sigma_2}(t, x_t)\| \leq b_{\sigma_2}\|x_t\|,$$
$$\|h_{\sigma_1}(t-s, x(s))\| \leq \widetilde{h}_{\sigma_1}(t-s, \|x(s)\|)\|x(s)\|, \qquad \|h_{\sigma_2}(t-s, x(s))\| \leq \widetilde{h}_{\sigma_2}(t-s)\|x(s)\|;$$

(ii) $\exists \epsilon_{\sigma_1}, \epsilon_{\sigma_2}, r > 0$ *such that*

$$\epsilon_{\sigma_1} + M_{\sigma_1}\widetilde{f}_{\sigma_1}(r)\mathrm{e}^{\epsilon_{\sigma_1}\tau} + M_{\sigma_1}\int_0^\infty \widetilde{h}_{\sigma_1}(s, r)\mathrm{e}^{\epsilon_{\sigma_1}s}\mathrm{d}s < \lambda_{\sigma_1},$$
$$\lambda_{\sigma_2} + M_{\sigma_2}b_{\sigma_2} + M_{\sigma_2}\int_0^\infty \widetilde{h}_{\sigma_2}(s)\mathrm{e}^{-\epsilon_{\sigma_2}s}\mathrm{d}s < \epsilon_{\sigma_2}.$$

Then switched delay systems (5.1) *are*

(1) *stable, if* $\exists T > 0$, *such that*

$$\frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}k_{\sigma_1}(t)\tau + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} \leq 1, \quad t \geq T \tag{5.12}$$

and

$$\limsup_{t\to\infty} \frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}k_{\sigma_1}(t)\tau + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} = 1; \tag{5.13}$$

(2) *locally exponentially stable, if*

$$\limsup_{t\to\infty} \frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}k_{\sigma_1}(t)\tau + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} < 1. \tag{5.14}$$

The proof is similar to that of Theorem 5.1.

As in Remark 1, in Theorems 5.1 and 5.2, (5.2)–(5.4) and (5.12)–(5.14) are actually switching laws which guarantee the switched delay system (5.1) to be stable or locally exponentially stable. We give an example to show how to design switching laws for the switched delay systems.

Example 5.1. Consider nonlinear switched delay systems

$$x'(t) = A_i x + f_i(t, x_t) + \int_{t_0}^t h_i(t-s, x(s))\mathrm{d}s, \quad i = 1, 2 \tag{5.15}$$

where $x = (x_1, x_2)^T \in R^2$,

$$A_1 = \begin{bmatrix} -4 & 1 \\ -1 & -5 \end{bmatrix}, \qquad f_1(t, x_t) = \begin{bmatrix} 0 \\ x_1^3(t-1) \end{bmatrix}, \qquad h_1(t-s, x(s)) = \begin{bmatrix} \mathrm{e}^{-2(t-s)}x_2^3(s) \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad f_2(t, x_t) = \begin{bmatrix} \sin x_1(t-1) \\ \sin x_2(t-1) \end{bmatrix}, \qquad h_2(t-s, x(s)) = \begin{bmatrix} \mathrm{e}^{t-s}x_1 \\ \mathrm{e}^{t-s}x_2 \end{bmatrix}.$$

With Theorem 5.2, we have $\lambda_1 = -4$, $\lambda_2 = 2$; $\widetilde{f}_1(\|x_t\|) = \|x_t\|^2$, $b_2 = 1$; $\widetilde{h}_1(t-s, x(s)) = \mathrm{e}^{-2(t-s)}\|x(s)\|^2$, $\widetilde{h}_2(t) = \mathrm{e}^t$. Take $\tau = 1$, $\epsilon_1 = 1/2$, $\epsilon_2 = 7/2$, $M_1 = M_2 = 1$, $r = 1$.

Then,

$$\epsilon_1 + M_1\widetilde{f}_1(r)\mathrm{e}^{\epsilon_1\tau} + M_1\int_0^\infty \widetilde{h}_1(s, r)\mathrm{e}^{\epsilon_1 s}\mathrm{d}s < 4 = \lambda_1,$$
$$\lambda_2 + M_2 b_2 + M_2\int_0^\infty \widetilde{h}_2(s)\mathrm{e}^{-\epsilon_2 s}\mathrm{d}s < 7/2 = \epsilon_2.$$

The switching laws are:

$(\mathrm{S}_1)^*$ $\pi_1(t) \geq k_1(t) + 7\pi_2(t)$ and $\limsup_{t\to\infty} \frac{k_1(t)+7\pi_2(t)}{\pi_1(t)} = 1$.

$(\mathrm{S}_2)^*$ $\limsup_{t\to\infty} \frac{k_1(t)+7\pi_2(t)}{\pi_1(t)} < 1$.

By Theorem 5.2,

- if we choose switching law $(\mathrm{S}_1)^*$, then the switched delay system

$$x'(t) = A_i x(t) + f_i(t, x_t) + \int_{t_0}^{t} h_i(t-s, x(s))\mathrm{d}s, \tag{5.16}$$

where $i = 1, 2$, $0 \leq \tau(t) \leq 1$, is stable;

- if we choose switching law $(\mathrm{S}_2)^*$, then the switched delay system (5.16) is locally exponentially stable.

Remark 2. (a) If $r = \infty$ in Theorems 5.1 and 5.2, then the results are globally exponential stability.

(b) If in some σ_1th subsystem, λ_{σ_1} is not large enough to satisfy the condition (ii) in Theorems 5.1 and 5.2, we can consider it as a σ_2 subsystem and use a similar approach to discuss switching laws so that the switched delay system is stable.

(c) If $\aleph_{\sigma_1} = \{1\}$ and $f_{\sigma_2} \equiv 0$, i.e., the switched system is composed by a stable delay system and some unstable non-delay systems, then we can obtain some more simple criteria.

Corollary 5.1. *Assume that all the conditions of Theorems 5.1 or 5.2 hold except $\widetilde{f_{\sigma_2}} \equiv 0$ and $\aleph_1 = \{1\}$. Then switched system (5.1) is*

(1) *stable, if $\exists T > 0$, such that*

$$\frac{\sum\limits_{\sigma_i\in\aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum\limits_{\sigma_2\in\aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} \leq 1, \quad t \geq T$$

and

$$\limsup_{t\to\infty} \frac{\sum\limits_{\sigma_i\in\aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum\limits_{\sigma_2\in\aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} = 1;$$

(2) *locally exponentially stable, if*

$$\limsup_{t\to\infty} \frac{\sum\limits_{\sigma_i\in\aleph_i} k_{\sigma_i}(t)\ln M_{\sigma_i} + \sum\limits_{\sigma_2\in\aleph_2} \epsilon_{\sigma_2}\pi_{\sigma_2}(t)}{\sum\limits_{\sigma_1\in\aleph_1} \epsilon_{\sigma_1}\pi_{\sigma_1}(t)} < 1.$$

6. Conclusion

We have studied some stability properties of linear and nonlinear switched delay systems consisting of both stable and unstable subsystems. Some switching laws have been developed, which show that if the average of total activation time of unstable subsystems is relatively small compared with that of stable subsystems, then global exponential stability for linear switched delay systems, and local exponential stability for nonlinear switched delay systems can be guaranteed. In the interests of brevity, only single delay is studied in this paper, but it can be extended to the case with multiple delays.

References

[1] A.A. Agrachev, Lie-algebraic conditions for exponential stability of switched systems, in: Proc. IEEE CDC, 1999, pp. 2679–2684.
[2] O. Beldiman, L. Bushnell, Stability, linearization and control of switched systems, in: Proc. IEEE ACC, 1999, pp. 2950–2954.
[3] M.S. Branicky, Stability of switched and hybrid systems, in: Proc. IEEE CDC, 1994, pp. 3498–3503.
[4] M.S. Branicky, Studies in hybrid systems: Modeling, analysis, control, ScD Thesis, MIT EECS, 1995.

[5] W.P. Dayawansa, C.F. Martin, A converse Lyapunov theorem for a class of dynamical systems which undergo switching, IEEE Trans. Automat. Control 44 (1999) 751–760.

[6] G. Davrazos, N.T. Koussoulas, A review of stability results for switched and hybrid systems, in: Proc. of 9th Mediterranean Conference on Control and Automation, MED, 2001.

[7] M. Dogruel, U. Ozguner, Stability of a set of matrices: An application to hybrid systems, in: Proc. IEEE Int. Symposium on Intelligent Control, 1995.

[8] R. Fierro, F.L. Lewis, A framework for hybrid control design, IEEE Trans. Syst. Man Cyb. A 27 (1997) 765–773.

[9] R. Fierro, F.L. Lewis, C.T. Abdallah, Common, multiple, and parametric Lyapunov functions for a class of hybrid dynamical systems, in: Proc. IEEE Mediterranean Symposium on New Directions in Control and Automation, June 1996, pp. 77–82.

[10] R. Fierro, A hybrid system approach to a class of intelligent control systems, Ph.D. Thesis, Univ. of Texas at Arlington, August 1997.

[11] B. Hu, X. Xu, A.N. Michel, J.P. Antsaklis, Stability analysis for a class of nonlinear switched systems, in: Proc. IEEE CDC, 1999, pp. 4374–4379.

[12] J. Hocherman-Frommer, S.R. Kulkarni, P.J. Ramadge, Controller switching based on output prediction errors, IEEE Trans. Automat. Control 43 (1998) 596–607.

[13] J. Hespanha, Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle, IEEE Trans. Automat. Control 49 (2004) 470–482.

[14] J. Hespanha, A.S. Morse, Stability of switched systems with average dwell-time, in: Proc. IEEE CDC, 1999, pp. 2655–2660.

[15] D. Liberzon, J.P. Hespanha, A.S. Morse, Stability of switched systems: A Lie-algebraic condition, Systems Control Lett. 37 (3) (1999) 117–122.

[16] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag. 19 (5) (1999) 59–70.

[17] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, IEEE Trans. Automat. Control 43 (4) (1998) 475–482.

[18] K.S. Narendra, J. Balakrishnan, A common Lyapunov function for stable LTI systems with commuting A-matrices, IEEE Trans. Automat. Control 39 (1994) 2469–2471.

[19] F.M. Pait, J. Kassab, Parallel algorithms for adaptive control: Robust stability, in: A.S. Morse (Ed.), Control Using Logic-Based Switching, Springer-Verlag, London, 1997.

[20] S. Pettersson, B. Lennartson, Stability and robustness for hybrid systems, in: Proc. IEEE CDC, 1996, pp. 1202–1207.

[21] S. Pettersson, B. Lennartson, Exponential stability of hybrid systems using piecewise quadratic Lyapunov functions resulting in LMIs, in: Proc. 14th Triennial World Congress, 1999.

[22] S. Pettersson, Analysis and design of hybrid systems, Ph.D. Thesis, Chalmers Univ. of Technology, 1999.

[23] R.N. Shorten, K.S. Naredra, On the stability and existence of common Lyapunov functions for stable linear switching systems, in: Proc. IEEE CDC, 1998, pp. 3723–3724.

[24] V. Solo, On the stability of slowly time-varying linear systems, Math. Control Signals Systems 7 (1994) 331–350.

[25] C.J. Tomlin, S.S. Sastry, Switching through singularities, in: Proc. IEEE CDC, 1997, pp. 1–6.

[26] M.A. Wick, P. Peleties, R.A. Decarlo, Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems, Eur. J. Control 4 (1998) 140–147.

[27] G. Zhai, B. Hu, K. Yasuda, A.N. Michel, Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach, J. Systems Sci. 8 (32) (2001) 1055–1061.