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On Hybrid Impulsive and Switching Systems and Application to Nonlinear Control

Zhi-Hong Guan, David J. Hill, and Xuemin (Sherman) Shen

***Abstract*—In this note, a new class of hybrid impulsive and switching models is introduced and their asymptotic stability properties are investigated. Using switched Lyapunov functions, some new general criteria for exponential stability and asymptotic stability with arbitrary and conditioned impulsive switching are established. In addition, a new hybrid impulsive and switching control strategy for nonlinear systems is developed. A typical example, the unified chaotic system, is given to illustrate the theoretical results.**

***Index Terms*—Chaos control, exponential stability, hybrid systems, impulsive and switching systems, switched Lyapunov function.**

I. Introduction

Hybrid systems consisting of interacting continuous and discrete dynamics under certain logic rules, have gained considerable attention recently in science and engineering [1], [4], [6], [7], [11], [15], [19], [22] since they provide a natural and convenient unified framework for mathematical modeling of many complex physical phenomena and practical applications. Examples include robotics, integrated circuit design, multimedia, manufacturing, power electronics, switched-capacitor networks, chaos generators, automated highway systems, and air traffic management systems. Hybrid control, which is based on switching between different models and controllers, has also received growing interest, due to its advantages, for instance, on achieving stability, improving transient response, and providing an effective mechanism to cope with highly complex systems and systems with large uncertainties. A substantial part of the literature on hybrid systems and hybrid control has been devoted to stability analysis and stabilization; see the survey papers [4], [16], [19], and the references therein. Most recently, on the basis of Lyapunov functions and other analysis tools, the stability and stabilization for linear or nonlinear switched systems have been further investigated and many valuable results have been obtained, see [1], [4], [6], [7], [11], [15], [19], [22], and some references therein.

In general, the most widely studied switching systems can be classified into two groups: continuous and discrete switching systems. However, these classes do not cover some useful switching systems existing in the real world which display a certain kind of dynamics with impulse effect at the switching points, i.e., the states jump. Examples of these systems include many evolutionary processes, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology. Other examples include optimal control models in economics, frequency-modulated signal processing systems, and flying object motions. All these systems are characterized by switches of states and abrupt changes at the switching instants, i.e.,

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the systems switch with impulse effect [9], [16], [21], which cannot be well described by using pure continuous or pure discrete models. Therefore, it is important and, in fact, necessary to study impulsive and switching systems.

From the control point of view, hybrid impulsive and switching control, based on the theory of impulsive and switching dynamic systems, is an effective method in the sense that it allows stabilization of a complex system by using only small control impulses in different modes, even though the complex system behaviors may follow unpredictable patterns [2], [3], [5], [9], [13]. In addition, a major advantage of combined impulsive and switching control can be seen from the fact that the impulsive time-invariant unperturbed system is always null-controllable [10]; this is not true for normal time-invariant unperturbed systems. Although the interest in impulsive control systems has grown in recent years due to its theoretical and practical significance [2], [3], [5], [7], [13], [21], but to our knowledge there are very few reports [1], [7], [11], [14], [15] dealing with hybrid impulsive and switching dynamical systems and the corresponding control problem.

This note studies a class of nonlinear hybrid impulsive and switching systems, and its application to nonlinear control. Using switched Lyapunov functions, the exponential stability and asymptotical stability of the class of hybrid impulsive and switching nonlinear systems are studied and some new general stability criteria are established. In addition, a new hybrid impulsive and switching control strategy for nonlinear system control is developed. A typical example, the unified chaotic system, is given to visualize the satisfactory control performance.

II. Problem Formulation

Let $R_+ = [0,+\infty), R^n$ denote the n-dimensional Euclidean space. For $x = \mathrm{col}(x_1,\ldots,x_n) \in R^n$, the norm of x is $\|x\|:=(\sum_{i=1}^n x_i^2)^{1/2}$. For $A = (a_{ij})_{n\times n} \in R^{n\times n}, \lambda_{\max}(A)$, and $\lambda_{\min}(A)$ are the maximum and the minimum eigenvalues of A, respectively. The identity matrix of order m is denoted as I_m (or, simply, I if no confusion arises).

In general, a nonlinear system can be written in the following form:

$$\dot{x} = Ax + f(t,x) \tag{2.1}$$

where $t \in R_+, x \in R^n$ is the state variable, A is an $n \times n$ matrix, and $f(t,x) : R_+ \times R^n \longmapsto R^n$ is a piecewise continuous vector-value function guaranteeing the existence and uniqueness of solutions for (2.1) with initial value problem. Correspondingly, the controlled nonlinear system can be described as

$$\dot{x} = Ax + f(t,x) + u(t,x) \tag{2.2}$$

where $u(t,x)$ is the control input. We can construct a hybrid impulsive and switching controller $u = u_1 + u_2$ for (2.2) as follows:

$$u_1(t) = \sum_{k=1}^{\infty} B_{1k}x(t)l_k(t) \quad u_2(t) = \sum_{k=1}^{\infty} B_{2k}x(t)\delta(t-t_k) \tag{2.3}$$

where B_{1k} and B_{2k} are $n \times n$ constant matrices, $\delta(\cdot)$ is the Dirac impulse, $l_k(t) = 1$ as $t_{k-1} < t \leq t_k$, and otherwise, $l_k(t) = 0$ with discontinuity points

$$t_1 < t_2 < \cdots < t_k < \cdots \quad \lim_{k\to\infty} t_k = \infty \tag{2.4}$$

where $t_1 > t_0, t_0 \geq 0$ is the initial time.

From (2.3), $u_1(t) = B_{1k}x(t), t \in (t_{k-1}, t_k], k = 1,2,\cdots$, which implies that the controller $u_1(t)$ switches its values at every instant t_k, and, without loss of generality [2], [21], it is assumed that $x(t_k) = x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$.

On the other hand, $u_2(t) = 0$ as $t \neq t_k$. Therefore, (2.2) and (2.3) together imply that

$$x(t_k+h) - x(t_k) = \int_{t_k}^{t_k+h} [Ax(s) + f(s,x(s)) + u_1(s) + u_2(s)]\, ds$$

where $h > 0$ is sufficiently small. As $h \to 0^+$, this reduces to $\triangle x(t)\,|_{t_k} := x(t_k^+) - x(t_k) = B_{2k}x(t_k)$, where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$. This implies that the controller $u_2(t)$ has the effect of suddenly changing the state of (2.2) at the points t_k; that is, $u_2(t)$ is an impulsive control, and $u_1(t)$ is a switching control.

Accordingly, under control (2.3), the closed-loop nonlinear system of (2.2) becomes

$$\begin{cases} \dot{x} = Ax + f(t,x) + B_{1k}x, & t \in (t_{k-1}, t_k] \\ \triangle x = B_{2k}x, & t = t_k \\ x(t_0^+) = x_0, & k = 1,2,\ldots. \end{cases} \tag{2.5}$$

System (2.5) is called *a hybrid impulsive and switching system*. In general, the hybrid impulsive and switching system has the following form:

$$\begin{cases} \dot{x} = A_{i_k}x + F_{i_k}(t,x), & t \in (t_{k-1}, t_k] \\ \triangle x = B_k x, & t = t_k \\ x(t_0^+) = x_0, & k = 1,2,\ldots \end{cases} \tag{2.6}$$

where $t \in R_+, x \in R^n$ is the state variable, $t_0 \geq 0$ is the initial time, A_{i_k} and B_k are $n \times n$ matrices, switching signal $\sigma : R_+ \longmapsto \{1,2,\ldots,m\}$, which is represented by $\{i_k\}$ according to $(t_{k-1}, t_k] \longmapsto i_k \in \{1,2,\ldots,m\}$, is a piecewise constant function, the time sequence $\{t_k\}$ satisfies (2.4), $\triangle x\,|_{t=t_k} = x(t_k^+) - x(t_k)$, and $F_{i_k}(t,x) : R_+ \times R^n \longmapsto R^n$ are piecewise continuous vector-value functions with $F_{i_k}(t,0) \equiv 0, t \in R_+$, and ensuring the existence and uniqueness of solutions for (2.6).

Obviously, (2.6) has m different modes, that is

$$\dot{x} = A_i x + F_i(t,x), \qquad i = 1,2,\ldots,m \tag{2.7}$$

switching in the interval R_+. For any switching signal $\sigma, t \in R_+$, and $t > t_0$, let $T_i(t_0,t)$ denote the total activation time of the ith subsystem (2.7) during $[t_0,t]$, which is the union of the corresponding switching intervals included in $[t_0,t]$. Furthermore, let $\mathrm{m}(T_i(t_0,t))$ denote the Lebesgue measure of the set $T_i(t_0,t)$. Then, the first equation of system (2.6) can be rewritten as

$$\dot{x} = A_i x + F_i(s,x), \qquad s \in T_i(t_0,t), \quad t \in R_+ \tag{2.8}$$

where $i = 1,2,\ldots,m, \bigcup_{i=1}^m T_i(t_0,t) = [t_0,t]$.

The characteristics of the nonlinear hybrid system (2.6) that differ from most existing models (see [4], [16], [19], and the references therein) are both its state discontinuity and its model diversity due to impulses and switches. Therefore, to ensure that it can be successfully used to describe and to deal with various impulsive and switching phenomena, especially some evolution processes [2], [16], [21], a detailed investigation of this new model is necessary.

In what follows, the global asymptotic and exponential stability of the hybrid model (2.6) is first studied, and then, an example of the controlled system (2.5) is investigated.

III. Stability of Hybrid Impulsive and Switching Systems

In the subsequent discussion, the following lemma will be needed.

Lemma 3.1 [8]: If $P \in R^{n\times n}$ is a positive–definite matrix, $Q \in R^{n\times n}$ is a symmetric matrix, then

$$\lambda_{\min}(P^{-1}Q)x^\top Px \leq x^\top Qx \leq \lambda_{\max}(P^{-1}Q)x^\top Px, x \in R^n.$$

We now consider the asymptotic properties of the hybrid system (2.6). For (2.6), assume that, for $t \in R_+, x \in R^n$, there exist continuous functions $\varphi_i(t) \geq 0$ and positive–definite matrices P_i, such that

$$F_i^\top(t,x)P_i x \leq \varphi_i(t)x^\top P_i x, \qquad i=1,2,\ldots,m. \tag{3.1}$$

Furthermore, for convenience, define the following locally integrable functions $\lambda_i(t)$ and parameters β_k and ρ by the inequalities and equalities

$$\lambda_{\max}\left[P_i^{-1}\left(A_i^\top P_i + P_i A_i\right)\right] + 2\varphi_i(t) \leq \lambda_i(t) \tag{3.2}$$

$$\lambda_{\max}[(I+B_k)^\top(I+B_k)] \leq \beta_k \tag{3.3}$$

$$\rho = \max_{1\leq i\leq m}\{\rho_i^2\} \quad \rho_i = (\lambda_{\max}(P_i)/\lambda_{\min}(P_i))^{1/2} \tag{3.4}$$

where $i=1,2,\ldots,m, k=1,2,\ldots$.

Remark 3.1: It is easy to see that the inequality (3.1) holds when the nonlinear function $F(t,x)$ satisfies the Lipschitz condition $\|F(t,x)\| \leq L(t)\|x\|, \forall x, t$. In fact, for any constant $\xi > 0, F^\top(t,x)Px \leq (1/2)[(F^\top(t,x)F(t,x))/(\xi) + \xi(Px)^\top(Px)] \leq (1/2)[(L^2(t))/(\xi)x^\top x + \xi x^\top P^\top Px] \leq (1/2)[(L^2(t))/(\xi\lambda_{\min}(P)) + \xi\lambda_{\max}(P)]x^\top Px$. But the converse situation is not true. For example, let $F(t,x)=(x_1,-x_1x_3,x_1x_2)^\top$ with $x=(x_1,x_2,x_3)^\top \in R^3$, then there exists a positive–definite matrix $P=\mathrm{diag}\{3,1,1\}$ such that $F^\top(t,x)\,Px \leq \varphi(t)x^\top Px$ with $\varphi(t)=1$. However, $\|F(t,x)\| \leq L(t)\|x(t)\|$ does not hold for any $x \in R^3$. In fact, as $x \to \infty$ along the trajectory $x=(x_1,x_1,0)^\top$, it follows that $(\|F(t,x)\|)/(\|x\|) \to +\infty$. Thus, for nonlinear function $F(t,x)$, the inequality (3.1) is less conservative than the Lipschitz condition which is usually assumed in literature; see, for instance, [7], [11], [13], and [22].

Theorem 3.1: Assume that (3.1) holds and the impulsive switching of (2.6) satisfies

$$\sum_{j=1}^{k-1}\ln(\rho\beta_j) + \sum_{i=1}^m \int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq \psi(t_0,t), \qquad t\in(t_{k-1},t_k] \tag{3.5}$$

where $k=1,2,\ldots,$ $T_i(t_0,t)$ is defined for (2.7), $\psi(t_0,t)$ is a continuous function on $R_+, \lambda_i(t), \beta_k$, and ρ are given by (3.2)–(3.4), respectively. Then

$$\lim_{t\to+\infty}\psi(t_0,t) = -\infty \tag{3.6}$$

implies that the trivial solution of (2.6) is globally asymptotically stable, and

$$\psi(t_0,t) \leq -c(t-t_0), \qquad t\geq t_0 \tag{3.7}$$

with $c>0$ being constant, implies that the trivial solution of (2.6) is globally exponentially stable.

Proof: Construct the switched Lyapunov function in the form of

$$V_{i_k}(x) = x^\top P_{i_k}x, \qquad i_k \in \{1,2,\ldots,m\} \tag{3.8}$$

where P_{i_k} is a positive–definite matrix given by (3.1), and let $V_{i_k}(t) =: V_{i_k}(x(t))$. Since (3.1) and (3.2) hold, from Lemma 3.1, the total derivative of $V_{i_k}(x)$, with respect to (2.6), is

$$\begin{aligned}
&\dot V_{i_k}(x(t))\,|_{(2.6)}\\
&= [A_{i_k}x + F_{i_k}(t,x)]^\top P_{i_k}x + x^\top P_{i_k}[A_{i_k}x + F_{i_k}(t,x)]\\
&= x^\top\left[A_{i_k}^\top P_{i_k} + P_{i_k}A_{i_k}\right]x + 2F_{i_k}^\top(t,x)P_{i_k}x\\
&\leq \left\{\lambda_{\max}\left[P_{i_k}^{-1}\left(A_{i_k}^\top P_{i_k} + P_{i_k}A_{i_k}\right)\right] + 2\varphi_i(t)\right\}x^\top P_{i_k}x\\
&\leq \lambda_{i_k}(t)V_{i_k}(t), \qquad t\in(t_{k-1},t_k]
\end{aligned} \tag{3.9}$$

which implies that $V_{i_k}(t) \leq V_{i_k}(t_{k-1}^+)\exp[\int_{t_{k-1}}^t \lambda_{i_k}(s)\,ds], t \in (t_{k-1},t_k]$, where $\lambda_{i_k}(t)$ is given by (3.2). Substituting (3.8) leads to $\lambda_{\min}(P_{i_k})x^\top(t)x(t) \leq \lambda_{\max}(P_{i_k})\, x^\top(t_{k-1}^+)x(t_{k-1}^+)\exp[\int_{t_{k-1}}^t \lambda_{i_k}(s)\,ds], t \in (t_{k-1},t_k]$ or

$$w(t) \leq \rho w(t_{k-1}^+)\exp\left[\int_{t_{k-1}}^t \lambda_{i_k}(s)\,ds\right], \qquad t\in(t_{k-1},t_k] \tag{3.10}$$

where ρ is defined in (3.4), and

$$w(t) = x^\top(t)x(t). \tag{311}$$

On the other hand, it follows from (2.6) that

$$\begin{aligned}
w(t_k^+) &= [(I+B_k)x(t_k)]^\top(I+B_k)x(t_k)\\
&\leq \lambda_{\max}[(I+B_k)^\top(I+B_k)]x^\top(t_k)x(t_k) \leq \beta_k w(t_k)
\end{aligned} \tag{3.12}$$

where $\beta_k \geq 0, k=1,2,\ldots,$ are given by (3.3).

Using (3.10) and (3.12) successively on each subinterval leads to the results. For $t\in(t_0,t_1], w(t) \leq \rho w(t_0^+)\exp[\int_{t_0}^t \lambda_{i_1}(s)\,ds]$, which leads to $w(t_1) \leq \rho w(t_0^+)\exp[\int_{t_0}^{t_1}\lambda_{i_1}(s)\,ds]$, and $w(t_1^+) \leq \beta_1 w(t_1) \leq \rho\beta_1 w(t_0^+)\exp[\int_{t_0}^{t_1}\lambda_{i_1}(s)\,ds]$. Similarly, for $t \in (t_1,t_2], w(t) \leq \rho w(t_1^+)\exp[\int_{t_1}^t \lambda_{i_2}(s)\,ds] \leq \rho^2\beta_1 w(t_0^+)\exp[\int_{t_0}^{t_1}\lambda_{i_1}(s)\,ds + \int_{t_1}^t \lambda_{i_2}(s)\,ds]$. In general, for $t\in(t_{k-1},t_k]$

$$\begin{aligned}
w(t) \leq w(t_0^+)\rho^k\beta_1\ldots\beta_{k-1}\exp\Bigg[&\int_{t_0}^{t_1}\lambda_{i_1}(s)\,ds\\
&+\int_{t_1}^{t_2}\lambda_{i_2}(s)\,ds + \cdots + \int_{t_{k-1}}^t \lambda_{i_k}(s)\,ds\Bigg].
\end{aligned} \tag{3.13}$$

Noticing the definition of $T_i(t_0,t)$ given in (2.7) and assumption (3.5), it follows from (3.13) that

$$\begin{aligned}
w(t) &\leq w(t_0^+)\rho\prod_{j=1}^{k-1}(\rho\beta_j)\exp\left[\sum_{i=1}^m\int_{T_i(t_0,t)}\lambda_i(s)\,ds\right]\\
&\leq w(t_0^+)\rho e^{\psi(t_0,t)}, \qquad t\in(t_{k-1},t_k]
\end{aligned}$$

and, therefore, $w(t) \leq w(t_0^+)\rho e^{\psi(t_0,t)}, t\geq t_0$, which implies from (3.6) and (3.7) that the trivial solution of (2.6) is globally asymptotically stable and globally exponentially stable, respectively. This completes the proof. ■

Remark 3.2: In Theorem 3.1, a general criteria for guaranteeing the global asymptotic stability of (2.6) is established. The inequality (3.5) characterizes the impulsive effect $\sum_{j=1}^{k-1}\ln(\rho\beta_j)$ and the switching effect $\sum_{i=1}^m\int_{T_i(t_0,t)}\lambda_i(s)\,ds$ in aggregate form, i.e., there is no special limit to $\ln(\rho\beta_j)$ and $\lambda_i(t)$, as well as to the switching model and switching interval. This is because for the Lyapunov functions $V_i(x)$ used in the proof of Theorem 3.1, there is no sign requirement on $\mathrm{D}V_i$ in interval $(t_{k-1},t_k]$ and at time instants t_k. Usually, such conditions are required to get the stability results, but there are exceptions [12].

Corollary 3.1: For (2.6), assume that (3.1) holds.

i) If $\lambda_i(t) \leq -\lambda_i < 0, \lambda_i > 0$ are constants, and there exists a constant $0<\alpha<\lambda_i, i=1,2,\ldots,m$, such that

$$\ln(\rho\,\beta_k) - \alpha(t_k - t_{k-1}) \leq 0, \qquad k=1,2,\ldots \tag{3.14}$$

then the trivial solution of (2.6) is globally exponentially stable, where $\lambda_i(t), \beta_k$, and ρ are given by (3.2)–(3.4), respectively.

ii) If $\lambda_i(t) \leq \lambda(t), \lambda(t)$ is locally integrable, $i=1,2,\ldots,m$, and there exists a constant $\alpha > 1$ such that

$$\ln(\alpha\rho\,\beta_k) + \int_{t_k}^{t_{k+1}}\lambda(s)\,ds \leq 0, \qquad k=1,2,\ldots \tag{3.15}$$

then, either $\lambda(t) \geq 0$ or $\sup_k\int_{t_{k-1}}^{t_k}|\lambda(s)|\,ds \leq M < +\infty$ implies that the trivial solution of (2.6) is globally asymptotically stable, where $\lambda_i(t), \beta_k$, and ρ are given by (3.2)–(3.4), respectively.

Proof: When $\lambda_i(t) \leq -\lambda_i < 0, i = 1, 2, \ldots, m$, let $\lambda = \min_{1\leq i\leq m}\{\lambda_i\}$, it follows from (3.14) that $\sum_{j=1}^{k-1}\ln(\rho\beta_j) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq \sum_{j=1}^{k-1}\ln(\rho\beta_j) - \sum_{i=1}^{m}\lambda_i \mathrm{m}(T_i(t_0,t)) \leq \sum_{j=1}^{k-1}\ln(\rho\beta_j) - \lambda(t-t_0) = \sum_{j=1}^{k-1}\ln(\rho\beta_j) - \alpha(t-t_0) - (\lambda-\alpha)(t-t_0) \leq \sum_{j=1}^{k-1}\ln(\rho\beta_j) - \alpha(t_{k-1}-t_0) - (\lambda-\alpha)(t-t_0) = \sum_{j=1}^{k-1}[\ln(\rho\beta_j) - \alpha(t_j - t_{j-1})] - (\lambda-\alpha)(t-t_0) \leq -(\lambda-\alpha)(t-t_0), t \in (t_{k-1}, t_k]$, namely, $\sum_{j=1}^{k-1}\ln(\rho\beta_j) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq -(\lambda-\alpha)(t-t_0), t \geq t_0$, which implies that (3.5) and (3.7) hold with $(\lambda - \alpha) > 0$ and therefore the trivial solution of system (2.6) is globally exponentially stable.

When $\lambda_i(t) \leq \lambda(t), i = 1, 2, \ldots, m, t \in (t_{k-1}, t_k]$, it leads to

$$\begin{aligned}\sum_{j=1}^{k-1}\ln(\rho\beta_j) &+ \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \\ &\leq \sum_{j=1}^{k-1}\ln(\rho\beta_j) + \int_{t_0}^{t}\lambda(s)\,ds \\ &\leq -\sum_{j=1}^{k-1}\ln(\alpha) + \sum_{j=1}^{k-1}\ln(\alpha\rho\beta_j) + \int_{t_0}^{t_k}\lambda(s)\,ds \\ &+ \int_{t_{k-1}}^{t_k}[|\lambda(s)| - \lambda(s)]\,ds. \end{aligned} \tag{3.16}$$

If $\lambda(t) \geq 0$, then it follows from (3.15) and (3.16) that $\sum_{j=1}^{k-1}\ln(\rho\beta_j) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq -\ln(\alpha^{k-1}) + \sum_{j=1}^{k-1}\ln(\alpha\rho\beta_j) + \int_{t_0}^{t_k}\lambda(s)\,ds = -\ln(\alpha^{k-1}) + \int_{t_0}^{t_1}\lambda(s)\,ds + \sum_{j=1}^{k-1}[\ln(\alpha\rho\beta_j) + \int_{t_j}^{t_{j+1}}\lambda(s)\,ds] \leq -\ln(\alpha^{k-1}) + \int_{t_0}^{t_1}\lambda(s)\,ds, t \in (t_{k-1}, t_k]$. Clearly, as $\alpha > 1$, $\lim_{t\to+\infty}[-\ln(\alpha^{k-1}) + \int_{t_0}^{t_1}\lambda(s)\,ds] = \lim_{k\to+\infty}[-\ln(\alpha^{k-1}) + \int_{t_0}^{t_1}\lambda(s)\,ds] = -\infty$.

Similarly, if $\sup_k \int_{t_{k-1}}^{t_k}|\lambda(s)|\,ds \leq M < +\infty$, then it follows from (3.15) and (3.16) that $\sum_{j=1}^{k-1}\ln(\rho\beta_j) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq -\ln(\alpha^{k-1}) + \int_{t_0}^{t_1}\lambda(s)\,ds + 2M, t \in (t_{k-1}, t_k]$, and $\lim_{t\to+\infty}[-\ln(\alpha^{k-1}) + \int_{t_0}^{t_1}\lambda(s)\,ds + 2M] = -\infty$.

Thus, in both cases, (3.5) and (3.6) hold, and from Theorem 3.1, it immediately leads to the conclusion of Corollary 3.1. This completes the proof. ∎

Remark 3.3: In the case of ii) in Corollary 3.1, the parameters $\lambda_i(t)$ may be positive, negative, or sign varying, which implies that stability or instability for switching subsystem (2.7) is not necessary. A special case, such as an autonomous impulsive system, i.e., $A_i \equiv A, F_i(t,x) \equiv F(x)$ satisfying the Lipschitz condition and with $\lambda > 0$ being constant was discussed in [13].

In the following discussion, the concept of "*average dwell-time*" introduced by Hespanha and Morse [6] will be used. That is, for each switching signal σ and each $t \geq t_0 \geq 0$, let $N_\sigma(t_0, t)$ denote the number of discontinuities of σ over the interval $[t_0, t)$. For given $N_0, \tau_a > 0$, let $S_a[\tau_a, N_0]$ denote the set of all switching signals satisfying $N_\sigma(t_0,t) \leq N_0 + ((t-t_0)/\tau_a)$. The constant τ_a is called the "average dwell-time" and N_0 the "chatter bound." This implies that, for a given switching signal $\sigma \in S_a[\tau_a, N_0]$ over $[t_0, t)$, there may exist some consecutive discontinuities with interval separated by less than τ_a, but the average interval between consecutive discontinuities is no less than τ_a.

Corollary 3.2: For (2.6), assume that (3.1) is satisfied and $\beta_k \leq \beta, \beta_k$ is defined by (3.3), $k = 1, 2, \ldots$.

i) If $\rho\beta \leq 1, \rho$ is defined by (3.4), and for $t \geq t_0$

$$\sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq \psi(t_0,t) \tag{3.17}$$

then the conclusion of Theorem 3.1 holds.

ii) If $\rho\beta > 1, \rho$ is defined by (3.4), and for $t \geq t_0$

$$\frac{\ln(\rho\beta)}{\tau_a}(t-t_0) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq \psi(t_0,t) \tag{3.18}$$

then the conclusion of Theorem 3.1 holds for any switching signal $\sigma = \{i_k\} \in S_a[\tau_a, N_0]$, where $N_0, \tau_a > 0$ are given constants satisfying $k - 1 \leq N_0 + (t-t_0)/(\tau_a)$ for any $t \in (t_{k-1}, t_k], k = 1, 2, \ldots$. Specifically, if $t_k - t_{k-1} \geq \delta > 0, k = 1, 2, \ldots$, and the average dwell time τ_a in (3.18) is replaced with δ, then the conclusion of Theorem 3.1 holds for arbitrary switching.

Proof: When $\rho\beta \leq 1$, then $\ln(\rho\beta_j) \leq \ln(\rho\beta) \leq 0, j = 1, 2, \ldots$. It follows from (3.17) that $\sum_{j=1}^{k-1}\ln(\rho\beta_j) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds \leq \psi(t_0,t), t \geq t_0$, which implies that (3.5) is satisfied and, therefore, the conclusion of Theorem 3.1 holds.

When $\rho\beta > 1$, since $k - 1 \leq N_0 + (t-t_0)/(\tau_a)$ for $t \in (t_{k-1}, t_k]$, it leads to $(\rho\beta)^{k-1} \leq (\rho\beta)^{N_0 + (t - t_0/\tau_a)}, t \in (t_{k-1}, t_k]$. Accordingly, as $t \in (t_{k-1}, t_k]$, it follows from (3.13) that $w(t) \leq w(t_0^+)\rho(\rho\beta)^{k-1}\exp[\sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds] \leq w(t_0^+)\rho(\rho\beta)^{N_0}(\rho\beta)^{(t-t_0)/(\tau_a)}\exp[\sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds] = w(t_0^+)\rho(\rho\beta)^{N_0}\exp[(\ln(\rho\beta))/(\tau_a)(t - t_0) + \sum_{i=1}^{m}\int_{T_i(t_0,t)}\lambda_i(s)\,ds], t \in (t_{k-1}, t_k]$. Moreover, by (3.18), it arrives at $w(t) \leq w(t_0^+)\rho(\rho\beta)^{N_0}e^{\psi(t_0,t)}, t \geq t_0$, which implies that the conclusion of Theorem 3.1 holds for any switching signal $\sigma = \{i_k\} \in S_a[\tau_a, N_0]$. For the special case, $t_k - t_{k-1} \geq \delta > 0$, the conclusion can be similarly proved and the details are omitted. This completes the proof. ∎

For (2.6), if $\{i_1, i_2, \cdots, i_m\} = \{1, 2, \ldots, m\}$ and $A_{i_{k+m}} = A_{i_k}, F_{i_{k+m}}(t,x) = F_{i_k}(t,x), k = 1, 2, \ldots$, then it is called *a hybrid impulsive and periodic switching system*, or (2.6) has a periodic switching law. In this case, one has the further results.

Corollary 3.3: Assume that (3.1) holds and (2.6) is a hybrid impulsive and periodic switching system with $t_k - t_{k-1} = \tau_k, \tau_{k+m} = \tau_k, \beta_k \leq \beta, k = 1, 2, \ldots$, and $\lambda_i(t) \leq \lambda_i, i = 1, 2, \ldots, m, \tau_k, \beta$, and λ_i are constants. Then

$$m\ln(\rho\beta) + \lambda_1\tau_1 + \cdots + \lambda_m\tau_m < 0 \tag{3.19}$$

implies that the trivial solution of (2.6) is globally asymptotically stable, where $\beta_k, \lambda_i(t)$, and ρ are given by (3.2), (3.3), and (3.4), respectively. In addition, if $B_{k+m} = B_k$, that is, $\beta_{k+m} = \beta_k, k = 1, 2, \ldots$, then the inequality (3.19) can be replaced by $\sum_{i=1}^{m}[\ln(\rho\beta_i) + \lambda_i\tau_i] < 0$.

Remark 3.4: Corollary 3.3 gives a compact criterion for a class of nonlinear impulsive and periodic switching system (2.6), which is independent of the construction of a Lyapunov function [14]. When $F_i(t,x) \equiv 0$ and $B_{k+m} \equiv B_k$, (2.6) becomes a linear impulsive and periodic switching system. For this special case, Corollary 3.3 immediately reduces to be similar to a result [15, Th. 1] with the switching interval being periodic. In addition, Corollary 3.3 is easily to verify since the parameters λ_i and β_k are given by inequality estimation and β_k is independent of the choice of positive matrix P_i.

IV. ILLUSTRATIVE EXAMPLE

As an application of Theorem 3.1 and Corollaries 3.1–3.3, we consider the control problem for a benchmark nonlinear chaotic system which includes the Lorenz system and Chen's system as special cases [17], i.e., we design an impulsive and switching controller to suppress the chaos.

Example: Consider the chaotic system [17] described by

$$\begin{cases} \dot{x}_1 = (25a+10)(x_2 - x_1) \\ \dot{x}_2 = (28-35a)x_1 - x_1x_3 + (29a-1)x_2 \\ \dot{x}_3 = x_1x_2 - \dfrac{a+8}{3}x_3 \end{cases} \tag{4.1}$$

where $a \in [0,1]$.

Rewrite system (4.1) as

$$\dot{x} = Ax + f(x) \tag{4.2}$$

where $x = (x_1, x_2, x_3)^\top$, $f(x) = (0, -x_1x_3, x_1x_2)^\top$, and

$$A = \begin{bmatrix} -(25a+10) & (25a+10) & 0 \\ 28-35a & 29a-1 & 0 \\ 0 & 0 & -\frac{a+8}{3} \end{bmatrix}.$$

The hybrid impulsive and switching controlled system based on (4.2) has the following form:

$$\begin{cases} \dot{x} = (A + B_{i_k})x + f(x), & t \in (t_{k-1}, t_k] \\ \triangle x = B_{2k}x, & t = t_k \\ x(t_0^+) = x_0, & k = 1, 2, \ldots \end{cases} \tag{4.3}$$

with B_{i_k} and B_{2k} being 3×3 matrices, $B_{i_k} \in \{B_1, B_2, \ldots, B_m\}$, $\triangle x(t_k) = x(t_k^+) - x(t_k)$, and $\{t_k\}$ satisfying (2.4). Obviously, (4.3) is a special case of the (2.6) with $A_{i_k} = A + B_{i_k}$, $F_{i_k}(t,x) \equiv f(x)$, and $B_k = B_{2k}$.

For (4.3), if $P_i \equiv I$, then $f^\top(x)x = 0$, i.e., in (3.1), $\varphi_i(t) = 0, i = 1, 2, \ldots, m$. Let $\lambda_{\max}\{(A+B_i)^\top + (A+B_i)\} \le \lambda_i, \lambda_{\max}\{(I + B_{2k})^\top(I+B_{2k})\} \le \beta_k, \tau = t_k - t_{k-1}, i = 1, 2, \ldots, m, k = 1, 2, \ldots$. Then, the corresponding results of Theorem 3.1 and Corollaries 3.1–3.3 hold.

It can be seen from the previous example that there are several ways to design the impulsive and switching time sequence $\{t_k\}$, control gain matrices $\{B_i\}$ and $\{B_{2k}\}$, such that the controlled system (4.3) is globally asymptotically stable. In addition, observe that it is not necessary to estimate the Lipschitz constant of the nonlinear term or bound the system state for (4.1) as is often used [3]–[20]. Because of the page limit, the further discussion and simulant results are omitted here.

V. CONCLUSION

In this note, some new criteria for exponential stability and asymptotic stability of a class of nonlinear hybrid impulsive and switching systems have been established using switched Lyapunov functions. As an application, a new hybrid impulsive and switching control strategy for chaos suppression has been developed. An illustrative example has been given to demonstrate the improved control performance.

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