

# HIGH ACCURACY TECHNIQUES FOR SINGULARLY PERTURBED CONTROL SYSTEMS—AN OVERVIEW

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In this paper, we give an overview of several approaches for optimal control and filtering of linear and bilinear singularly perturbed systems. Two main approaches that are extensively demonstrated in the control literature, the recursive fixed-point iterations and the Hamiltonian matrix block diagonalization approach are presented pretty much in detail. The remaining two approaches, the approach based on the slow-fast manifold theory and the approach based on recursive calculations of the coefficients for the Taylor series expansions of the corresponding optimal control and filtering equations are only outlined.

**Key Words:** Optimal Control; Singularly Perturbed Control Systems; Chang Transformation; Recursive Fixed-point Iterations; Hamiltonian Matrix Block Diagonalization; Slow-fast Manifold Theory

## 1 Introduction

It is well documented in the control systems literature that theory of singular perturbations has been a very fruitful control engineering research area in the last thirty five years<sup>1-6</sup>. The singularly perturbed control systems have been studied using Taylor series, asymptotic expansions, and power-series methods—techniques traditionally used in mathematics for studying singularly perturbed systems of differential equations. Being nonrecursive in nature, these expansion methods become very cumbersome and computationally very expensive (the size of computations required can be considerable) when a higher order of accuracy,  $O(\epsilon^k)$ ,  $k \geq 2$ , where  $\epsilon$  represents a small positive singular perturbation parameter, is required. In such cases, the advantage of using the expansion methods (important theoretical tools to

remove ill-conditioning of the original problems and produce well-conditioned, approximate, reduced-order subproblems) is questionable from the numerical point of view, and sometimes these methods are almost not applicable in practice as was demonstrated on practical examples<sup>7-10</sup>. It can be said, in general, that until the middle of the eighties, the singular perturbation methods used in control engineering were efficient for solving control problems for which only the accuracy of  $O(\epsilon)$  was sufficient. In the era of an increased application of modern control theory results to real physical systems this is a serious problem. Even more, the standard statement of singular perturbation theory that the approximate results obtained are valid under the assumption that “it exists  $\epsilon$  small enough” limits the practical implementation of  $O(\epsilon)$ -theory of singular perturbations to real physical systems.

One of the most important results of modern mathematical theory of singular perturbations is the Chang transformation, which is developed for exact pure-slow and pure-fast decomposition of linear singularly perturbed systems<sup>11</sup>. As a matter of fact, this transformation was derived as a by-product of a more general problem studied by

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<sup>\*</sup> $O(\epsilon^k)$  is defined by  $O(\epsilon^k) < C(\epsilon)^k$ , where  $c$  is a bounded constant and  $k$  stands for any real number.

Chang. The results of Chang are of fundamental importance for the high accuracy techniques of singularly perturbed linear control systems. Note that the discrete-time version of the original Chang's paper was considered in Borno<sup>12</sup>.

In order to broaden the class of real physical systems for which theory of singular perturbations can be successfully applied, the development of  $O(\varepsilon^k)$ -theory is a necessary requirement. The high accuracy approach to singularly perturbed control systems started in 1984 in the works of Gajic<sup>13</sup> and Sobolev<sup>14</sup>. The work of Gajic, based on the fixed point iterations, originally known under the name of the recursive approach to singular perturbations<sup>15,16,6</sup> culminated in the so-called Hamiltonian approach for the exact pure-slow and pure-fast decomposition of the linear-quadratic optimal regulator<sup>17</sup> and Kalman filter<sup>18</sup>. In Su *et al.*<sup>17</sup>, the algebraic Riccati equation of the singularly perturbed control problem has been completely and exactly decomposed into the reduced-order pure-slow and pure-fast algebraic Riccati equations. This result facilitates exact optimal regulation and filtering from subsystem levels as demonstrated in Su *et al.*<sup>17</sup>, Gajic and Lim<sup>18</sup> and Lim<sup>19</sup>. The corresponding discrete-time results of Su *et al.*<sup>17</sup> and Gajic and Lim<sup>18</sup> are obtained in Lim<sup>19</sup> and Lim *et al.*<sup>20</sup> and Gajic *et al.*<sup>21</sup>.

The work of Sobolev<sup>14</sup> based on slow-fast manifold theory resulted also in the exact pure slow and pure-fast decomposition of the linear-quadratic optimal control problems as demonstrated in Sobolev<sup>14</sup> and Fridman<sup>22,23</sup>. It should be pointed out that the results of Fridman<sup>22,23</sup> hold for both finite-time and steady state optimization problems. The closed-loop decomposition results of Su *et al.*<sup>17</sup> are valid only at the steady state. In addition, for the finite time optimization, the corresponding open-loop exact decomposition result is obtained in Su *et al.*<sup>24</sup>.

In the recent results of Derbel *et al.*<sup>25</sup> the coefficients for the Taylor series expansions of some singularly perturbed control problems were obtained in a recursive manner. The corresponding applications to synchronous machines were considered in Derbel *et al.*<sup>26</sup>, Djemel *et al.*<sup>27</sup>. The results of Derbel and his coworkers might help to make the classical approach to singularly perturbed

linear control systems, based on Taylor series, asymptotic expansions, and power-series methods, a high accurate technique. However, a lot of work in that direction is needed.

In this paper, the high accuracy approaches to singularly perturbed linear (and bilinear) control systems, named above as the recursive fixed-point method and the Hamiltonian approach are reviewed in detail, both developed by Gajic and his coworkers. In addition, the main results of the slow-fast manifold decomposition of Sobolev, and the recursive approach of Derbel and his coworkers for calculating coefficients of Taylor series of the corresponding singularly perturbed control problems are being indicated.

Since the recursive approach is an integral part of the Hamiltonian approach, in the following, the first thing being reviewed are the main results obtained using the recursive approach to singularly perturbed linear and bilinear control systems. In addition, the recursive fixed-point algorithms remain powerful tools for some classes of singularly perturbed linear and bilinear control systems, especially for finite time linear-quadratic optimization problems, output feedback, and steady state Nash and Stackelberg differential games.

## 2 The Recursive Approach

Singularly perturbed systems display multiple time scale phenomena, hence they are parallel in nature and very well suited for parallel computations and parallel processing of information. The recursive methods for singularly perturbed linear and bilinear control systems were developed in Gajic<sup>13,15</sup>; Gajic *et al.*<sup>16</sup>, Gajic and Shen<sup>6</sup>, Aganovic and Gajic<sup>28</sup> in the spirit of parallel and distributed computations<sup>29</sup> and parallel processing of information in terms of reduced-order, independent, approximate slow and fast filters. The recursive techniques are applicable to almost all important areas of optimal linear control theory, in the context of continuous and discrete, deterministic and stochastic, singularly perturbed systems<sup>6</sup>. A generalization of the recursive methods to the optimal control of singularly perturbed bilinear systems is done in Aganovic<sup>30</sup>, Aganovic *et al.*<sup>31</sup>.

The development of the recursive techniques based on the fixed-point reduced-order parallel algorithms that produce  $O(\varepsilon^k)$ ,  $k=1, 2, 3, \dots$ , accuracy for singularly perturbed linear-quadratic optimal *steady state* control problems has been done in Gajic<sup>13</sup>, Gajic<sup>15</sup>, Grodt and Gajic<sup>7</sup>, Gajic *et al.*<sup>16</sup>, Shen<sup>32</sup>, Gajic and Shen<sup>33,34</sup>, Qureshi<sup>35</sup>, Qureshi *et al.*<sup>36</sup>, Su *et al.*<sup>17</sup>, Skataric and Gajic<sup>9</sup>, Borno and Gajic<sup>37</sup>. The corresponding methods for *finite time* optimal linear-quadratic singularly perturbed control systems have been developed in Grodt and Gajic<sup>7</sup>, Su *et al.*<sup>24</sup>, Shen<sup>32,38</sup>. A special class of singularly perturbed systems known as quasi singularly perturbed systems is considered in Skataric and Gajic<sup>9</sup>. The application of the above recursive approach to the linear-quadratic regulator for loop shaping for high frequency compensation is considered by Geray and Looze<sup>39</sup>.

The main algebraic equations of linear steady state optimal continuous-time control theory for singularly perturbed systems, the algebraic Lyapunov and Riccati equations have been studied in Gajic<sup>15</sup>, Shen *et al.*<sup>40</sup>, Gajic and Shen<sup>6</sup>, Gajic and Qureshi<sup>41</sup>. The recursive algorithms for the solution of these equations have been obtained in the most general case when the problem matrices are functions of a small positive singular perturbation parameter. The numerical decomposition has been achieved in those algorithms so that only low-order systems are involved in algebraic computations. The introduced recursive methods are of the mixed point type and can be implemented as parallel synchronous algorithms. Here, we present only the final results, that is, we present the corresponding fixed-point algorithms for solving the continuous-time algebraic filter type Lyapunov and the continuous-time regulator-type algebraic Riccati equations.

The algebraic Lyapunov equations of singularly perturbed systems has the form

$$KA^T + AK + GG^T = 0 \quad \dots (1)$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\varepsilon}A_3 & \frac{1}{\varepsilon}A_4 \end{bmatrix}, G = \begin{bmatrix} G_1 \\ \frac{1}{\varepsilon}G_2 \end{bmatrix}, K = \begin{bmatrix} K_1 & K_2 \\ K_2^T & \frac{1}{\varepsilon}K_3 \end{bmatrix} \quad \dots (2)$$

with the following dimensions  $A_1^{n_1 \times n_1}, A_4^{n_2 \times n_2}, G_1^{n_1 \times m}, G_2^{n_2 \times m}, K_1^{n_1 \times n_1}, K_3^{n_2 \times n_2}, n = n_1 + n_2,$

compatible to the system decomposition into  $n_1$  slow and  $n_2$  fast state space variables. In addition, the matrix  $A_4$  is assumed to be nonsingular, which is the standard assumption in the theory of singularly perturbed linear systems<sup>4</sup>.

The small positive singular perturbation parameter  $\varepsilon$  affects the coefficient matrices in (1) in such a way that it makes the problem of solving the algebraic Lyapunov equation (1) numerically ill-conditioned. Using the recursive fixed-point algorithm, the numerical ill-conditioning can be removed and the solution of eq. (1) can be obtained in terms of reduced-order well conditioned algebraic Lyapunov equations corresponding, respectively, to the slow and fast system state variables as follows

$$\left. \begin{aligned} K_j &= K_j^{(0)} + \varepsilon E_j, j = 1, 2, 3 \\ K_1^{(0)} A_0^T + A_0 K_1^{(0)} + G_0 G_0^T &= 0 \\ A_0 &= A_1 - A_2 A_4^{-1} A_3, G_0 = G_1 - A_2 A_4^{-1} G_2 \\ K_3^{(0)} A_4^T + A_4 K_3^{(0)} + G_2 G_2^T &= 0 \\ K_2^{(0)} &= -(A_2 K_3^{(0)} + K_1^{(0)} A_3^T + G_1 G_2^T) A_4^{-T} \dots (3) \\ E_1^{(i+1)} A_0^T + A_0 E_1^{(i+1)} &= L_1(E_2^{(i)}) \\ E_3^{(i+1)} A_4^T + A_4 E_3^{(i+1)} &= L_3(E_2^{(i)}) \\ E_2^{(i+1)} &= L_3(E_1^{(i+1)}, E_2^{(i)}, E_3^{(i+1)}) \\ E_2^{(0)} &= 0, i = 0, 1, 2, \dots \end{aligned} \right\}$$

where  $L_1, L_2, L_3$  are linear matrix functions. Note that this decomposition requires stability of matrices  $A_0$  and  $A_4$ , which guarantees the existence of the unique solutions for  $K_1^{(0)}$  and  $K_3^{(0)}$ <sup>41</sup>. The above fixed point algorithm has the rate of convergence of  $O(\varepsilon)$ , which indicates that after  $k$  iterations the accuracy of  $O(\varepsilon^k)$  is achieved<sup>16,6</sup>.

The continuous-time, regulator type, algebraic Riccati equation, whose positive semidefinite stabilizing solution solves the linear-quadratic optimization problem of singularly perturbed systems, is defined by

$$PA + A^T P - P + Q - P \varepsilon P = 0 \quad \dots (4)$$

where

$$P = \begin{bmatrix} P_1 & \varepsilon P_2 \\ \varepsilon P_2^T & \varepsilon P_3 \end{bmatrix},$$

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, S = \begin{bmatrix} S_1 & \frac{1}{\varepsilon} S_2 \\ \frac{1}{\varepsilon} S_2^T & \frac{1}{\varepsilon^2} S_3 \end{bmatrix} \quad \dots (5)$$

The  $O(\varepsilon)$ -approximate slow and fast algebraic Riccati equations of (4)-(5) are derived in Chow and Kokotovic<sup>42</sup>

$$\begin{aligned} P_1^{(0)} A_s + A_s^T P_1^{(0)} + Q_s - P_1^{(0)} S_s P_1^{(0)} &= 0 \\ P_3^{(0)} A_4 + A_4^T P_3^{(0)} + Q_3 - P_3^{(0)} S_3 P_3^{(0)} &= 0 \end{aligned} \quad \dots (6)$$

In addition

$$P_2^{(0)} = F_2(P_1^{(0)}, P_3^{(0)}) \quad \dots (7)$$

where  $F_2$  is a quadratic matrix function. The slow subsystem matrices introduced in eq. (6) can be obtained by using the corresponding formulas of Chow and Kokotovic<sup>42</sup> or even in a simpler manner by using the results of Wang and Frank<sup>43</sup>. The required solutions of approximate slow and fast Riccati equations (6) exist under the standard stabilizability-detectability conditions imposed on the slow and fast subsystems. It has been known since the work of Chow and Kokotovic<sup>42</sup>, Gajic<sup>15</sup> that

$$P_j = P_j^{(0)} + \varepsilon E_j, j = 1, 2, 3 \quad \dots (8)$$

The derivations of the error equations,  $E_j$ , and the development of the fixed-point algorithm for their efficient numerical solution were obtained by Gajic<sup>15</sup>. The corresponding algorithm has the form

$$\left. \begin{aligned} E_1^{(i+1)} D_1 + D_1^T E_1^{(i+1)} &= H_1(E_1^{(i)}, E_2^{(i)}, E_3^{(i)}, \varepsilon) \\ E_3^{(i+1)} D_3 + D_3^T E_3^{(i+1)} &= H_3(E_2^{(i)}, E_3^{(i)}, \varepsilon) \\ E_2^{(i+1)} &= H_2(E_1^{(i+1)}, E_2^{(i)}, E_3^{(i+1)}, \varepsilon) \\ E_1^{(0)} = 0, E_2^{(0)} = 0, E_3^{(0)} = 0, &i = 0, 1, 2, \dots \end{aligned} \right\} \quad \dots (9)$$

Hence, this algorithm requires solution of the reduced-order algebraic Lyapunov equations, where  $H_j, j=1, 2, 3$ , are quadratic matrix functions. The newly defined matrices in eqs.(6)-(9) and the corresponding matrix function can be found in Gajic<sup>15</sup> Gajic and Shen<sup>6</sup>. The algorithm of eq. (9) converges to the exact solution for the error equations with the rate of convergence  $O(\varepsilon)$ .

The discrete-time versions of the algebraic Lyapunov and Riccati equations are studied in Shen *et al.*<sup>40</sup>, Gajic and Shen<sup>34</sup>. It should be pointed out that the partitioned form of the singularly

perturbed algebraic Riccati equation is very complicated in the discrete-time domain. That problem can be overcome by using a bilinear transformation of Kondo and Furuta<sup>44</sup>, that is applicable under a mild assumption, so that the solution of the discrete algebraic Riccati equation of perturbed systems is obtained by using results derived for the corresponding continuous-time algebraic Riccati equation<sup>34</sup>. It is shown that in the discrete-time domain the recursive methods converge also with the rate of convergence of  $O(\varepsilon)$ , hence each fixed-point iteration improves (theoretically) the order of accuracy by  $O(\varepsilon)$ .

It is interesting to point out that the fixed-point algorithm for solving the discrete-time domain algebraic Lyapunov equation, defined by

$$A^T P A - P = -Q \quad \dots (10)$$

with the problem matrices having the singularly perturbed structure established in Litkouhi and Khalil<sup>45,46</sup> as

$$\begin{aligned} A &= \begin{bmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \\ P &= \begin{bmatrix} \frac{1}{\varepsilon} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \end{aligned} \quad \dots (11)$$

leads to the *continuous-time* reduced-order well-conditioned algebraic Lyapunov equations for  $P_1^{(0)}$  and  $E_1$  with  $P_1 = P_1^{(0)} + \varepsilon E_1$ . The reduced-order fast subsystem algebraic Lyapunov equations (for  $P_3^{(0)}$  and  $E_3$ ) remain the discrete-time ones, and the equations for  $P_2^{(0)}$  and  $E_2$  are linear, easily solvable, reduced-order algebraic equations (see Gajic and Shen<sup>6</sup>; Gajic and Qureshi<sup>41</sup>).

The celebrated Chang transformation decouples *exactly linear* singularly perturbed systems into independent slow and fast subsystems<sup>11</sup>. This transformation plays the fundamental role in modern theory of singularly perturbed control systems, and it is an essential part of the high accuracy techniques based on the recursive fixed-point and Hamiltonian approaches. The linear singularly perturbed system defined by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) \\ \varepsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) \end{aligned} \quad \dots (12)$$

is transformed via the Chang transformation into pure-slow and pure-fast subsystems

$$\begin{aligned}\eta_1(t) &= (A_1 - A_2 L) \eta_1(t) \\ \varepsilon \eta_2(t) &= (A_4 + \varepsilon L A_2) \eta_2(t)\end{aligned} \quad \dots (13)$$

with

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \varepsilon H L & -\varepsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \dots (14)$$

where the matrices  $L$  and  $H$  satisfy

$$\begin{aligned}A_4 L - A_3 - \varepsilon L (A_1 - A_2 L) &= 0 \\ H A_4 - A_2 + \varepsilon (H L A_2 - A_1 H + A_2 L H) &= 0\end{aligned} \quad \dots (15)$$

The unique solution of eq. (15) exist, by the implicit function theorem, for sufficiently small values of  $\varepsilon$  under the assumption that the matrix  $A_4$  is nonsingular. The fixed point algorithm for solving eq. (15) is derived in Kokotovic *et al.*<sup>47</sup>. It has the following form

$$\begin{aligned}A_4 L^{(i+1)} &= A_3 + \varepsilon L^{(i)} (A_1 - A_2 L^{(i)}) \\ H^{(i+1)} A_4 &= A_2 - \varepsilon (H^{(i)} L^{(i)} A_2 - A_1 H^{(i)} + A_2 L^{(i)} H^{(i)})\end{aligned}$$

$$L^{(0)} = A_4^{-1} A_3, H^{(0)} = A_2 A_4^{-1}, i = 0, 1, 2, \dots \quad \dots (16)$$

This algorithm converges with the rate of convergence of  $O(\varepsilon)$ . The Newton method for iterative solution of equation (15) is derived in Grodt and Gajic<sup>7</sup> as follows:

$$\begin{aligned}D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} &= Q^{(i)} \\ D_1^{(i)} &= A_4 + \varepsilon L^{(i)} A_2, D_2^{(i)} = -\varepsilon (A_1 - A_2 L^{(i)}) \\ Q^{(i)} &= A_3 + \varepsilon L^{(i)} A_2 L^{(i)}, L^{(0)} = A_4^{-1} A_3\end{aligned} \quad \dots (17)$$

This algorithm has quadratic convergence of  $O(\varepsilon^2)$ . Having obtained the solution for  $L$  with the required accuracy, the  $H$ -equation can be solved directly as a linear Sylvester equation

$$H^{(i+1)} D_1^{(i+1)} + D_2^{(i+1)} H^{(i+1)} = A_2 \quad \dots (18)$$

Note that the  $L$ - and  $H$ -equations have to be solved sequentially, first the  $L$ -equation and then the  $H$ -equation. A new version of the Chang transformation is developed in Qureshi and Gajic<sup>48</sup> in which equations for  $L$  and  $H$  are completely decoupled, hence it can be solved in parallel. The transformation of Qureshi and Gajic<sup>48</sup> is given by

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\varepsilon L_{new} \\ -H_{new} & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \dots (19)$$

where the matrices  $L_{new}$  and  $H_{new}$  satisfy

$$\begin{aligned}L_{new} A_4 - A_2 - \varepsilon (A_1 - L_{new} A_3) L_{new} &= 0 \\ H_{new} A_4 - A_3 - \varepsilon H_{new} (A_1 + A_2 H_{new}) &= 0\end{aligned} \quad \dots (20)$$

These equations can be also solved efficiently either by using the fixed point iterations or the Newton method as demonstrated above.

The discrete-time versions of the Chang transformation can be found in several papers (see Gajic and Shen<sup>33</sup>). The new version of the discrete-time Chang transformation is presented in Gajic and Shen<sup>6</sup>, Borno<sup>12</sup>. Note that the Chang transformation also exactly decouples the singularly perturbed algebraic, differential, and difference Lyapunov equations into the corresponding reduced-order, independent, pure-slow and pure-fast, Lyapunov-type equations<sup>6</sup>.

Based on the previously established results on the Chang transformation and algebraic Lyapunov and Riccati equations, the linear-quadratic Gaussian control problem of singularly perturbed systems has been solved in Khalil and Gajic<sup>49</sup>, Gajic<sup>15</sup>. The approach of Khalil and Gajic<sup>49</sup> was based on the Taylor series expansions, and the approach of Gajic<sup>15</sup> on the fixed point iterations for calculating the solutions of the Chang decoupling equations, solutions of the regulator and filter algebraic Riccati equations, and the coefficients of the optimal (and approximate) filter and controller. The linear stochastic system with the corresponding measurements is defined by

$$\begin{aligned}x_1'(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) + G_1 w(t) \\ \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t) + G_2 w(t) \\ y(t) &= C_1 x_1(t) + C_2 x_2(t) + v(t)\end{aligned} \quad \dots (21)$$

where  $u(t)$  represents an  $m$ -dimensional control vector,  $y(t)$  are  $r_2$ -dimensional system measurements,  $w(t) \in R^{r_1}$  and  $v(t) \in R^{r_2}$  are system and measurement disturbances assumed to be zero-mean, stationary, mutually uncorrelated, Gaussian white noise stochastic processes with intensities  $W > 0$  and  $V > 0$ . The quadratic performance criterion to be minimized is

$$J = \lim_{t_f \rightarrow \infty} \frac{1}{t_f} E \left\{ \int_0^{t_f} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T Q \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t) R u(t) dt \right\} \quad \dots (22)$$

It has been shown in Khalil and Gajic<sup>49</sup>, Gajic<sup>15</sup> that the optimal solution to the above linear-quadratic stochastic optimization problem can be

obtained in terms of reduced-order slow and fast Kalman filters as follows:

$$\begin{aligned} u_{opt}(t) &= -f_1 \hat{\eta}_1(t) - f_2 \hat{\eta}_2(t) \\ \dot{\hat{\eta}}_s(t) &= -a_{s1} \hat{\eta}_1(t) + g_s v(t) \\ \varepsilon \dot{\hat{\eta}}_f(t) &= a_f \hat{\eta}_1(t) + g_s v(t) \\ v(t) &= y(t) - c_1 \hat{\eta}_s(t) - c_2 \hat{\eta}_f(t) \end{aligned} \quad \dots (23)$$

Note that the slow and fast Kalman filters are driven by the innovation process  $v(t)$ , hence communications of optimal slow and fast estimates are needed in order to form the innovation process. The corresponding singularly perturbed discrete stochastic problem is considered in Shen<sup>32</sup>, Gajic and Shen<sup>33</sup>.

In Skataric and Gajic<sup>9</sup>; Skataric<sup>50</sup> a special class of linear control systems represented by the standard singularly perturbed system matrix and with the control input matrix having three different nonstandard forms is studied. The obtained results are quite simplified (compared to the standard singularly perturbed control systems), and in one case the optimal solution of the algebraic Riccati equation is completely determined in terms of reduced-order algebraic Lyapunov equations. The proposed method is successfully applied to the reduced-order design of optimal controllers for a hydro power plant<sup>9</sup>. It is important to point out that the solutions to the real 11<sup>th</sup>- and 14<sup>th</sup>-order hydro power control systems are obtained by the presented reduced-order parallel algorithms, but the global method fails to produce the answers in both cases (computations were performed using MATLAB).

The problem of high gain feedback and cheap control is studied in Huey *et al.*<sup>51</sup> via the fixed-point iterations. The singular perturbation methodology is used to describe the problems under consideration<sup>4,5</sup>. The reduced-order parallel algorithm producing any arbitrary order of accuracy is obtained under the control oriented assumptions. It is important to point out that in the presented methodology there is no need to study the high gain feedback and cheap control problems in the limit when a small parameter  $\varepsilon$  tends to zero. This avoids the impulsive behaviour and the presence of singular controls. The efficiency of the

algorithm obtained is demonstrated on an example of a flexible space structure.

The recursive approach to deterministic output feedback control of singularly perturbed linear systems is considered in Gajic *et al.*<sup>8</sup>. The well-defined recursive numerical technique for the solution of nonlinear algebraic matrix equations, associated with the output feedback control problem of singularly perturbed systems has been developed. The numerical slow-fast decomposition is achieved so that only low-order systems are involved in algebraic computations. The paper shows that each iteration step of the fixed-point algorithm improves the accuracy by an order of magnitude, that is, the accuracy of  $O(\varepsilon^k)$  can be obtained by performing only  $k$  iterations. This represents the significant improvement since all results on the output feedback control problems for singularly perturbed systems have been previously obtained with the accuracy of  $O(\varepsilon)$  only. As an example, an industrial important reactor—fluid catalytic cracker—demonstrates the efficiency of the proposed algorithm and the failure of  $O(\varepsilon)$  theory. The static output feedback control problem for discrete linear singularly perturbed stochastic systems is studied in Qureshi *et al.*<sup>36</sup>, where a recursive algorithm is presented to solve the corresponding nonlinear algebraic equations. The algorithm removes the ill-conditioning by decomposing the higher order equations into lower order equations corresponding to the fast and slow time scales.

In Borno<sup>52</sup>, Borno and Gajic<sup>37</sup> the fixed-point iterations were used to solve a system of coupled algebraic Riccati equations coming from the optimal control problem of jump parameter linear systems<sup>53</sup>. The algorithm yields an arbitrary order of accuracy and operates on reduced-order algebraic equations.

The recursive approach to singularly perturbed linear control systems is extended in the work of Aganovic<sup>30</sup>, Aganovic and Gajic<sup>28</sup> to bilinear control systems. The composite near-optimal control of singularly perturbed bilinear systems is obtained in Aganovic and Gajic<sup>28</sup> by combining the ideas from Chow and Kokotovic<sup>42</sup> and Cebuhar and Constanza<sup>54</sup>. Obtained results are demonstrated on a fourth-order induction motor drives. The extension of the near-optimal composite control to

the optimal reduced-order control is also considered. The reduced-order open-loop optimal control of singularly perturbed bilinear systems is presented in Aganovic and Gajic<sup>28</sup>.

### 3 The Hamiltonian Approach

The Hamiltonian approach represents the most efficient method for exact solution of optimal control and filtering problems of singularly perturbed linear systems. This approach removes numerical ill-conditioning of the original problems and produces well-conditioned, reduced-order, exact pure-slow and pure-fast subproblems. The class of problems solvable by the Hamiltonian approach are steady state linear-quadratic optimal control and filtering problems whose Hamiltonian matrices under appropriate scaling and permutation preserve singularly perturbed forms such that they can be block diagonalized into pure-slow and pure-fast Hamiltonian matrices. That is why, this method is called the Hamiltonian approach to singularly perturbed linear control systems. The problems presently solvable by the Hamiltonian method are: linear-quadratic optimal regulator and Kalman filter in continuous- and discrete-time domains, optimal open-loop control of continuous- and discrete-time linear systems, multimodeling estimation and control,  $H_\infty$  optimal control and filtering of linear systems, linear-quadratic zero-sum differential games, linear-quadratic high gain, cheap control, and small measurement noise problems, sampled data control systems, and nonstandard linear singularly perturbed systems. Some other classes of linear-quadratic type optimal control problems that can be solved by the Hamiltonian approach may emerge in the near future.

The Hamiltonian approach to singularly perturbed linear optimal control systems is based on block diagonalization of the Hamiltonian matrix compatibility to its slow-fast structure. One of the main results of this method is the complete and exact decomposition of the corresponding algebraic Riccati equations into the reduced-order, completely independent, pure-slow and pure-fast, algebraic Riccati equations. It is well known that the algebraic Riccati equations can be studied in terms of corresponding Hamiltonian matrices. The Hamiltonian matrices of singularly perturbed linear optimal control systems retain the singularly per-

turbed form by interchanging and appropriately scaling some of the state and costate variables, hence they can be block diagonalized via the decoupling transformation of Chang<sup>11</sup> and Qureshi and Gajic<sup>48</sup>. The block diagonalization procedure produces the pure-slow and pure-fast Hamiltonian matrices, each corresponding to the pure-slow and pure-fast nonsymmetric algebraic Riccati equations.

The algebraic Riccati equation of singularly perturbed *continuous-time control systems* given by

$$A^T P_c + P_c A + Q - P_c S P_c = 0, \dim \{P_c\} = n_1 + n_2$$

$$A = O\left(\frac{1}{\varepsilon}\right), S = O\left(\frac{1}{\varepsilon^2}\right), Q = O(1) \quad \dots (24)$$

is numerically ill-conditioned due to the special structures of matrices  $A$  and  $S$ . By using the Hamiltonian approach, this equation is completely and exactly decomposed into two reduced-order algebraic Riccati equations corresponding to slow and fast time scales as

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0, \dim \{P_s\} = n_1$$

$$P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0, \dim \{P_f\} = n_2 \dots (25)$$

with  $a_i, b_i = O(1), i=1, 2, 3, 4$ , see Su *et al.*<sup>17</sup>. Eqs (25) are well-conditioned, reduced-order, pure-slow and pure-fast, algebraic Riccati equations. The pure-slow and pure-fast algebraic Riccati equations are nonsymmetric, but their  $O(\varepsilon)$  perturbations are symmetric, that is

$$P_s^{(0)} a_1^{(0)} + a_1^{(0)T} P_s^{(0)} - a_3^{(0)} + P_s^{(0)} a_2^{(0)} P_s^{(0)} = 0$$

$$\dim \{P_s^{(0)}\} = n_1$$

$$a_i = a_i^{(0)} + O(\varepsilon), a_4^{(0)} = -a_1^{(0)T}, a_3^{(0)} = a_3^{(0)T}, a_2^{(0)} = a_2^{(0)T}$$

$$P_f^{(0)} b_1^{(0)} + b_1^{(0)T} P_f^{(0)} - b_3^{(0)} + P_f^{(0)} b_2^{(0)} P_f^{(0)} = 0 \dots (26)$$

$$\dim \{P_f^{(0)}\} = n_2$$

$$b_i = b_i^{(0)} + O(\varepsilon), b_4^{(0)} = -b_1^{(0)T}, b_3^{(0)} = b_3^{(0)T}, b_2^{(0)} = b_2^{(0)T}$$

Interestingly enough, algebraic Riccati equations (26) are identical to approximate slow and fast algebraic Riccati equations of Chow and Kokotovic<sup>42</sup>, which are defined in eq. (6), that is  $a_1^{(0)} = A_s, a_2^{(0)} = -S_2, a_3^{(0)} = -Q_s$  and  $b_1^{(0)} = A_4, b_2^{(0)} = -S_3, b_3^{(0)} = -Q_3$ . The unique positive semi-

definite stabilizing solutions of eqs. (26) exist under standard stabilizability-detectability conditions imposed on slow and fast subsystems. These solutions  $(P_s^{(0)}, P_f^{(0)})$  can be easily obtained by using any standard method for solving the symmetric algebraic Riccati equation. It is shown in Su *et al.*<sup>17</sup> that the Newton method is very efficient for solving the pure-slow and pure-fast nonsymmetric algebraic Riccati equations. The Newton method for solving eq. (25) is given in terms of Lyapunov iterations as follows:

$$P_s^{(i+1)}(a_1 + a_2 P_s^{(i)}) - (a_4 - P_s^{(i)} a_2) P_s^{(i+1)} \\ = a_3 + P_s^{(i)} a_2 P_s^{(i)}, i = 0, 1, 2, \dots$$

$$P_f^{(i+1)}(b_1 + b_2 P_f^{(i)}) - (b_4 - P_f^{(i)} b_2) P_f^{(i+1)} \\ = b_3 + P_f^{(i)} b_2 P_f^{(i)}, i = 0, 1, 2, \dots \quad \dots (27)$$

Having found the solutions for  $P_s$  and  $P_f$ , the required solution of eq. (24) is obtained as a simple matrix function of  $P_s$  and  $P_f$ , (Su *et al.*<sup>17</sup>) that is

$$P_c = F_c(P_s, P_f) \quad \dots (28)$$

The above results about the exact pure-slow and pure-fast decomposition of the algebraic Riccati equation applied to the linear-quadratic optimal control problem defined by

$$\dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) \\ \varepsilon \dot{x}_2(t) = A_3 x_1(t) + A_4 x_2(t) + B_2 u(t) \quad \dots (29)$$

and

$$J = \min_u \int_0^\infty \left[ \begin{array}{c} \left( \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right)^T Q \left( \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right) \\ + u^T(t) R u(t) \end{array} \right] dt \quad \dots (30)$$

where

$$u(x(t)) = -F_1 x_1(t) - F_2 x_2(t) \quad \dots (31)$$

lead to the following lemma, which can be deduced from the results of Su *et al.*<sup>17</sup>.

**Lemma 1:** Consider the optimal closed-loop linear system

$$\dot{x}_1(t) = (A_1 - B_1 F_1) x_1(t) + (A_2 - B_1 F_2) x_2(t) \\ \varepsilon \dot{x}_2(t) = (A_3 - B_2 F_1) x_1(t) + (A_4 - B_2 F_2) x_2(t) \quad \dots (32)$$

Under standard stabilizability-detectability conditions imposed on the slow and fast

subsystems, there exists a nonsingular transformation  $\mathbf{T}$

$$\begin{bmatrix} \xi_s(t) \\ \xi_f(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \dots (33)$$

such that

$$\dot{\xi}_s(t) = (a_1 + a_2 P_s) \xi_s(t) \\ \varepsilon \dot{\xi}_f(t) = (b_1 + b_2 P_f) \xi_f(t) \quad \dots (34)$$

where  $P_s$  and  $P_f$  are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati eq. (25). The nonsingular transformation  $\mathbf{T}$  is given by

$$\mathbf{T} = (\Pi_1 + \Pi_2 P) \quad \dots (35)$$

Known matrices  $\Pi_1, \Pi_2$  are given in terms of solutions of Chang decoupling equations. Even more, the global solution  $P$  can be obtained from the reduced-order exact pure-slow and pure-fast algebraic Riccati equations using eqn. (28).

In a similar manner, the numerically ill-conditioned algebraic Riccati equation of singularly perturbed discrete-time control systems, given by

$$P_d = A_d^T P_d A_d + Q_d \\ - A_d^T P_d B_d (R_d + B_d^T P_d B_d)^{-1} B_d^T P_d A_d = 0 \quad \dots (36) \\ \dim\{P_d\} = n_1 + n_2$$

is exactly solved in terms of two reduced-order algebraic *continuous-time* Riccati equations corresponding to slow and fast time scales having the form of eq. (25), that is

$$P_s^d a_1^d - a_4^d P_s^d - a_3^d + P_s^d a_2^d P_s^d = 0, \dim\{P_s^d\} = n_1 \\ P_f^d b_1^d - b_4^d P_f^d - b_3^d + P_f^d b_2^d P_f^d = 0, \dim\{P_f^d\} = n_2 \quad \dots (37)$$

The sought solution of eq (36) is obtained, under the standard stabilizability-detectability assumptions imposed on the slow and fast subsystems, as a simple matrix function of the solutions of the pure-slow and pure-fast algebraic Riccati equations<sup>19-21</sup>.

$$P_d = F_D(P_s^d, P_f^d) \quad \dots (38)$$

The decomposition of the discrete-time algebraic Riccati equation in terms of independent, reduced-



order, continuous-time algebraic Riccati equations represents a pretty powerful result since the continuous-time algebraic Riccati equation is much better understood and easier for solving than the discrete-time algebraic Riccati equation.

The above decomposition of the algebraic Riccati equations and their variants produce new insights into the slow-fast time scale optimal filtering and control for several important problems of linear singularly perturbed systems. The results obtained are characterized by well-conditioning, complete and exact decoupling of the slow and fast time scale phenomena, reduction of off-line and on-line computational requirements, and parallel processing of information. In the Kalman filtering problem, the Hamiltonian approach produces a new filtering scheme that provides full parallelism between the slow and fast filters. In the old filtering scheme the filters are driven by the innovation process, hence additional communications of slow and fast estimates are needed.

The new pure-slow and pure-fast filter decomposition scheme is used in Lim<sup>19</sup>, Lim *et al.*<sup>20</sup> to solve the linear-quadratic optimal Gaussian control problem defined in eqs. (21)-(22). The optimal solution is obtained in the form

$$\begin{aligned} u_{opt}(t) &= -F_s \hat{\eta}_s(t) - F_f \hat{\eta}_f(t) \\ \dot{\hat{\eta}}_s(t) &= (a_{1F} + a_{2F} P_{sF})^T \hat{\eta}_s(t) + B_s u(t) + K_s y(t) \\ \dot{\hat{\eta}}_f(t) &= (b_{1F} + b_{2F} P_{fF})^T \hat{\eta}_f(t) + B_f u(t) + K_f y(t) \end{aligned} \quad \dots (39)$$

where  $a_{1F}$ ,  $a_{2F}$ ,  $b_{1F}$ ,  $b_{2F}$ ,  $P_{sF}$ ,  $P_{fF}$  come from the pure-slow and pure-fast filter algebraic equations dual to eq. (25); (see Gajic and Lim<sup>18</sup>):

$$\begin{aligned} P_{sF} a_{1F} - a_{4F} P_{sF} - a_{3F} + P_{sF} a_{2F} P_{sF} &= 0, \\ \dim\{P_{sF}\} &= n_1 \\ P_{fF} b_{1F} - b_{4F} P_{fF} - b_{3F} + P_{fF} b_{2F} P_{fF} &= 0, \\ \dim\{P_{fF}\} &= n_2 \end{aligned} \quad \dots (40)$$

The finite time optimal open-loop control problems (linear two-point boundary value problem) for singularly perturbed control systems can be studied from the Hamiltonian approach point of view also. The original, numerically ill-conditioned, two-point boundary value problem is

transformed into the pure-slow and pure-fast reduced-order completely decoupled initial value problems. By doing this, the stiffness of the singularly perturbed two-point boundary value problem is converted into the problem of an ill-defined system of linear algebraic equations<sup>24</sup>. The study of the open-loop control problem presented for singularly perturbed continuous-time systems is extended to the corresponding discrete-time domain in Qureshi *et al.*<sup>55</sup> and Qureshi<sup>55</sup>.

The optimal control and filtering of the multimodeling structures<sup>56</sup> via the Hamiltonian approach has been studied in Coumarbatch and Gajic<sup>57</sup> and Coumarbatch<sup>58</sup>. The preliminary results obtained give important fundamentals that can be extended to the development of Pareto multimodeling strategies<sup>56</sup> and the quasi-decentralized multimodeling estimation<sup>59</sup>.

The results on the Hamiltonian approach to continuous-time  $H_\infty$  optimal filtering and control of linear singularly perturbed systems have been obtained in Hsieh and Gajic<sup>60</sup> and Lim and Gajic<sup>61</sup>. The papers indicate the difficulties encountered in the  $H_\infty$  optimization of singularly perturbed linear systems, and the necessity for an additional transformation to exactly decouple the slow and fast Kalman filters—in contrast to the classic optimal singularly perturbed linear Kalman filter, where the same transformation decouples both the algebraic filter Riccati equation and the corresponding Kalman filter.

The open-loop cheap (and high gain) control problem in continuous-time and the problem of complete decomposition of the corresponding algebraic “cheap (high gain)” Riccati equation into the reduced-order pure-slow and pure-fast Riccati equations are presented in Gajic and Shen<sup>6</sup>. The results for the special class of discrete-time cheap optimal control problems, sampled data control systems, are obtained in Popescu and Gajic<sup>62</sup>. The dual problem to the continuous-time cheap control problem is the small measurement noise optimal continuous-time Kalman filtering problem. It is interesting to point out that the small measurement noise under certain assumptions induces the slow-fast time scale separation of the system state space variables. This problem is solved in Aganovic *et al.*<sup>31</sup> where it has been shown how to get and exactly decouple the corresponding pure-slow and pure-fast Kalman filters.

The most recent developments in the field of the Hamiltonian approach to singularly perturbed linear control systems are presented in Kecman *et al.*<sup>63</sup> where the eigenvector method is introduced for simultaneous pure-fast/pure-slow block diagonalization of the Hamiltonian matrix and the solution of Chang's algebraic equations required for such a decomposition.

#### 4 Other Approaches

The work of Sobolev<sup>14</sup> based on slow-fast manifold theory resulted in the exact pure-slow and pure-fast decomposition of the linear-quadratic optimal control problem. The results of Sobolev<sup>14</sup> have been extended in Fridman<sup>22,23</sup> to the  $H_\infty$ -optimal linear-quadratic control problem of singularly perturbed systems. It should be pointed out that the results of Fridman<sup>22,23</sup> hold for both finite time and steady state optimization problems. It remains an open question whether or not the integral manifold approach to decomposition of singularly perturbed linear-quadratic control problems leads to the same results as obtained by using the Hamiltonian approach. The slow-fast manifold theory is also applied to decomposition of nonlinear singularly perturbed control systems as demonstrated in Sobolev<sup>14</sup> and Fridman<sup>64</sup>.

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