Game Theory Approach to Discrete H_∞ Filter Design

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***Abstract*—In this correspondence, a finite-horizon discrete H_∞ filter design with a linear quadratic (LQ) game approach is presented. The exogenous inputs composed of the "hostile" noise signals and system initial condition are assumed to be finite energy signals with unknown statistics. The design criterion is to minimize the worst possible amplification of the estimation error signals in terms of the exogenous inputs, which is different from the classical minimum variance estimation error criterion for the modified Wiener or Kalman filter design. The approach can show how far the estimation error can be reduced under an existence condition on the solution to a corresponding Riccati equation. A numerical example is given to compare the performance of the H_∞ filter with that of the conventional Kalman filter.**

I. Introduction

The celebrated Wiener and/or Kalman estimators have been widely used in noise signal processesing. This type of estimation assumes that signal generating processes have known dynamics and that the noise sources have known statistical properties. However, these assumptions may limit the application of the estimators since in many situations, only approximate signal models are available and/or the statistics of the noise sources are not fully known or are unavailable. In addition, both Wiener and Kalman estimators may not be robust against parameter uncertainty of the signal models. Recent developments in optimal filtering have focused on the H_∞ estimation methods [1]–[10]. The optimal H_∞ estimator is designed to guarantee that the operator relating the noise signals to the resulting estimation errors should possess an H_∞ norm less than a prescribed positive value. In the H_∞ estimation, the noise sources can be arbitrary signals with only a requirement of bounded noise. Since the H_∞ estimation problem involves the minimization of the worst possible amplification of the error signal, it can be viewed as a dynamic, two-person, zero sum game. In the game, the H_∞ filter (the designer) is a player prepared for the worst strategy that the other player (the nature) can provide, i.e., the goal of the filter is to provide an uniformly small estimation error for any processes and measurement noises and any initial states. In this correspondence, we define a difference game in which the state estimator and the disturbance signals (processes noise, initial condition and measurement noise) have the conflicting objectives of, respectively, minimizing and maximizing the estimation error. The minimizer picks the optimal filtered estimate, and the maximizer picks the worst-case disturbance and initial condition. We give a detailed derivation to solve the game that directly produces the solution for the discrete H_∞ filtering problem. A similar design approach has been proposed in [1] and [2] for the continuous case. We then give a numerical example to compare the H_∞ filter with the Kalman filter. The comparison includes the magnitudes of the transfer functions from processes and measurement noises to estimation errors, which are the estimations of the true signals. It is shown that the H_∞ filter is more robust compared with those of Wiener and Kalman filters in terms of model uncertainty and gives better estimates.

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II. Discrete H_∞ Filter Design

Consider the following discrete-time system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k \\ y_k &= C_k x_k + v_k \\ k &= 0, 1, \cdots, N-1; \qquad x_0 = x_0 \end{aligned} \tag{1}$$

where

$x_k \in \mathcal{R}^n$ state vector,
$w_k \in \mathcal{R}^m$ process noise vector,
$y_k \in \mathcal{R}^p$ measurement vector,
$v_k \in \mathcal{R}^p$ measurement noise vector.

A_k, B_k, and C_k are matrices of the appropriate dimensions. Assume that (A_k, B_k) is controllable and (C_k, A_k) is detectable. Define the measurement history as $Y_k = (y_k, 0 \le k \le N-1)$. The estimate of the state $\hat{x}_k$ at time k is computed based on the measurement history up to $N-1$. We are not necessarily interested in the estimation of x_k but in the estimation of a linear combination of x_k

$$z_k = L_k x_k. \tag{2}$$

The H_∞ filter is required to provide an uniformly small estimation error $e_k = z_k - \hat{z}_k$ for any $w_k, v_k \in l_2$ and $x_0 \in \mathcal{R}^n$. The measure of performance is then given by

$$J = \frac{\sum_{k=0}^{N-1} ||z_k - \hat{z}_k||^2_{Q_k}}{||x_0 - \hat{x}_0||^2_{p_0^{-1}} + \sum_{k=0}^{N-1} \{||w_k||^2_{W_k^{-1}} + ||v_k||^2_{V_k^{-1}}\}} \tag{3}$$

where $((x_0 - \hat{x}_0), w_k, v_k) \ne 0, \hat{x}_0$ is an *priori* estimate of $x_0, Q_k \ge 0, p_0^{-1} > 0, W_k > 0$ and $V_k > 0$ are the weighting matrices, and $||s_k||^2_{R_k} = s_k^T R_k s_k$. The optimal estimate z_k among all possible $\hat{z}_k$ (i.e., the worse-case performance measure) should satisfy

$$\sup J < 1/\gamma \tag{4}$$

where "sup" stands for supremum, and $\gamma > 0$ is a prescribed level of noise attenuation. The matrices $Q_k \ge 0, W_k > 0, V_k > 0$ and $p_0 > 0$ are left to the choice of the designer and depend on performance requirements. Discrete H_∞ filtering can be interpreted as a *minimax* problem where the estimator strategy $\hat{z}_k$ play against the exogenous inputs w_k, v_k and the initial state x_0. The performance criterion becomes

$$\begin{aligned} &\min_{\hat{z}_k} \max_{(v_k, w_k, x_0)} J \\ &\quad = -\frac{1}{2\gamma} ||x_0 - \hat{x}_0||^2_{p_0^{-1}} + \frac{1}{2} \\ &\quad\quad \cdot \sum_{k=0}^{N-1} \left[||z_k - \hat{z}_k||^2_{Q_k} - \frac{1}{\gamma} (||w_k||^2_{W_k^{-1}} + ||v_k||^2_{V_k^{-1}}) \right] \end{aligned} \tag{5}$$

where "min" stands for minimization and "max" maximization. Note that unlike the traditional minimum variance filtering approach (Wiener and/or Kalman filtering), the H_∞ filtering deals with deterministic disturbances, and no *a priori* knowledge of the noise statistics is required. Since the observation y_k is given, v_k can be uniquely determined by (1) once the optimal values of w_k and x_0 are found. Letting $\hat{z}_k = L_k\hat{x}_k$, we can rewrite the performance criterion (5) as

$$\min_{\hat{x}_k} \max_{(y_k, w_k, x_0)} J = -\frac{1}{2\gamma}||x_0 - \hat{x}_0||^2_{p_0^{-1}} + \frac{1}{2}\sum_{k=0}^{N-1}[||x_k - \hat{x}_k||^2_{\overline{Q}_k} - \frac{1}{\gamma}(||w_k||^2_{W_k^{-1}} + ||y_k - C_k x_k||^2_{V_k^{-1}})] \quad (6)$$

where $\overline{Q}_k = L_k^T Q_k L_k$. The following theorem presents a complete solution to the H_∞ filtering problem for the system (1) with the performance criterion (6).

Theorem: Let $\gamma > 0$ be a prescribed level of noise attenuation. Then, there exists an H_∞ filter for z_k if and only if there exists a stabilizing symmetric solution $P_k > 0$ to the following discrete-time Riccati equation:

$$\begin{aligned} P_{k+1} &= A_k P_k (I - \gamma\overline{Q}_k P_k + C_k^T V_k^{-1} C_k P_k)^{-1} \\ &\quad \cdot A_k^T + B_k W_k B_k^T \\ P_0 &= p_0. \end{aligned} \quad (7)$$

The H_∞ filter is given by

$$\hat{z}_k = L_k \hat{x}_k, \qquad k = 0, 1, \cdots, N-1 \quad (8)$$

where

$$\hat{x}_{k+1} = A_k\hat{x}_k + K_k(y_k - C_k\hat{x}_k), \hat{x}_0 = \hat{x}_0. \quad (9)$$

K_k is the gain of the H_∞ filter and is given by

$$K_k = A_k P_k (I - \gamma\overline{Q}_k P_k + C_k^T V_k^{-1} C_k P_k)^{-1} C_k^T V_k^{-1}. \quad (10)$$

Proof: By using a set of Lagrange multiplier to adjoin the constraint (1) to the performance criterion (6), the resulting *Hamiltonian* is

$$\begin{aligned} M &= \frac{1}{2}\left[||x_k - \hat{x}_k||^2_{\overline{Q}_k} - \frac{1}{\gamma}(||w_k||^2_{W_k^{-1}} + ||y_k - C_k x_k||^2_{V_k^{-1}})\right] \\ &\quad + \frac{\lambda_{k+1}^T}{\gamma}[A_k x_k + B_k w_k - x_{k+1}] \\ &\quad + [A_k x_k + B_k w_k - x_{k+1}]^T \frac{\lambda_{k+1}}{\gamma}. \end{aligned} \quad (11)$$

Taking the first variation, the necessary conditions for a maximum are

$$x_0 = \hat{x}_0 + p_0\lambda_0, \qquad \lambda_N = 0 \quad (12)$$

$$w_k = W_k B_k^T \lambda_{k+1} \quad (13)$$

$$\lambda_k = A_k^T\lambda_{k+1} + \gamma\overline{Q}_k(x_k - \hat{x}_k) + C_k^T V_k^{-1}(y_k - C_k x_k). \quad (14)$$

These first-order necessary conditions result in a two-point boundary value problem

$$\begin{pmatrix} x_{k+1} \\ \lambda_k \end{pmatrix} = \begin{pmatrix} A_k & B_k W_k B_k^T \\ \gamma\overline{Q}_k - C_k^T V_k^{-1} C_k & A_k^T \end{pmatrix} \begin{pmatrix} x_k \\ \lambda_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ -\gamma\overline{Q}_k\hat{x}_k + C_k^T V_k^{-1} y_k \end{pmatrix}, \quad k = 0, 1, \cdots, N-1 \quad (15)$$

with boundary conditions

$$x_0 = \hat{x}_0 + p_0\lambda_0, \qquad \lambda_N = 0. \quad (16)$$

Since the two-point boundary value problem is linear, the solution is assumed to be of the form

$$x_k^* = \overline{x}_k + P_k\lambda_k^* \quad (17)$$

where $\overline{x}_k$ and P_k are undetermined variables. x_k^* and λ_k^* represent the optimal value of x_k and λ_k, respectively, for any fixed admissible functions of $\overline{x}_k$ and y_k. The optimal values for w_k and x_0 are

$$w_k^* = W_k B_k^T \lambda_{k+1}^*, \quad x_0^* = \hat{x}_0 + p_0\lambda_0^*. \quad (18)$$

Substituting (17) into (15) results in

$$\overline{x}_{k+1} + P_{k+1}\lambda_{k+1}^* = A_k\overline{x}_k + A_k P_k \lambda_k^* + B_k W_k B_k^T \lambda_{k+1}^* \quad (19)$$

and

$$\begin{aligned} \lambda_k^* &= (I - \gamma\overline{Q}_k P_k + C_k^T V_k^{-1} C_k P_k)^{-1} \\ &\quad \cdot [\gamma\overline{Q}_k(\overline{x}_k - \hat{x}_k) + C_k^T V_k^{-1}(y_k - C_k\overline{x}_k) \\ &\quad + A_k^T\lambda_{k+1}^*]. \end{aligned} \quad (20)$$

From (19) and (20), we have

$$\begin{aligned} &\overline{x}_{k+1} - A_k\overline{x}_k - A_k P_k (I - \gamma\overline{Q}_k P_k + C_k^T V_k^{-1} C_k P_k)^{-1} \\ &\quad \cdot [\gamma\overline{Q}_k(\overline{x}_k - \hat{x}_k) + C_k^T V_k^{-1}(y_k - C_k\overline{x}_k)] \\ &= [-P_{k+1} + A_k P_k (I - \gamma\overline{Q}_k P_k + C_k^T V_k^{-1} C_k P_k)^{-1} \\ &\quad \cdot A_k^T + B_k W_k B_k^T]\lambda_{k+1}^*. \end{aligned} \quad (21)$$

For (21) to hold true for arbitrary λ_k^*, both sides are set identically to zero, resulting in

$$\begin{aligned} \overline{x}_{k+1} &= A_k\overline{x}_k + A_k P_k[(I - (\gamma\overline{Q}_k - C_k^T V_k^{-1} C_k)P_k]^{-1} \\ &\quad \cdot [\gamma\overline{Q}_k(\overline{x}_k - \hat{x}_k) + C_k^T V_k^{-1}(y_k - C_k\overline{x}_k)] \\ \overline{x}_0 &= \hat{x}_0 \end{aligned} \quad (22)$$

and

$$\begin{aligned} P_{k+1} &= A_k P_k (I - \gamma\overline{Q}_k P_k + C_k^T V_k^{-1} C_k) P_k)^{-1} A_k^T \\ &\quad + B_k W_k B_k^T \\ P_0 &= p_0. \end{aligned} \quad (23)$$

Equation (23) is the well-known Riccati difference equation. It has been prooven that if the solution P_k to the Riccati equation (23) exists $\forall k \in [0, N-1]$. Then, $P_k > 0 \forall k \in [0, N-1]$.

Now, substituting the optimal strategies (18) into the performance (6), we obtain

$$\begin{aligned} \min_{\hat{x}_k}\max_{y_k} J &= -\frac{1}{2\gamma}||\lambda_0^*||^2_{p_0} + \frac{1}{2}\sum_{k=0}^{N-1}[||\overline{x}_k + P_k\lambda_k^* - \hat{x}_k||^2_{\overline{Q}_k} \\ &\quad - \frac{1}{\gamma}(||W_k B_k^T \lambda_{k+1}^*||^2_{W_k^{-1}} + ||y_k - C_k\overline{x}_k \\ &\quad - C_k P_k \lambda_k^*||^2_{V_k^{-1}})]. \end{aligned} \quad (24)$$

In the sequel, we will perform the *min-max* optimization of J with respect to $\hat{x}_k$ and y_k, respectively. Adding to (24) the identically zero term

$$\frac{1}{2\gamma}[||\lambda_0^*||^2_{p_0} - ||\lambda_N^*||^2_{P_N} + \sum_{k=0}^{N-1}(||\lambda_{k+1}^*||^2_{P_{k+1}} - ||\lambda_k^*||^2_{P_k})] = 0 \quad (25)$$

results in the following *min-max* problem

$$\min_{\hat{x}_k}\max_{y_k} J = \frac{1}{2}\sum_{k=0}^{N-1}[||\overline{x}_k - \hat{x}_k||^2_{\overline{Q}_k} - \frac{1}{\gamma}||y_k - C_k\overline{x}_k||^2_{V_k^{-1}}] \quad (26)$$

subject to the dynamic constraints (22) and (23).

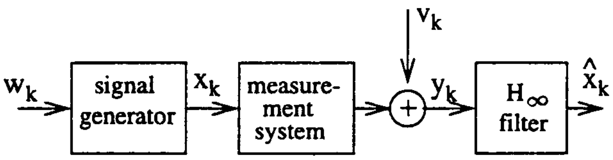

Fig. 1. Signal generating mechanism.

Letting

$$r_k = \overline{x}_k - \hat{x}_k, \quad q_k = y_k - C_k\overline{x}_k \tag{27}$$

(26) becomes

$$\min_{r_k}\max_{q_k} J = \frac{1}{2}\sum_{k=0}^{N-1}\left[||r_k||^2_{\overline{Q}_k} - \frac{1}{\gamma}||q_k||^2_{V_k^{-1}}\right]. \tag{28}$$

The two independent players r_k and q_k in (28) affect the variables $\overline{x}_k$, but $\overline{x}_k$ does not appear in the performance index, and therefore, the optimal strategies of r_k and q_k are

$$r_k^* = 0, \quad q_k^* = 0 \tag{29}$$

i.e.

$$\overline{x}_k = \hat{x}_k^*, \quad y_k^* = C_k\overline{x}_k. \tag{30}$$

The value of the game is the value of the cost function (6). When the optimal strategies $\hat{x}_k^*, y_k^*, w_k^*,$ and x_0^* in (18) and (30) are substituted into the (6)

$$J(\hat{x}_k^*, y_k^*, w_k^*, x_0^*) = 0 \tag{31}$$

giving a zero value game.

Thus far, the strategies of $\hat{x}_k^*, y_k^*, w_k^*,$ and x_0^* have been assumed to be optimal, based on the satisfaction of the necessary conditions for optimality. If the strategies can also satisfy a saddle-point inequality, they represent optimal strategies. A saddle point strategy can be obtained by solving two optimization problems:

$$\min_{\hat{x}_k}\max_{y_k}\max_{w_k}\max_{x_0} J = J^* \tag{32}$$

$$\max_{y_k}\max_{w_k}\max_{x_0}\min_{\hat{x}_k} J = J_*. \tag{33}$$

When $J^* = J_*$, the solutions to (32) and (33) produce saddle point strategies. It can be easily shown that if P_k exists $\forall k \in [0, N-1]$, the optimal strategies $\hat{x}_k^*, y_k^*, w_k^*,$ and x_0^* satisfy a saddle point inequality

$$J(\hat{x}_k^*, y_k, w_k, x_0) \leq J(\hat{x}_k^*, y_k^*, w_k^*, x_0^*) \leq J(\hat{x}_k, y_k^*, w_k^*, x_0^*). \tag{34}$$

Note that the notation $J_1 \geq J_2$ means that $J_1 - J_2$ is a positive semi-definite matrix.

The right inequality can be checked by adding the identically zero term

$$\frac{1}{2\gamma}[||x_0^* - \hat{x}_0||^2_{P_0^{-1}} - ||x_N^* - \hat{x}_N||^2_{P_N^{-1}} + \sum_{k=0}^{N-1}(||x_{k+1}^* - \hat{x}_{k+1}||^2_{P_{k+1}^{-1}} - ||x_k^* - \hat{x}_k||^2_{P_k^{-1}})] \tag{35}$$

to $J(\hat{x}_k, y_k^*, w_k^*, x_0^*)$, and the left inequality can be checked by adding the identically zero term

$$\frac{1}{2\gamma}[||x_0 - \hat{x}_0^*||^2_{P_0^{-1}} - ||x_N - \hat{x}_N^*||^2_{P_N^{-1}} + \sum_{k=0}^{N-1}(||x_{k+1} - \hat{x}_{k+1}^*||^2_{P_{k+1}^{-1}} - ||x_k - \hat{x}_k^*||^2_{P_k^{-1}})] \tag{36}$$

to $J(\hat{x}_k^*, y_k, w_k, x_0)$.

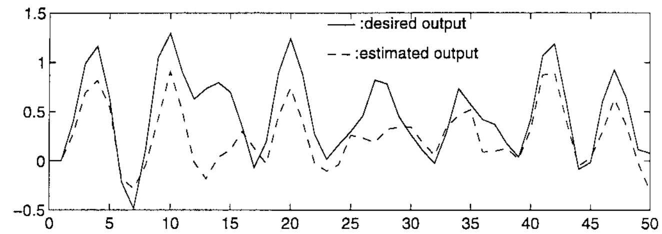

Fig. 2. Kalman filter estimate.

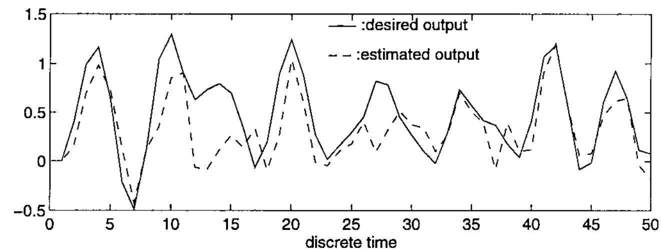

Fig. 3. H_∞ filter estimate.

The optimal strategy of the measurement noise can be obtained by

$$v_k^* = y_k^* - C_k\hat{x}_k^* = C_k\overline{x}_k - C_k\hat{x}_k^* = 0. \tag{37}$$

With (22) and (30), the optimal H_∞ filter is given by

$$\hat{z}_k^* = L_k\hat{x}_k^*, \qquad k = 0, 1, \cdots, N-1 \tag{38}$$

where

$$\hat{x}_{k+1}^* = A_k\hat{x}_k^* + K_k(y_k - C_k\hat{x}_k^*), \qquad \overline{x}_0 = \hat{x}_0 \tag{39}$$

$$K_k = A_kP_k(I - \gamma\overline{Q}_kP_k + C_k^TV_k^{-1}C_kP_k)^{-1}C_k^TV_k^{-1} \tag{40}$$

and P_k is given by (23).

It is important to note that the optimal H_∞ filter depends on the weighting on the estimation error in the performance criterion, i.e., the designer choses the weighting matrices based on the performance requitements, whereas both Wiener and Kalman filters are dependent on the variance of the noises.

For the time-invariant case $(N \to \infty)$, the optimal steady-state H_∞ filter is given by

$$\hat{z}_k^* = L\hat{x}_k^*, \qquad k = 0, 1, \cdots, \infty \tag{41}$$

where

$$\hat{x}_{k+1}^* = A\hat{x}_k^* + K(y_k - C\hat{x}_k^*), \overline{x}_0 = \hat{x}_0 \tag{42}$$

$$K = AP(I - \gamma\overline{Q}P + C^TV^{-1}CP)^{-1}C^TV^{-1} \tag{43}$$

and the Riccati equation becomes

$$P = AP(I - \gamma\overline{Q}P + C^TV^{-1}CP)^{-1}A^T + BWB^T. \tag{44}$$

The solution of the Riccati equation (44) can be obtained by the following [9]. Let

$$H = \begin{pmatrix} A^{-T} & A^{-T}[C^TV^{-1}C - \gamma\overline{Q}] \\ BWB^TA^{-T} & A + BWB^TA^{-T}[C^TV^{-1}C - \gamma\overline{Q}] \end{pmatrix}. \tag{45}$$

Assume that matrix H has no eigenvalues on the unit circle and that the eigenvector S corresponds to the outer circle (unstable) eigenvalues of the matrix H. Spanning S as $S = [S_1^T \quad S_2^T]^T$, the solution of Riccati equation P is given as

$$P = S_2S_1^{-1}. \tag{46}$$

Details of the last result can be found in [11]. Note that in the limiting case, where the parameter $\gamma \to 0$, the H_∞ filter given by (41)–(44) reduces to a steady-state Kalman filter.

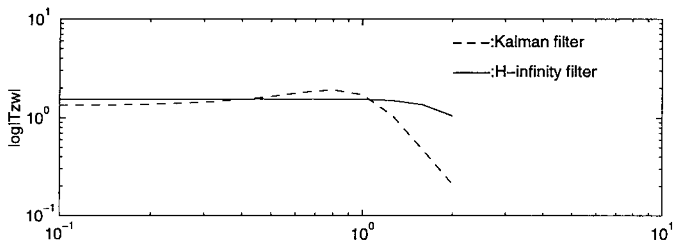

Fig. 4. Estimation error power spectra—without disturbance.

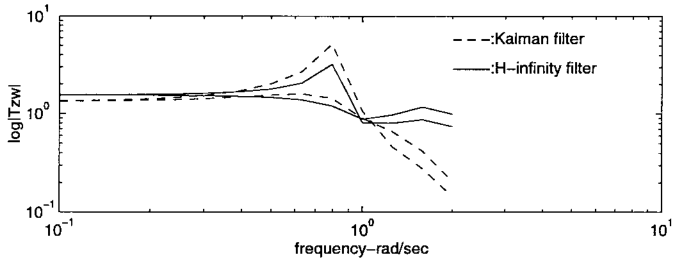

Fig. 5. Estimation error power spectra—with disturbance.

III. Numerical Example

A signal generating system (Fig. 1) is the damped harmonic oscillator with velocity measurements described by

$$\dot{x} = \begin{pmatrix} 0 & w_n \\ -w_n & -2\xi w_n \end{pmatrix} x + \begin{pmatrix} 0 \\ w_n \end{pmatrix} w$$
$$y = (0 \quad 1)x + v$$

where the state $x = (x_1 \quad x_2)^T$ with x_1 as the position and x_2 as the velocity. The natural frequency $w_n = 1.1$, and the damping coefficient $\xi = 0.15$. w is a driving signal, and v is the measurement noise. It is assumed that w and v are uncorrelated, stationary, zero-mean, white noise processes of unit intensity. Assuming a zero-order hold on the input, this system is converted to the following discrete system with sample time $T = 1$:

$$x_{k+1} = \begin{pmatrix} 0.5079 & 0.7594 \\ -0.7594 & 0.2801 \end{pmatrix} x_k + \begin{pmatrix} 0.4921 \\ 0.7594 \end{pmatrix} w_k$$
$$y_k = (0 \quad 1)x_k + v_k$$

where it is desired to estimate

$$z_k = (1 \quad 0)x_k. \tag{47}$$

Using (45), (46), and (43), the following Kalman filter gain $(\gamma = 0)$ and H_∞ filter gain $(\gamma = 1.24)$ are obtained.

$$G_K = \begin{pmatrix} 0.4236 \\ 0.0873 \end{pmatrix}, \quad G_H = \begin{pmatrix} 0.1792 \\ 1.1321 \end{pmatrix}. \tag{48}$$

The performance of the two filters is compared by simulating their estimate of z_k and the magnitudes of the transfer function T_{zw} from noises $[w_k^T \quad v_k^T]^T$ to the estimation error e_k. The results are depicted in Figs. 2 to 5. From Figs. 2 and 3, it is observed that the H_∞ filter gives the estimate z_k relatively better than that of the corresponding Kalman filter, even though the statistics of the noises w_k and v_k are known. Figs. 4 and 5 give the magnitudes of the estimation error power spectra using both H_∞ filter and Kalman filter when the system parameters (w_n, ξ) vary from (0.9, 0.08) to (1.1, 0.3). It is shown that the error spectra of the H_∞ filter have lower peaks, which means that the error spectra of the H_∞ filter are less sensitive to exact knowledge of the parameters of the system. This can be explained as follows: H_∞ filters guarantee the smallest estimation error energy over all possible disturbances of finite energy; therefore, they are overconservative, resulting in a better robust behavior to disturbance variations. All the simulation results are obtained by using MATLAB [12].

IV. Conclusions

A difference game has been formulated and solved for the discrete H_∞ filter design. The existence of a solution to the difference Riccati equation, over the time interval, is a necessary and sufficient condition for the existence of the optimal discrete H_∞ filter. Since the design criterion is based on the worst-case disturbance, the H_∞ filter is less sensitive to uncertainty in the exogenous signals statistics and dynamical model.

References

[1] R. N. Banavar and J. L. Speyer, "A linear quadratic game theory approach to estimation and smoothing," in *Proc. IEEE ACC*, 1991, pp. 2818–2822.

[2] C. E. de Souza, U. Shaked, and M. Fu, "Robust H_∞ filtering for continuous time varying uncertain systems with deterministic signal," *IEEE Trans. Signal Processing*, vol. 43, pp. 709–719, 1995.

[3] M. J. Grimble and A. Elsayed, "Solution of the H_∞ optimal linear filtering problem for discrete-time systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1092–1104, 1990.

[4] U. Shaked and Y. Theodor, "H_∞-optimal estimation: A tutorial," in *Proc. 31st IEEE CDC*, 1992, pp. 2278–2286.

[5] K. M. Nagpal and P. P. Khargonekar, "Filtering and smoothing in an H_∞ setting," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 152–166, 1991.

[6] W. M. Haddad, D. S. Berstein, and D. Mustafa, "Mixed-norm H_2/H_∞-regulation and estimation: The discrete-time case," *Syst. Contr. Lett.*, vol. 16, pp. 235–247, 1991.

[7] B. Hassibi and T. Kailath, "H_∞ adaptive filtering," in *Proc. IEEE ICASSP'95*, Detroit, MI, 1995, pp. 949–952.

[8] X. Shen and L. Deng, "Discrete H_∞ filter design with application to speech enhancement," in *Proc. IEEE ICASSP'95*, Detroit, MI, 1995, pp. 1504–1507.

[9] I. Yaesh and U. Shaked, "A transfer function approach to the problem of discrete-time systems: H_∞-optimal linear control and filtering," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1264–1271, 1991.

[10] T. Basar, "Optimum performance levels for H_∞ filters, predictors, and smoothers," *Syst. Contr. Lett.*, vol. 16, pp. 309–317, 1991.

[11] D. R. Vaughan, "A nonrecursive algebraic solution for the discrete riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 597–599, 1970.

[12] C. Moler, J. Little, and S. Bamgert, *PC-MATLAB*. Sherborn, MD: The Math Works, 1987.