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Brief Paper

New Method for Optimal Control and Filtering of Weakly Coupled Linear Discrete Stochastic Systems*

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Key Words—Order reduction; linear–quadratic regulators; Kalman filters; decoupling problems.

Abstract—The algebraic regulator and filter Riccati equations of weakly coupled *discrete-time* stochastic linear control systems are completely and exactly decomposed into reduced-order *continuous-time* algebraic Riccati equations corresponding to the subsystems. That is, the exact solution of the global discrete algebraic Riccati equation is found in terms of the reduced-order subsystem nonsymmetric continuous-time algebraic Riccati equations. In addition, the optimal global Kalman filter is decomposed into local optimal filters both driven by the system measurements and the system optimal control inputs. As a result, the optimal linear-quadratic Gaussian control problem for weakly coupled linear discrete systems takes decomposition and parallelism between subsystem filters and controllers.

1. *Introduction*

The theory of weakly coupled control systems has been very well documented in the control literature, (Kokotovic *et al.*, 1969; Gajic *et al.*, 1990; Gajic and Shen, 1993; and references therein). Discrete-time linear control systems have been the subject of recent research (Shen and Gajic, 1990a, b). In this paper we introduce a completely new approach that is pretty much different than all other methods used so far in the theory of weak coupling. The new approach is *based on a closed-loop decomposition* technique that guarantees complete decomposition of the optimal filters and regulators and distribution of all required off-line and on-line computations.

In the regulation problem (optimal linear–quadratic control problem), we show how to decompose exactly the weakly coupled discrete algebraic Riccati equation into *two reduced-order continuous-time algebraic Riccati equations.* Note that the reduced-order continuous-time algebraic Riccati equations are nonsymmetric, but their $O(\epsilon^2)$ approximations are symmetric. The Newton method is very efficient for solving these nonsymmetric Riccati equations, since initial guesses $O(\epsilon^2)$ close to the exact solutions can be easily obtained. It is important to note that *it is easier to solve the continuous-time algebraic Riccati equation than the discrete-time algebraic Riccati equation.*

In the filtering problem, in addition to using duality between filter and regulator to solve the discrete-time filter algebraic Riccati equation in terms of the reduced-order continuous-time algebraic Riccati equations, we have obtained completely independent *reduced-order Kalman filters both driven by the system measurements and the system optimal control inputs.* In the literature on linear stochastic weakly coupled systems it is possible to find exactly decomposed reduced-order Kalman filters for continuous-time systems (Shen and Gajic, 1990c) and for discrete-time systems (Shen and Gajic, 1990b), but these filters are driven by the innovation processes, so that additional communication channels have to be formed in order to construct the innovation processes. In the last part of this paper we use the separation principle to solve the corresponding linear-quadratic Gaussian control problem.

2. *Linear–quadratic control problem*

Consider a linear time-invariant discrete system

$$x(k+1) = Ax(k) + Bu(k) + Gw(k), \tag{1}$$

with the corresponding quadratic performance criterion

$$J = \tfrac{1}{2}\sum_{k=0}^{\infty}[x(k)^{\mathrm{T}}Qx(k) + u(k)^{\mathrm{T}}Ru(k)]. \tag{2}$$

The weakly coupled structure of (1) and (2) gives the following partitions (Gajic and Shen, 1993):

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix},$$
$$Q = \begin{bmatrix} Q_1 & \epsilon Q_2 \\ \epsilon Q_2^T & Q_3 \end{bmatrix} \geq 0, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} > 0, \tag{3}$$

$$A = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix},$$
$$S = BR^{-1}B^{\mathrm{T}} = \begin{bmatrix} S_1 & \epsilon S_2 \\ \epsilon S_2^T & S_3 \end{bmatrix}, \tag{4}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are state variables corresponding to two weakly coupled subsystems, $u_i \in \mathbb{R}^{m_i}$, $i = 1, 2$, are control inputs, and ϵ is a small coupling parameter. In addition, it is assumed that $\det A = O(1) \Leftrightarrow \det A_1 = O(1)$ and $\det A_4 = O(1)$ (Gajic and Shen, 1993).

It is well known that the solution to the above optimal regulation probem is given by

$$u(k) = -R^{-1}B^{\mathrm{T}}\lambda(k+1) = -(R + B^{\mathrm{T}}P_{\mathrm{r}}B)^{-1}B^{\mathrm{T}}P_{\mathrm{r}}Ax(k)$$
$$= -F(\epsilon)x(k) = -\begin{bmatrix} F_1(\epsilon) & \epsilon F_2(\epsilon) \\ \epsilon F_3(\epsilon) & F_4(\epsilon) \end{bmatrix} x(k), \tag{5}$$

where $\lambda(k)$ is a costate variable and P_{r} is the positive-semidefinite stabilizing solution of the discrete algebraic Riccati equation given by (Dorato and Levis, 1971; Lewis, 1986):

$$P_{\mathrm{r}} = Q + A^{\mathrm{T}}P_{\mathrm{r}}(I + SP_{\mathrm{r}})^{-1}A$$
$$= Q + A^{\mathrm{T}}P_{\mathrm{r}}A - A^{\mathrm{T}}P_{\mathrm{r}}B(R + B^{\mathrm{T}}P_{\mathrm{r}}B)^{-1}B^{\mathrm{T}}P_{\mathrm{r}}A. \tag{6}$$

Such a solution exists under the standard stabilizability–detectability assumption imposed on the triple (A, B, Q).

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The Hamiltonian form of (1) and (2) can be written as the forward recursion (Lewis, 1986)

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \mathbf{H}_r \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}, \tag{7}$$

where

$$\mathbf{H}_r = \begin{bmatrix} A + BR^{-1}B^{T}A^{-T}Q & -BR^{-1}B^{T}A^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \tag{8}$$

is a symplectic matrix that has the property that its eigenvalues can be grouped into two disjoint subsets Γ_1 and Γ_2, such that for every $\lambda_c \in \Gamma_1$ there exists $\lambda_d \in \Gamma_2$ satisfying $\lambda_c\lambda_d = 1$, and we can choose either Γ_1 or Γ_2 to contain only the stable eigenvalues (Salgado *et al.*, 1988).

Partitioning the vector $\lambda(k)$ such that $\lambda(k) = [\lambda_1^T(k)\ \lambda_2^T(k)]^T$, with $\lambda_1(k) \in \mathbb{R}^{n_1}$ and $\lambda_2(k) \in \mathbb{R}^{n_2}$, we get

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \mathbf{H}_r \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}. \tag{9}$$

It has been shown in Gajic and Shen (1993, p. 181) that the symplectic matrix (7) has the form

$$\mathbf{H}_r = \begin{bmatrix} \overline{A_{1r}} & \epsilon\overline{A_{2r}} & \overline{S_{1r}} & \epsilon\overline{S_{2r}} \\ \epsilon\overline{A_{3r}} & \overline{A_{4r}} & \epsilon\overline{S_{3r}} & \overline{S_{4r}} \\ \overline{Q_{1r}} & \epsilon\overline{A_{2r}} & \overline{A_{11r}^{T}} & \epsilon\overline{A_{21r}^{T}} \\ \epsilon\overline{Q_{3r}} & \overline{Q_{4r}} & \epsilon\overline{A_{12r}^{T}} & \overline{A_{22r}^{T}} \end{bmatrix}. \tag{10}$$

Note that in the following there is no need for analytical expressions for matrices with an overbar. These matrices have to be formed by the computer in the process of calculations, which can be done easily. Interchanging subvectors in (9) yields

$$\begin{bmatrix} x_1(k+1) \\ \lambda_1(k+1) \\ x_2(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \begin{bmatrix} \overline{A_{1r}} & \overline{S_{1r}} & \epsilon\overline{A_{2r}} & \epsilon\overline{S_{2r}} \\ \overline{Q_{1r}} & \overline{A_{11r}^{T}} & \epsilon\overline{Q_{2r}} & \epsilon\overline{A_{21r}^{T}} \\ \epsilon\overline{A_{3r}} & \epsilon\overline{S_{3r}} & \overline{A_{4r}} & \overline{S_{4r}} \\ \epsilon\overline{Q_{3r}} & \epsilon\overline{A_{12r}^{T}} & \overline{Q_{4r}} & \overline{A_{22r}^{T}} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} T_{1r} & \epsilon T_{2r} \\ \epsilon T_{3r} & T_{4r} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix}, \tag{11}$$

where

$$T_{1r} = \begin{bmatrix} \overline{A_{1r}} & \overline{S_{1r}} \\ \overline{Q_{1r}} & \overline{A_{11r}^{T}} \end{bmatrix}, \quad T_{2r} = \begin{bmatrix} \overline{A_{2r}} & \overline{S_{2r}} \\ \overline{Q_{2r}} & \overline{A_{21r}^{T}} \end{bmatrix},$$
$$T_{3r} = \begin{bmatrix} \overline{A_{2r}} & \overline{S_{3r}} \\ \overline{Q_{3r}} & \overline{A_{12r}^{T}} \end{bmatrix}, \quad T_{4r} = \begin{bmatrix} \overline{A_{4r}} & \overline{S_{4r}} \\ \overline{Q_{4r}} & \overline{A_{22r}^{T}} \end{bmatrix}. \tag{12}$$

Introducing the notation

$$U(k) = \begin{bmatrix} x_1(k) \\ \lambda_1(k) \end{bmatrix}, \quad V(k) = \begin{bmatrix} x_2(k) \\ \lambda_2(k) \end{bmatrix}, \tag{13}$$

we have the weakly coupled discrete system

$$U(k+1) = T_{1r}U(k) + \epsilon T_{2r}V(k),$$
$$V(k+1) = \epsilon T_{3r}U(k) + T_{4r}V(k). \tag{14}$$

Applying the transformation (Gajic and Shen, 1989, 1993)

$$\mathbf{T}_r = \begin{bmatrix} I - \epsilon^2 L_r H_r & -\epsilon L_r \\ \epsilon H_r & I \end{bmatrix}, \quad \mathbf{T}_r^{-1} = \begin{bmatrix} I & \epsilon L_r \\ -\epsilon H_r & I - \epsilon^2 H_r L_r \end{bmatrix},$$
$$\begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} = \mathbf{T}_r \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} \tag{15}$$

to (14) produces two completely decoupled subsystems:

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \eta(k+1) = (T_{1r} - \epsilon^2 T_{2r} H_r)\eta(k), \tag{16}$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \xi(k+1) = (T_{4r} + \epsilon^2 H_r T_{2r})\xi(k), \tag{17}$$

where L_r and H_r satisfy

$$H_r T_{1r} - T_{4r} H_r + T_{3r} - \epsilon^2 H_r T_{2r} H_r = 0, \tag{18}$$

$$L_r(T_{4r} + \epsilon^2 H_r T_{2r}) - (T_{1r} - \epsilon^2 T_{2r} H_r) L_r - T_{2r} = 0. \tag{19}$$

By assuming that ϵ is sufficiently small, the unique solutions of (18) and (19) exist under the condition that the matrices T_{1r} and $-T_{4r}$ have no eigenvalues in common (Gajic and Shen, 1989). The algebraic equations (18) and (19) can be solved using the Newton method (Gajic and Shen, 1989), which converges quadratically in the neighborhood of the sought solution. The good initial guess required in the Newton recursive scheme is easily obtained, with an accuracy $O(\epsilon^2)$, by setting $\epsilon = 0$ in those equations, which requires solution of linear algebraic Lyapunov equations. Note that (18) and (19) could have been obtained in completely decoupled form if, instead of the transformation of Gajic and Shen (1989), we had used the transformation developed in Qureshi (1992) (see also Gajic and Shen, 1993, p. 74).

The rearrangement and modification of variables in (11) is done by using a permutation matrix E of the form

$$\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = E \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}. \tag{20}$$

From (13), (15)–(17) and (20), we obtain the relationship between the original coordinates and the new ones:

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix} = E^T\mathbf{T}_r E \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \Pi_r \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \begin{bmatrix} \Pi_{1r} & \Pi_{2r} \\ \Pi_{3r} & \Pi_{4r} \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}. \tag{21}$$

Since $\lambda(k) = P_r x(k)$, where P_r satisfies the discrete algebraic Riccati equation (5), it follows from (21) that

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} = (\Pi_{1r} + \Pi_{2r}P_r)x(k),$$
$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = (\Pi_{3r} + \Pi_{4r}P_r)x(k). \tag{22}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same characteristic for the new systems (16) and (17); that is (Su and Gajic, 1992)

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix}. \tag{23}$$

Then (22) and (23) yield

$$\begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} = (\Pi_{3r} + \Pi_{4r}P_r)(\Pi_{1r} + \Pi_{2r}P_r)^{-1}. \tag{24}$$

It can be shown from (21) that $\Pi_r = I + O(\epsilon) \Rightarrow \Pi_{1r} = I + O(\epsilon)$, $\Pi_{2r} = O(\epsilon)$, which implies that the matrix inversion defined in (24) exists for sufficiently small ϵ.

Following the same logic, we can find P_r by introducing

$$E^T\mathbf{T}_r^{-1}E = \Omega_r = \begin{bmatrix} \Omega_{1r} & \Omega_{2r} \\ \Omega_{3r} & \Omega_{4r} \end{bmatrix}, \tag{25}$$

and this yields

$$P_r = \left(\Omega_{3r} + \Omega_{4r}\begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix}\right)\left(\Omega_{1r} + \Omega_{2r}\begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix}\right)^{-1}. \tag{26}$$

The required matrix in (26) is invertible for small values of ϵ, since from (25) we have $\Omega_r = I + O(\epsilon) \Rightarrow \Omega_{1r} = I + O(\epsilon)$, $\Omega_{2r} = O(\epsilon)$. Partitioning (16) and (17) as

$$\begin{bmatrix}\eta_1(k+1)\\ \eta_2(k+1)\end{bmatrix} = \begin{bmatrix} a_{1r} & a_{2r}\\ a_{3r} & a_{4r}\end{bmatrix}\begin{bmatrix}\eta_1(k)\\ \eta_2(k)\end{bmatrix} = (T_{1r} - \epsilon^2 T_{2r}H_r)\begin{bmatrix}\eta_1(k)\\ \eta_2(k)\end{bmatrix}, \quad (27)$$

$$\begin{bmatrix}\xi_1(k+1)\\ \xi_2(k+1)\end{bmatrix} = \begin{bmatrix} b_{1r} & b_{2r}\\ b_{3r} & b_{4r}\end{bmatrix}\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix} = (T_{4r} + \varepsilon^2 H_r T_{2r})\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix}, \quad (28)$$

and using (23) yields two reduced-order nonsymmetric algebraic Riccati equations.

$$P_{ra}a_{1r} - a_{4r}P_{ra} - a_{3r} + P_{ra}a_{2r}P_{ra} = 0, \quad (29)$$

$$P_{rb}b_{1r} - b_{4r}P_{rb} - b_{3r} + P_{rb}b_{2r}P_{rb} = 0. \quad (30)$$

It is very interesting that the algebraic Riccati equation of weakly coupled discrete systems is completely and exactly decomposed into two reduced-order nonsymmetric *continuous* algebraic Riccati equations (29) and (30). These are much easier to solve. The existence of unqiue solutions for the *continuous* algebraic Riccati equations (29) and (30) is guaranteed for sufficiently small ϵ by the implicit function theorem and by assuming stabilizability–detectabiltiy of the subsystems (A_1, B_1, Q_1) and (A_4, B_4, Q_4) (see Su and Gajic, 1992).

It can be shown that $O(\epsilon^2)$ perturbations of (29) and (30) lead to the symmetric reduced-order discrete-time algebraic Riccati equations obtained in Shen and Gajic (1990b). The solutions of these equations can be used as very good initial guesses for the Newton method for solving the obtained nonsymmetric algebraic Riccati equations (29) and (30). Another way to find initial guesses $O(\epsilon^2)$ close to the exact solutions is simply to perturb the coefficients in (29) and (30) by $O(\epsilon^2)$, which leads to the reduced-order nonsymmetric algebraic Riccati equations

$$\begin{aligned} &P_{ra}^{(0)}\overline{A_{1r}} - \overline{A_{11r}^{T}}P_{ra}^{(0)} - \overline{Q_{1r}} + P_{ra}^{(0)}\overline{S_{1r}}P_{ra}^{(0)} = 0,\\ &P_{rb}^{(0)}\overline{A_{4r}} - \overline{A_{22r}^{T}}P_{rb}^{(0)} - \overline{Q_{4r}} + P_{rb}^{(0)}\overline{S_{4r}}P_{rb}^{(0)} = 0. \end{aligned} \quad (31)$$

The nonsymmetric algebraic Riccati equations have been studied in Medanic (1982). An efficient algorithm for solivng the general nonsymmetric algebraic Riccati equation is derived in Avramovic *et al.* (1980).

The Newton algorithm for (29) is given by

$$P_{ra}^{(i+1)}(a_{1r} + a_{2r}P_{ra}^{(i)}) - (a_{4r} - P_{ra}^{(i)}a_{2r})P_{ra}^{(i+1)} = a_{3r} + P_{ra}^{(i)}a_{2r}P_{ra}^{(i)}, \quad i = 0, 1, 2, \ldots \quad (32)$$

The Newton algorithm for (30) is similarly obtained as

$$P_{rb}^{(i+1)}(b_{1r} + b_{2r}P_{rb}^{(i)}) - (b_{4r} - P_{rb}^{(i)}b_{2r})P_{rb}^{(i+1)} = b_{3r} + P_{rb}^{(i)}b_{2r}P_{rb}^{(i)}, \quad i = 0, 1, 2, \ldots . \quad (33)$$

The proposed method is very suitable for parallel computations. The reduced-order subsystems in the new coordinates are given by

$$\eta_1(k+1) = (a_{1r} + a_{2r}P_{ra})\eta_1(k), \quad (34)$$

$$\xi_1(k+1) = (b_{1r} + b_{2r}P_{rb})\xi_1(k). \quad (35)$$

The importance of the reduced-order techniques for solving algebraic Riccati equations for systems containing small parameters is demonstrated in Skataric and Gajic (1992), where the global eigenvector method for solving the algebraic Riccati equation failed to produce the answer for a 14th-order hydropower plant. However, on decomposing this global algebraic Riccati equation into two reduced-order algebraic Riccati equations of orders six and eight, the method proposed in Skataric and Gajic (1992) has produced the desired solution. Note that the 'Schur method', according to Laub and his co-workers (Kenney *et al.*, 1989, pp. 110), 'is relatively efficient and reliable'. Thus reduced-order techniques for solving the algebraic Riccati equations are desirable.

3. *New filtering method for weakly coupled linear discrete systems*

The continuous-time filtering problem of weakly coupled linear stochastic systems has been studied by Shen and Gajic (1990c). In this section we solve the filtering problem of linear discrete-time weakly coupled systems using the problem formulation from Shen and Gajic (1990b). The new method is based on exact decomposition of the global weakly coupled discrete algebraic Riccati equation into reduced-order local agebraic Riccati equations. The optimal filter gain will be completely determined in terms of the exact reduced-order continuous-time algebraic Riccati equations, based on the duality property between the optimal filter and regulator. Furthermore, we have obtained *the exact expressions for the optimal reduced-order local filters both driven by the system measurements.* This is an important advantage over the results of Shen and Gajic (1990b, c), where the local filters are driven by the innovation process, so that additional communication channels have to be used in order to construct the innovation process.

Consider the linear discrete-time invariant weakly coupled stochastic system

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + \epsilon A_2x_2(k) + G_1w_1(k) + \epsilon G_2w_2(k),\\ x_2(k+1) &= \epsilon A_3x_1(k) + A_4x_2(k) + \epsilon G_3w_1(k) + G_4w_2(k),\\ x_1(0) &= x_{10}, \quad x_2(0) = x_{20}, \end{aligned} \quad (36)$$

with corresponding measurements

$$y(k) = \begin{bmatrix} y_1(k)\\ y_2(k)\end{bmatrix} = \begin{bmatrix} C_1 & \epsilon C_2\\ \epsilon C_3 & C_4\end{bmatrix}\begin{bmatrix} x_1(k)\\ x_2(k)\end{bmatrix} + \begin{bmatrix} v_1(k)\\ v_2(k)\end{bmatrix}, \quad (37)$$

where $x_i \in \mathbb{R}^{n_i}$ are state vectors, $w_i \in \mathbb{R}^{r_i}$ and $v_i \in \mathbb{R}^{l_i}$ are independent zero-mean stationary white Gaussian noise stochastic processes with intensities $W_i > 0$ and $V_i > 0$, respectively, and $y_i \in \mathbb{R}^{l_i}$, $i = 1, 2$, are the system measurements. In the following A_i, G_i and C_i, $i = 1, \ldots, 4$, are constant matrices.

The optimal Kalman filter, driven by the innovation process, is given by (Kwakernaak and Sivan, 1972)

$$\hat{x}(k+1) = A\hat{x}(k) + K[y(k) - C\hat{x}(k)], \quad (38)$$

where

$$A = \begin{bmatrix} A_1 & \epsilon A_2\\ \epsilon A_3 & A_4\end{bmatrix}, \quad C = \begin{bmatrix} C_1 & \epsilon C_2\\ \epsilon C_3 & C_4\end{bmatrix}, \quad K = \begin{bmatrix} K_1(\epsilon) & \epsilon K_2(\epsilon)\\ \epsilon K_3(\epsilon) & K_4(\epsilon)\end{bmatrix}. \quad (39)$$

The filter gain K is obtained from

$$K = AP_fC^T(V + CP_fC^T)^{-1}, \quad V = \begin{bmatrix} V_1 & 0\\ 0 & V_2\end{bmatrix}, \quad (40)$$

where P_f is the positive-semidefinite stabilizing solution of the discrete-time filter algebraic Riccati equation and is given by

$$P_f = AP_fA^T - AP_fC^T(V + CP_fC^T)^{-1}CP_fA^T + GWG^T, \quad (41)$$

with

$$G = \begin{bmatrix} G_1 & \epsilon G_2\\ \epsilon G_3 & G_4\end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0\\ 0 & W_2\end{bmatrix}. \quad (42)$$

Owing to the weakly coupled structure of the problem matrices, the required solution P_f has the form

$$P_f = \begin{bmatrix} P_{f1}(\epsilon) & \epsilon P_{f2}(\epsilon)\\ \epsilon P_{f2}^T(\epsilon) & P_{f3}(\epsilon)\end{bmatrix}. \quad (43)$$

Partitioning the discrete-time filter Riccati equation given by (41), in the sense of weak-coupling methodology, will produce a lot of terms and make the corresponding problem numerically inefficient, even though the problem order reduction is achieved. Using the decomposition procedure given in the previous section and the duality property between the optimal filter and regulator, we propose a new decomposition scheme such that the subsystem filters of the weakly coupled discrete systems are completely decoupled and both are driven by the system measurements.

The desired decomposition of the Kalman filter (38) will be obtained by using duality between the optimal filter and regulator, and the decomposition method developed in Section 2. Consider the optimal *closed-loop* Kalman filter (38) driven by the system measurements; that is,

$$\begin{aligned}\hat{x}_1(k+1) &= (A_1 - K_1C_1 - \epsilon^2K_2C_3)\hat{x}_1(k) \\ &\quad + \epsilon(A_2 - K_1C_2 - K_2C_4)\hat{x}_2(k) \\ &\quad + K_1y_1(k) + \epsilon K_2y_2(k), \\ \hat{x}_2(k+1) &= \epsilon(A_3 - K_3C_1 - K_4C_3)\hat{x}_1(k) \\ &\quad + (A_4 - K_4C_4 - \epsilon^2K_3C_2)\hat{x}_2(k) \\ &\quad + \epsilon K_3y_1(k) + K_4y_2(k).\end{aligned} \tag{44}$$

By using (36) and duality between the optimal filter and regulator, that is,

$$A \to A^{\mathrm{T}},\quad Q \to GWG^{\mathrm{T}},\quad B \to C^{\mathrm{T}},\quad BR^{-1}B^{\mathrm{T}} \to C^{\mathrm{T}}V^{-1}C, \tag{45}$$

the filter 'state–costate equation' can be defined as

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1)\end{bmatrix} = \mathbf{H}_f\begin{bmatrix} x(k) \\ \lambda(k)\end{bmatrix}, \tag{46}$$

where

$$\mathbf{H}_f = \begin{bmatrix} A^{\mathrm{T}} + C^{\mathrm{T}}V^{-1}CA^{-1}GWG^{\mathrm{T}} & -C^{\mathrm{T}}V^{-1}CA^{-1} \\ -A^{-1}GWG^{\mathrm{T}} & A^{-1}\end{bmatrix}. \tag{47}$$

Partitioning $\lambda(k)$ as $\lambda(k) = [\lambda_1^{\mathrm{T}}(k)\ \lambda_2^{\mathrm{T}}(k)]^{\mathrm{T}}$, with $\lambda_1(k) \in \mathbb{R}^{n_1}$ and $\lambda_2(k) \in \mathbb{R}^{n_2}$, (46) can be rewritten as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1)\end{bmatrix} = \begin{bmatrix} \overline{A_{1f}^{\mathrm{T}}} & \epsilon\overline{A_{3f}^{\mathrm{T}}} & \overline{S_{1f}} & \epsilon\overline{S_{2f}} \\ \epsilon\overline{A_{2f}^{\mathrm{T}}} & \overline{A_{4f}^{\mathrm{T}}} & \epsilon\overline{S_{3f}} & \overline{S_{4f}} \\ \overline{Q_{1f}} & \epsilon\overline{Q_{2f}} & \overline{A_{11f}} & \epsilon\overline{A_{12f}} \\ \epsilon\overline{Q_{3f}} & \overline{Q_{4f}} & \epsilon\overline{A_{21f}} & \overline{A_{22f}}\end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k)\end{bmatrix}. \tag{48}$$

Interchanging the second and third rows yields

$$\begin{bmatrix} x_1(k+1) \\ \lambda_1(k+1) \\ x_2(k+1) \\ \lambda_2(k+1)\end{bmatrix} = \begin{bmatrix} \overline{A_{1f}^{\mathrm{T}}} & \overline{S_{1f}} & \epsilon\overline{A_{3f}^{\mathrm{T}}} & \epsilon\overline{S_{2f}} \\ \overline{Q_{1f}} & \overline{A_{11f}} & \epsilon\overline{Q_{2f}} & \epsilon\overline{A_{12f}} \\ \epsilon\overline{A_{2f}^{\mathrm{T}}} & \epsilon\overline{S_{3f}} & \overline{A_{4f}^{\mathrm{T}}} & \overline{S_{4f}} \\ \epsilon\overline{Q_{3f}} & \epsilon\overline{A_{21f}} & \overline{Q_{4f}} & \overline{A_{22f}}\end{bmatrix}\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k)\end{bmatrix} = \begin{bmatrix} T_{1f} & \epsilon T_{2f} \\ \epsilon T_{3f} & T_{4f}\end{bmatrix}\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k)\end{bmatrix}, \tag{49}$$

where

$$T_{1f} = \begin{bmatrix} \overline{A_{1f}^{\mathrm{T}}} & \overline{S_{1f}} \\ \overline{Q_{1f}} & \overline{A_{11f}}\end{bmatrix},\quad T_{2f} = \begin{bmatrix} \overline{A_{3f}^{\mathrm{T}}} & \overline{S_{2f}} \\ \overline{Q_{2f}} & \overline{A_{12f}}\end{bmatrix},$$
$$T_{3f} = \begin{bmatrix} \overline{A_{2f}^{\mathrm{T}}} & \overline{S_{3f}} \\ \overline{Q_{3f}} & \overline{A_{21f}}\end{bmatrix},\quad T_{4f} = \begin{bmatrix} \overline{A_{4f}^{\mathrm{T}}} & \overline{S_{4f}} \\ \overline{Q_{4f}} & \overline{A_{22f}}\end{bmatrix}. \tag{50}$$

These matrices comprise the system matrix of a standard weakly coupled discrete system, so that the reduced-order decomposition can be achieved by applying the decoupling transformation from Section 2 to (49), which yields two completely decoupled subsystems:

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1)\end{bmatrix} = \begin{bmatrix} a_{1f} & a_{2f} \\ a_{3f} & a_{4f}\end{bmatrix}\begin{bmatrix} \eta_1(k) \\ \eta_2(k)\end{bmatrix} = (T_{1f} - \epsilon^2T_{2f}H_f)\begin{bmatrix} \eta_1(k) \\ \eta_2(k)\end{bmatrix}, \tag{51}$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1)\end{bmatrix} = \begin{bmatrix} b_{1f} & b_{2f} \\ b_{3f} & b_{4f}\end{bmatrix}\begin{bmatrix} \xi_1(k) \\ \xi_2(k)\end{bmatrix} = (T_{4f} + \epsilon^2H_fT_{2f})\begin{bmatrix} \xi_1(k) \\ \xi_2(k)\end{bmatrix}. \tag{52}$$

Note that the decoupling transformation has the form of (15), with H_f and L_f matrices obtained from (18) and (19) with the T_{if} taken from (50). By duality, the following reduced-order nonsymmetric algebraic Riccati equations hold:

$$P_{fa}a_{1f} - a_{4f}P_{fa} - a_{3f} + P_{fa}a_{2f}P_{fa} = 0, \tag{53}$$

$$P_{fb}b_{1f} - b_{4f}P_{fb} - b_{3f} + P_{fb}b_{2f}b_{2f}P_{fb} = 0. \tag{54}$$

Using the permutation matrix

$$\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k)\end{bmatrix} = E\begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k)\end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2}\end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k)\end{bmatrix}, \tag{55}$$

we can define

$$\Pi_f = \begin{bmatrix} \Pi_{1f} & \Pi_{2f} \\ \Pi_{3f} & \Pi_{4f}\end{bmatrix} = E^{\mathrm{T}}\begin{bmatrix} I - \epsilon^2L_fH_f & -\epsilon L_f \\ \epsilon H_f & I\end{bmatrix}E. \tag{56}$$

Then the desired transformation is given by

$$\mathbf{T}_f = \Pi_{1f} + \Pi_{2f}P_f. \tag{57}$$

The transformation $\mathbf{T}_f$ applied to the filter variables (44) as

$$\begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2\end{bmatrix} = \mathbf{T}_f^{-\mathrm{T}}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2\end{bmatrix} \tag{58}$$

produces

$$\begin{aligned}\begin{bmatrix} \hat{\eta}_1(k+1) \\ \hat{\eta}_2(k+1)\end{bmatrix} &= \mathbf{T}_f^{-\mathrm{T}}\begin{bmatrix} A_1 - K_1C_1 - \epsilon^2K_2C_3 & \epsilon(A_2 - K_1C_2 - K_2C_4) \\ \epsilon(A_3 - K_3C_1 - K_4C_3) & A_4 - K_4C_4 - \epsilon^2K_3C_2\end{bmatrix} \\ &\quad \times \mathbf{T}_f^{\mathrm{T}}\begin{bmatrix} \hat{\eta}_1(k) \\ \hat{\eta}_2(k)\end{bmatrix} + \mathbf{T}_f^{-\mathrm{T}}\begin{bmatrix} K_1 & \epsilon K_2 \\ \epsilon K_3 & K_4\end{bmatrix}y(k),\end{aligned} \tag{59}$$

such that the complete *closed-loop* decomposition is achieved; that is,

$$\begin{aligned}\hat{\eta}_1(k+1) &= (a_{1f} + a_{2f}P_{fa})^{\mathrm{T}}\hat{\eta}_1(k) + \mathbf{K}_1y(k), \\ \hat{\eta}_2(k+1) &= (b_{1f} + b_{2f}P_{fb})^{\mathrm{T}}\hat{\eta}_2(k) + \mathbf{K}_2y(k),\end{aligned} \tag{60}$$

where

$$\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2\end{bmatrix} = \mathbf{T}_f^{-\mathrm{T}}K. \tag{61}$$

It is important to point out that the matrix P_f in (57) can be obtained in terms of P_{fa} and P_{fb} by using an expression dual to (26); that is,

$$P_f = \left(\Omega_{3f} + \Omega_{4f}\begin{bmatrix} P_{fa} & 0 \\ 0 & P_{fb}\end{bmatrix}\right)\left(\Omega_{1f} + \Omega_{2f}\begin{bmatrix} P_{fa} & 0 \\ 0 & P_{fb}\end{bmatrix}\right)^{-1}, \tag{62}$$

with Ω_{1f}, Ω_{2f}, Ω_{3f} and Ω_{4f} obtained from

$$\Omega_f = \begin{bmatrix} \Omega_{1f} & \Omega_{2f} \\ \Omega_{3f} & \Omega_{4f}\end{bmatrix} = E^{\mathrm{T}}\begin{bmatrix} I & \epsilon L_f \\ -\epsilon H_f & I - \epsilon^2H_fL_f\end{bmatrix}E. \tag{63}$$

The results obtained can be summarized in the following lemma.

Lemma 1. Given the *closed-loop* optimal Kalman filter (44) of a linear discrete weakly coupled system, there exists a nonsingular transformation matrix (57) that completely decouples (44) into reduced-order local filters (60), both driven by the system measurements. Furthermore, the decoupling transformation (57) and the filter coefficients given in (51) and (52) can be obtained in terms of the exact reduced-order completely decoupled continuous-time Riccati equations (53) and (54).

4. *Linear–quadratic Gaussian optimal control problem*

This section presents a new approach in the study of the LQG control problem of weakly coupled discrete systems when the performance index is defined on an infinite-time period. The discrete-time LQG problem of weakly coupled systems has been studied in Shen and Gajic (1990b). We shall solve the LQG problem by using the results obtained in previous sections. That is, the discrete algebraic Riccati equation is completely and exactly decomposed into two reduced-order continuous-time algebraic Riccati equations. In addition, the local filters will be driven by the system measurements, in contrast to the work of Shen and Gajic (1990b), where the local filters are driven by the innovation process.

Consider the weakly coupled discrete-time linear stochastic control system represented by (Shen and Gajic, 1990b)

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + \epsilon A_2x_2(k) + B_1u_1(k) \\ &\quad + \epsilon B_2u_2(k) + G_1w_1(k) + \epsilon G_2w_2(k), \\ x_2(k+1) &= \epsilon A_3x_1(k) + A_4x_2(k) + \epsilon B_3u_1(k) \\ &\quad + B_4u_2(k) + \epsilon G_3w_1(k) + G_4w_2(k), \\ y(k) &= \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 & \epsilon C_2 \\ \epsilon C_3 & C_4 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}, \end{aligned} \tag{64}$$

with the performance criterion

$$J = \tfrac{1}{2}E\left\{\sum_{k=0}^{\infty}[z^T(k)z(k) + u^T(k)Ru(k)]\right\}, \quad R > 0, \tag{65}$$

where $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$, are the state vectors, $u_i \in \mathbb{R}^{m_i}$, $i = 1, 2$, are the control inputs, $y_i \in R^{l_i}$, $i = 1, 2$, are the observed outputs, $w_i \in R^{r_i}$, $i = 1, 2$, and $v_i \in R^{l_i}$, $i = 1, 2$ are independent zero-mean stationary Gaussian white noise processes with intensities $W_i > 0$ and $V_i > 0$, $i = 1, 2$, respectively, and $z \in \mathbb{R}^{s_i}$, $i = 1, 2$, are the controlled outputs given by

$$z(k) = D_1x_1(k) + D_2x_2(k). \tag{66}$$

All matrices are of appropriate dimensions and are assumed to be constant. The optimal control law of the system (64) with performance criterion (65) is given by (Kwakernaak and Sivan, 1972)

$$u(k) = -F\hat{x}(k), \tag{67}$$

with the time-invariant filter

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K[y(k) - C\hat{x}(k)], \tag{68}$$

where

$$A = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & \epsilon C_2 \\ \epsilon C_3 & C_4 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & \epsilon K_2 \\ \epsilon K_3 & K_4 \end{bmatrix}. \tag{69}$$

The regulator gain F and filter gain K are obtained from

$$F = (R + B^TP_rB)^{-1}B^TP_rA, \tag{70}$$

$$K = AP_fC^T(V + CP_fC^T)^{-1}, \tag{71}$$

where P_r and P_f are positive-semidefinite stabilizing solutions of the discrete-time algebraic regulator and filter Riccati equations respectively, given by

$$P_r = D^TD + A^TP_rA - A^TP_rB(R + B^TP_rB)^{-1}B^TP_rA, \tag{72}$$

$$P_f = AP_fA^T - AP_fC^T(V + CP_fC^T)^{-1}CP_fA^T + GWG^T, \tag{73}$$

with

$$D = \begin{bmatrix} D_1 & \epsilon D_2 \\ \epsilon D_3 & D_4 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & \epsilon G_2 \\ \epsilon G_3 & G_4 \end{bmatrix}. \tag{74}$$

The required solutions P_r and P_f have the forms

$$P_r = \begin{bmatrix} P_{r1}(\epsilon) & \epsilon P_{r2}(\epsilon) \\ \epsilon P_{r2}^T(\epsilon) & P_{r3}(\epsilon) \end{bmatrix}, \quad P_f = \begin{bmatrix} P_{f1}(\epsilon) & \epsilon P_{f2}(\epsilon) \\ \epsilon P_{f2}^T(\epsilon) & P_{f3}(\epsilon) \end{bmatrix}. \tag{75}$$

In obtaining the required solutions of (72) and (73) in terms of the reduced-order problems, Shen and Gajic (1990b) have used a bilinear transformation technique introduced in Kondo and Furuta (1986) to transform the discrete-time algebraic Riccati equation into the continuous-time algebraic Riccati equation. In this case the exact decomposition method of the discrete algebraic regulator and filter Riccati equations produces two sets of two reduced-order nonsymmetric algebra Riccati equations; that is, for the regulator

$$P_{ra}a_{1r} - a_{4r}P_{ra} - a_{3r} + P_{ra}a_{2r}P_{ra} = 0, \tag{76}$$

$$P_{rb}b1_r - b_{4r}P_{rb} - b_{3r} + P_{rb}b_{2r}P_{rb} = 0, \tag{77}$$

and for the filter

$$P_{fa}a_{1f} - a_{4f}P_{fa} - a_{3f} + P_{fa}a_{2f}P_{fa} = 0, \tag{78}$$

$$P_{fb}b_{1f} - b_{4f}P_{fb} - b_{3f} + P_{fb}b_{2f}P_{fb} = 0, \tag{79}$$

where the unknown coefficients are obtained from the results in the previous sections. The Newton algorithm can be used efficiently in solving the reduced-order nonsymmetric Riccati equations (76)–(79) (Su and Gajic, 1992; Gajic and Shen, 1993).

It was shown in the previous section that the optimal global Kalman filter, based on the exact decomposition technique, is decomposed into reduced-order local optimal filters both driven by the system measurements. These local filters can be implemented independently, and are given by

$$\begin{aligned} \hat{\eta}_1(k+1) &= (a_{1f} + a_{2f}P_{fa})^T\hat{\eta}_1(k) + \mathbf{K}_1y(k) + \mathbf{B}_1u(k), \\ \hat{\eta}_2(k+1) &= (b_{1f} + b_{2f}P_{fb})^T\hat{\eta}_2(k) + \mathbf{K}_2y(k) + \mathbf{B}_2u(k), \end{aligned} \tag{80}$$

where

$$\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{T}_2^{-T}B = (\Pi_{1f} + \Pi_{2f}P_f)^{-T}B. \tag{81}$$

The optimal control in the new coordinates can be obtained as

$$u(k) = -F\hat{x}(k) = -F\mathbf{T}_f^T\begin{bmatrix} \hat{\eta}_1(k) \\ \hat{\eta}_2(k) \end{bmatrix} = -[\mathbf{F}_1 \quad \mathbf{F}_2]\begin{bmatrix} \hat{\eta}_1(k) \\ \hat{\eta}_2(k) \end{bmatrix}, \tag{82}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are obtained from

$$[\mathbf{F}_1 \quad \mathbf{F}_2] = F\mathbf{T}_f^T = (R + B^TP_rB)^{-1}B^TP_rA(\Pi_{1f} + \Pi_{2f}P_f)^T. \tag{83}$$

The optimal value of J is given by the well-known form (Kwakernaak and Sivan, 1972)

$$J_{opt} = \tfrac{1}{2}\,\mathrm{tr}\,[D^TDP_f + P_rK(CP_fC^T + V)K^T], \tag{84}$$

where F, K, P_r and P_r are obtained from (70)–(73).

5. *Conclusions*

A new approach to solving the LQG optimal control for linear weakly coupled discrete systems has been proposed. The importance of the proposed method lies in the fact that the optimal control and filtering can be completely and exactly decomposed into local level subproblems, which reduces both off-line and on-line required computations and allows parallelism of the filtering and control tasks. In addition, a very important feature of the obtained results is that the natural filter configuration of being driven by the system measurements and optimal control is preserved for the local filter design. The obtained results are also applicable to the weakly coupled linear control systems having off-diagonal blocks of zeros in the matrices R, V and W replaced by $O(\epsilon)$ quantities. In that case the procedure is exactly the same, although the derivations are a little bit more complicated.

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