sented by the pair polynomials given in Lemma 2 with

$$\tau_k = \max\{\delta_{1k}, \delta_{2k}, \delta_{3k}, \delta_{4k}\}$$
$$\sigma_k = \min\{\delta_{1k}, \delta_{2k}, \delta_{3k}, \delta_{4k}\}$$
$$\xi_k = \max\{\delta_{5k}, \delta_{6k}, \delta_{7k}, \delta_{8k}\}$$
$$\Upsilon_k = \min\{\delta_{5k}, \delta_{6k}, \delta_{7k}, \delta_{8k}\}$$

where

$$\delta_{1k} = \lambda_k \cos k\phi - \chi_k \sin k\phi, \quad \delta_{2k} = \lambda_k \cos k\phi - \rho_k \sin k\phi$$
$$\delta_{3k} = \mu_k \cos k\phi - \rho_k \sin k\phi, \quad \delta_{4k} = \mu_k \cos k\phi - \chi_k \sin k\phi$$
$$\delta_{5k} = \chi_k \cos k\phi + \lambda_k \sin k\phi, \quad \delta_{6k} = \rho_k \cos k\phi + \lambda_k \sin k\phi$$
$$\delta_{7k} = \rho_k \cos k\phi + \mu_k \sin k\phi, \quad \delta_{8k} = \chi_k \cos k\phi + \mu_k \sin k\phi$$

and another four polynomials represented by the pair polynomials given in Lemma 1 with

$$\tau_k = \max\{\zeta_{1k}, \zeta_{2k}, \zeta_{3k}, \zeta_{4k}\}$$
$$\sigma_k = \min\{\zeta_{1k}, \zeta_{2k}, \zeta_{3k}, \zeta_{4k}\}$$
$$\xi_k = \max\{\zeta_{5k}, \zeta_{6k}, \zeta_{7k}, \zeta_{8k}\}$$
$$\Upsilon_k = \min\{\zeta_{5k}, \zeta_{6k}, \zeta_{7k}, \zeta_{8k}\}$$

where

$$\zeta_{1k} = \lambda_k \cos k\phi + \chi_k \sin k\phi, \quad \zeta_{2k} = \lambda_k \cos k\phi + \rho_k \sin k\phi$$
$$\zeta_{3k} = \mu_k \cos k\phi + \rho_k \sin k\phi, \quad \zeta_{4k} = \mu_k \cos k\phi + \chi_k \sin k\phi$$
$$\zeta_{5k} = \chi_k \cos k\phi - \lambda_k \sin k\phi, \quad \zeta_{6k} = \rho_k \cos k\phi - \lambda_k \sin k\phi$$
$$\zeta_{7k} = \rho_k \cos k\phi - \mu_k \sin k\phi, \quad \zeta_{8k} = \chi_k \cos k\phi - \mu_k \sin k\phi$$

are Hurwitz polynomials.

Proof: The proof is similar to the proof of Theorem 1 using Lemma 1, Lemma 2, and Lemma 4.

This completes the proof.

IV. Conclusion

We have shown that the results given by Soh and Berger [1] for a family of interval polynomials to have only roots in the sector defining the damping ratio of linear continuous time systems can be simplified. The number of polynomials required to be Hurwitz is half the number of polynomials given by Soh and Berger [1].

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The Recursive Reduced-Order Solution of an Open-Loop Control Problem of Linear Singularly Perturbed Systems

W. Su, Z. Gajic, and X. Shen

Abstract—**A reduced-order method with an arbitrary degree of accuracy is obtained for solving the linear-quadratic optimal open-loop control problem. The original two-point boundary value problem is transformed in the pure-slow and pure-fast reduced-order completely decoupled initial value problems. By doing this, the stiffness of the singularly perturbed two-point boundary value problem is converted in the problem of an ill-defined linear system of algebraic equations.**

I. Introduction

A linear singularly perturbed control system is given by [1], [2]

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + B_1 u \qquad x_1(t_0) = x_{10}$$
$$\epsilon \dot{x}_2 = A_3 x_1 + A_4 x_2 + B_2 u \qquad x_2(t_0) = x_{20} \tag{1}$$

where $x_i \in R^{n_i}$, $i = 1, 2$, $u \in R^m$ are state and control variables, respectively, and ϵ is a small positive parameter. As the parameter ϵ tends to zero, the solution behaves nonuniformly, producing a so-called stiff problem [1], [2].

Since the recursive reduced-order numerical solution for finite-time closed-loop control has been solved in [3], this note will concentrate on the finite-time open-loop control.

With (1), consider the performance criterion

$$J = \frac{1}{2}\int_{t_0}^{T}\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^T R u\right\} dt + \frac{1}{2}\begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix}^T F \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} \tag{2}$$

with positive definite R and positive semidefinite Q and F.

The open-loop optimal control problem has the solution given by

$$u(t) = -R^{-1}B^T p(t) \tag{3}$$

where $p(t) \in R^{n_1+n_2}$ is a costate variable satisfying, [5]

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} \tag{4}$$

with boundary conditions expressed in the standard form as

$$M\begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} + N\begin{bmatrix} x(T) \\ p(T) \end{bmatrix} = c \tag{5}$$

where

$$M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ -F & I \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix} \tag{6}$$

for the free endpoint problem, or

$$M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ x(T) \end{bmatrix} \tag{7}$$

for the fixed endpoint problem.

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Matrices A, Q, S, and F have the forms

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}$$

$$S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{Z}{\epsilon} \\ \frac{Z^T}{\epsilon} & \frac{S_2}{\epsilon^2} \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & \epsilon F_2 \\ \epsilon F_2^T & \epsilon F_3 \end{bmatrix}. \tag{8}$$

The approximate optimal solution of the open-loop control for linear singularly perturbed systems has been studied in [7], where the problem order was reduced and the stiff problem was avoided successfully by using the classic approach based on the power series expansions. The theory developed in [7] was based on the dichotomy transformation [8] which requires the positive definite and negative definite solutions of the corresponding algebraic Riccati equation. It was concluded in [7] that the developed method is efficient for an $O(\epsilon)$ accuracy only. In this note the solution to the optimal open-loop control problem of singularly perturbed systems with an arbitrary order of accuracy is presented.

The optimal open-loop control problem is a two-point boundary value problem with the associated state–costate equations forming the Hamiltonian matrix. For singularly perturbed systems, after modifying some costate variables, the Hamiltonian matrix retains the singularly perturbed form by interchanging some state and costate variables so that it can be block diagonalized via the nonsingular transformation introduced in [4].

The idea of the note is to exploit the reduced subsystems to find the optimal open-loop control in the new coordinates. The proposed method is very suitable for parallel computations since it allows complete parallelism in both slow and fast time scales.

II. The Recursive Reduced-Order Solution of an Open-Loop Optimal Control Problem

Partitioning vector p as $p = [p_1^T \quad \epsilon p_2^T]^T$ with $p_1 \in R^{n_1}$ and $p_2 \in R^{n_2}$, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{T_3}{\epsilon} & \frac{T_4}{\epsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{9}$$

where

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix}. \tag{10}$$

Note that (9) retains the singular perturbation form as (1).

Introduce a notation

$$\begin{bmatrix} x_1 \\ p_1 \end{bmatrix} = w, \quad \begin{bmatrix} x_2 \\ p_2 \end{bmatrix} = \lambda \tag{11}$$

and apply the following transformation [4] defined by

$$K^{-1} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix}, \quad K = \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon LH \end{bmatrix} \tag{12}$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{13}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0. \tag{14}$$

The transformation (12) applied to (9) produces two completely decoupled subsystems

$$\dot{\eta} = (T_1 - T_2 L)\eta \tag{15}$$

and

$$\epsilon\dot{\xi} = (T_4 + \epsilon L T_2)\xi \tag{16}$$

where

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = K^{-1}\begin{bmatrix} w \\ \lambda \end{bmatrix}. \tag{17}$$

The algebraic equations (13) and (14) can be solved by using any of the recursive algorithms presented in [3].

The boundary conditions are changed due to an interchange of p_1 and x_2, which modifies matrices in (6) as follows

$$M_1\begin{bmatrix} w(t_0) \\ \lambda(t_0) \end{bmatrix} + N_1\begin{bmatrix} w(T) \\ \lambda(T) \end{bmatrix} = c_1 \tag{18}$$

where

$$M_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -F_1 & I_{n_1} & -\epsilon F_2 & 0 \\ 0 & 0 & 0 & 0 \\ -F_2^T & 0 & -F_3 & I_{n_2} \end{bmatrix},$$

$$c_1 = \begin{bmatrix} x_{10} \\ 0 \\ x_{20} \\ 0 \end{bmatrix}. \tag{19}$$

The nonsingular transformation (12) applied to (18) produces

$$M_2\begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} + N_2\begin{bmatrix} \eta(T) \\ \xi(T) \end{bmatrix} = c_1 \tag{20}$$

where

$$M_2 = M_1 K, \quad N_2 = N_1 K. \tag{21}$$

Since solutions of (15) and (16) are given by

$$\eta(t) = e^{(T_1 - T_2 L)(t - t_0)}\eta(t_0) \tag{22}$$

$$\xi(t) = e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t - t_0)}\xi(t_0). \tag{23}$$

We can eliminate $\eta(T)$ and $\xi(T)$ from (20) such that

$$\left\{M_2 + N_2\begin{bmatrix} e^{(T_1 - T_2 L)(T - t_0)} & 0 \\ 0 & e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(T - t_0)} \end{bmatrix}\right\} \cdot \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1. \tag{24}$$

Equation (24) can be represented in the form

$$\alpha(\epsilon)\begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1. \tag{25}$$

It is shown in Appendix that $\alpha(\epsilon)$ is invertible, hence $\eta(t_0)$ and $\xi(t_0)$ can be obtained.

Now we are able to find $\eta(t)$ and $\xi(t)$ from (15) and (16). Using (12), we can find $w(t)$ and $\lambda(t)$. Partitioning $w(t)$ and $\lambda(t)$

according to (11), we get values for $p_1(t)$, $p_2(t)$. The costate variables $p(t)$ and the optimal control law are therefore found.

The only difficulty we have encountered in the procedure is to compute $\alpha(\epsilon)$ in (25) where an ill-defined problem occurs when ϵ is extremely small or $(T-t_0)$ is very large because the matrix T_4 contains both stable and unstable modes. In that case we refer to [7].

III. Numerical Example

In order to illustrate the proposed method, we shall consider a real world problem—a magnetic control system [9]. Problem matrices are given by

$$A = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.345 & 0 \\ 0 & \dfrac{-0.524}{\epsilon} & \dfrac{-0.465}{\epsilon} & \dfrac{0.262}{\epsilon} \\ 0 & 0 & 0 & \dfrac{-1}{\epsilon} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\epsilon} \end{bmatrix}$$

$$Q = \operatorname{diag}\{1\ 0\ 1\ 0\}, \quad R = 1, \quad \epsilon = 0.1$$

with the initial condition

$$x^T(t_0) = [\,-1.3702 \quad 0.10686 \quad -0.53307 \quad 0.83467\,]$$

and time interval specified by $t_0 = 0$ and $T = 1$. Obtained results are presented in Table I.

The approximate control is defined as

$$u^{(k)}(t) = -R^{-1}B^T p^{(k)}(t)$$

where k stands for the number of iterations used to solve recursively equation (13). Values for $p^{(k)}(t)$ are obtained by following steps (14)–(25), with $p^{(k)}(t)$ obtained directly from (17) and (11). Note that steps (13)–(25) can be performed by using the method of asymptotic expansions, but since it is not recursive in its nature, it can be efficient for an $O(\epsilon)$ accuracy only, as was pointed out in [7].

Appendix

Transition matrices of (22) and (23) can be denoted $\Phi(t-t_0)$ and $\Psi(t-t_0)$, respectively, and partitioned as

$$\Phi(t-t_0) = \begin{bmatrix} \Phi_{11}(t-t_0) & \Phi_{12}(t-t_0) \\ \Phi_{21}(t-t_0) & \Phi_{22}(t-t_0) \end{bmatrix} \tag{A1}$$

$$\Psi(t-t_0) = \begin{bmatrix} \Psi_{11}(t-t_0) & \Psi_{12}(t-t_0) \\ \Psi_{21}(t-t_0) & \Psi_{22}(t-t_0) \end{bmatrix}. \tag{A2}$$

From (24) we have

$$\alpha(\epsilon) = \left(M_2 + N_2 \begin{bmatrix} \Phi(T-t_0) & 0 \\ 0 & \Psi(T-t_0) \end{bmatrix} \right). \tag{A3}$$

Using expressions for M_2 and N_2 given by (18) and (20) we get

$$\alpha(\epsilon) = \begin{bmatrix} I & 0 & 0 & 0 \\ * & \Phi_{22} - F_1\Phi_{12} & 0 & 0 \\ * & * & I & 0 \\ * & * & * & \Psi_{22} - F_3\Psi_{12} \end{bmatrix} + O(\epsilon) \tag{A4}$$

where asterisks denote terms which are not important for the nonsingularity of $\alpha(\epsilon)$.

Since matrices $\Phi_{22}-F_1\Phi_{12}$ and $\Psi_{22}-F_3\Psi_{12}$ are invertible [6], the matrix $\alpha(\epsilon)$ is invertible for sufficiently small values of ϵ.

TABLE I

Values of an Approximate Control at Certain Time Instants

approximate control	$t = 0.25$	$t = 0.5$	$t = 1$
optimal	3.1719 *E*-1	3.0299 *E*-1	−8.2827 *E*-2
$k = 3$	3.1719 *E*-1	3.0299 *E*-1	−8.2827 *E*-2
$k = 2$	3.1720 *E*-1	3.0299 *E*-1	−8.2825 *E*-2
$k = 1$	3.1712 *E*-1	3.0287 *E*-1	−8.2758 *E*-2
$k = 0$	3.3244 *E*-1	3.1350 *E*-1	−7.6749 *E*-2

However, in the case of singularly perturbed systems, due to the nature of the fast subsystem transition matrix (23), which contains unstable modes, we can observe that $\alpha(0)$ is singular. Thus, $\alpha(\epsilon)$ is invertible for $0 < \epsilon < \epsilon_1$ and ϵ_1 sufficiently small. In other words, the stiffness of the singularly perturbed system of differential equations is carried over to the stiffness of the linear system of algebraic equations. However, the latter problem is much easier to handle.

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Balance Realization of Stable Transfer Function Matrices

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Abstract—**Simple formulas are presented to compute the internally balanced minimal realization and the singular decomposition of the Hankel operator of a given continuous-time $p \times m$ stable transfer function matrix $E(s)/d(s)$. The proposed formulas involve the Schwarz numbers of $d(s)$ and the singular eigenvalues-eigenmatrices of a suitable finite matrix. Similar results are also obtained for a given discrete-time transfer function matrix.**

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