Compatibility of matrices for correlation-based measures of concordance

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A motivating example

- Given a 3×3 matrix

$$P = \begin{pmatrix} 1 & -0.95 & 0.5 \\ -0.95 & 1 & -0.4 \\ 0.5 & -0.4 & 1 \end{pmatrix},$$

  how to check whether P is a correlation matrix?

- For a correlation matrix P, one can always find a r.v. $\boldsymbol{X}$ (for e.g., $\mathrm{N}(\mathbf{0}, P)$) s.t. $\rho(\boldsymbol{X}) = P$.

- What about matrices of pairwise Spearman's rho, Kendall's tau... or other pairwise measures of concordance (MOC)?

Definitions

Definition 1.1 (κ-compatibility)

For a given $d \times d$ matrix R and a bivariate MOC

$$\kappa : (X, Y) \mapsto [-1, 1],$$

R is called κ-compatible if there exists a continuous d-random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ such that

$$\kappa_d(\boldsymbol{X}) := (\kappa(X_i, X_j))_{i,j=1,\dots,d} = R.$$

Definition 1.2 (κ-compatible set)

A set of all κ-compatible matrices is called a κ-compatible set.

Our main questions

1. Does there exist a class of MOCs whose compatibility is easy to study?

   ⇒ We introduce a correlation-based transformed rank measures of concordance.

2. Can we characterize κ-compatible sets for some paticular κ, such as Spearman's rho and Kendall's tau?

   ⇒ Positive answers for **Spearman's rho, Blomqvist's beta and van der Waerden's coefficient**.

   ⇒ For **Kendall's tau and Gini's gamma**, their characterizations are left open problems.

For ρ: Pearson's linear correlation and two functions g_1, g_2, consider the bivariate measure

$$\kappa_{g_1,g_2}(X_1, X_2) = \rho(g_1(X_1), g_2(X_2)).$$

Definition 2.1 (Seven axioms for MOC; Scarsini, 1984)

1. **Domain**: $\kappa(X, Y)$ is defined for any continuous random variables X, Y.
2. **Symmetry**: $\kappa(X, Y) = \kappa(Y, X)$.
3. **Coherence**: if $C_{X,Y} \preceq C_{X',Y'}$, then $\kappa(X, Y) \leq \kappa(X', Y')$.
4. **Range**: $-1 \leq \kappa(X, Y) \leq 1$.
5. **Independence**: if X and Y are independent, then $\kappa(X, Y) = 0$.
6. **Change of sign**: $\kappa(-X, Y) = -\kappa(X, Y)$.
7. **Continuity**: $\lim_{n\to\infty} \kappa(X_n, Y_n) = \kappa(X, Y)$ if $\lim_{n\to\infty} H_n = H$ pointwise for $(X_n, Y_n) \sim H_n$ and $(X, Y) \sim H$.

What are admissible g_1, g_2?

- The seven axioms imply that (c.f. Scarsini, 1984)

$$\kappa(X_1, X_2) = \kappa(f_1(X_1), f_2(X_2))$$

for any f_1, f_2: strictly increasing (or decreasing) functions.

$\Rightarrow$ $\kappa(X_1, X_2)$ is forced to be independent of the marginal distributions of X_1, X_2 but be dependent only on the copula of (X_1, X_2), which is the joint distribution of

$$(U_1, U_2) := (F_1(X_1), F_2(X_2)) \sim C_{X_1,X_2}.$$

- Therefore, we consider the following form of κ_{g_1,g_2};

$$\begin{aligned}\kappa_{g_1,g_2}(X_1, X_2) &= \rho\big(g_1(F_1(X_1)), g_2(F_2(X_2))\big)\\ &= \rho\big(g_1(U_1), g_2(U_2)\big) =: \kappa_{g_1,g_2}(C_{X_1,X_2}).\end{aligned}$$

- For κ_{g_1,g_2} to satisfy the coherence axiom, we want

$$C_{X,Y} \preceq C_{X',Y'} \Rightarrow C_{g_1(X),g_2(Y)} \preceq C_{g_1(X'),g_2(Y')}$$

since its (RHS) implies $\kappa_{g_1,g_2}(X,Y) \leq \kappa_{g_1,g_2}(X',Y')$ by coherence of ρ.

Theorem 2.1 (Monotonicity of g_1 and g_2)

Let g_1, g_2 be two continuous functions. If κ_{g_1,g_2} satisfies the seven axioms, then

$$(g_1(x) - g_1(y))(g_2(x) - g_2(y)) \geq 0 \text{ for any } x > y \in [0,1].$$

- Without the loss of generality, we can assume g_1, g_2 are both increasing functions by invariance of ρ under linear transform.

- Under the assumption of left-continuity of g_1, g_2, they are quantiles of some distribution functions. Consequently, we consider the following class:

Definition 2.2 ((G_1, G_2)–transformed rank correlations)

For two distribution functions G_1 and G_2,
(G_1, G_2)-transformed rank correlation coefficient is defined by

$$\kappa_{G_1,G_2}(X_1, X_2) = \rho\left(G_1^{-1}(F_1(X_1)), G_2^{-1}(F_2(X_2))\right),$$

where G_j^{-1} is a generalized inverses of G_j for $j = 1, 2$. We call the pair (G_1, G_2) concordance inducing if κ_{G_1,G_2} is a measure of concordance (i.e., κ_{G_1,G_2} satisfies the seven Scarsini's axioms).

Examples of κ_{G_1,G_2}

1. Spearman's rho: Let $G_1 = G_2 = G$ for G being the cdf of the uniform distribution on $[0,1]$. Then κ_{G_1,G_2} is called the Spearman's rho ρ_S:

$$\rho_S(C) \propto \iint_{[0,1]^2} (C(u,v) - \Pi(u,v))\mathrm{d}u\mathrm{d}v.$$

2. Blomqvist's beta: Let $G_1 = G_2 = G$ for G being the cdf of $\mathrm{Bern}(1/2)$. Then κ_{G_1,G_2} yields the Blomqvist's beta β:

$$\beta(C) = 4C(1/2,1/2) - 1.$$

3. Van der Waerden's coefficient: Let $G_1 = G_2 = G$ for G being the cdf of $\mathrm{N}(0,1)$. Then κ_{G_1,G_2} is called the van der Waerden's ζ.

Theorem 2.2 (Characterization of concordance-inducing G)

Let G_1 and G_2 be distribution functions. The (G_1, G_2)-transformed rank correlation coefficient κ_{G_1,G_2} is a measure of concordance if and only if

1. G_1 and G_2 are of the same type with G, where
2. G is a distribution function of a (i) non-degenerated (ii) radially symmetric distribution with (iii) finite second moment.

Remark: If G_1, G_2, G are all of the same type, then

$$\kappa_{G_1,G_2}(X_1, X_2) = \kappa_{G,G}(X_1, X_2) =: \kappa_G(X_1, X_2),$$

by invariance of ρ under location-scale transform. Therefore, w.l.o.g., we can assume $G_1 = G_2 = G$.

Properties of the compatible set $\mathcal{K}_G$

- Recall the notation of the κ_G-compatible set:

$$\mathcal{K}_G = \{R \in \mathcal{M}^{d\times d} : \exists \boldsymbol{X}\text{: a continuous } d\text{-r.v. s.t. } \kappa_G(\boldsymbol{X}) = R\}.$$

Proposition 3.1 (Properties of $\mathcal{K}_G$)

1. **Convexity**: $\mathcal{K}_G$ is convex,
2. **Bounds**: For any concordance inducing G, we have

$$\mathcal{P}_d^{\mathsf{B}}(1/2) \subseteq \mathcal{K}_G \subseteq \mathcal{P}_d,$$

where $\mathcal{P}_d$ is the set of all $d \times d$ correlation matrices, and $\mathcal{P}_d^{\mathsf{B}}(1/2)$ is the symmetric Bernoulli compatible set:

$$\mathcal{P}_d^{\mathsf{B}}(1/2) = \{\rho(\boldsymbol{B}) : B_j \sim \mathrm{Bern}(1/2),\ j = 1, \ldots, d\}.$$

We summarize the results of Devroye & Letac (2015), Huber & Maric (2017), Wang et al. (2018), and Hofert & Koike (2019):

Proposition 3.2 (Characterizations of some compatible sets)

1. **Normal variance mixture**: If $\sqrt{W}Z \sim G$ with $W \geq 0$, $\mathbb{E}W = 1$ and $Z \sim \mathrm{N}(0,1)$, then
$$\mathcal{K}_G = \mathcal{P}_d.$$
2. **Spearman's rho**: For the ρ_{S}-compatible set $\mathcal{S}_d$,
$$\mathcal{S}_d \begin{cases} = \mathcal{P}_d & d \leq 9, \\ \subset \mathcal{P}_d & d \geq 12 \text{ (strictly).} \end{cases}$$
3. **Blomqvist's beta**: For the β-compatible set $\mathcal{B}_d$, we have
$$\mathcal{B}_d = \mathcal{P}_d^{\mathsf{B}}(1/2) = \mathrm{conv}\{\boldsymbol{c}\boldsymbol{c}^\top : \boldsymbol{c} \in \{\pm 1\}^d\}$$

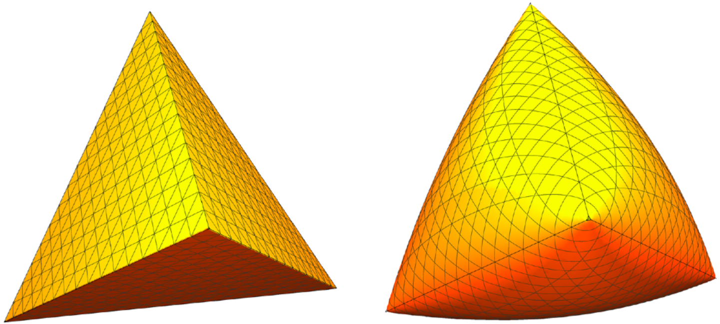

Figure: The set $\mathcal{P}_d^{\mathsf{B}}(1/2)$ (left) and $\mathcal{P}_d$ (right) when $d = 3$. $d(d-1)/2 = 3$ off-diagonal entries are projected onto the Euclidean space (The figure is retrieved from Tropp, 2018).

Other topics and future work

- **Attainability**: $R \in \mathcal{K}_G$ is κ_G-attainable if one can simulate a r.v. $\boldsymbol{X}$ s.t. $k_G(\boldsymbol{X}) = R$.
  - $\Rightarrow$ **Thm**: If $R \in \mathcal{P}_d^{\mathsf{B}}(1/2)$, then it is κ_G-attainable for any G (Hofert & Koike, 2019).
- **Dimension reduction for block matrices**: When a given matrix has a block/hierarchical structure, then the compatibility and attainability problems can be reduced to lower dimensional.

The main open problem:

- **Kendall/Gini-compatibility**: Their compatible sets might lose convexity. We only know $\mathcal{T}_d \subseteq \mathcal{P}_d^B(1/2)$ and they are equal when $d = 3$.

Compatibility of wider classes of MOCs

Edwards et al. (2005) proposed the class of MOC

$$\kappa_\mu(C) \propto \iint C \mathrm{d}\mu$$

where μ is a D_4-invariant measure on $[0,1]^2$, i.e.,

$$\mu(A) = \mu(\sigma(A)),\ A \in \mathfrak{B}(0,1)^2,$$

for σ: 90° & 180° rotations.

- κ_G corresponds to $\mu = \lambda_{G,G}$: pushforward Lebesgue measure by $G \otimes G$.
- Gini's gamma is $\mu = (M + W)/2$.
- Kendall's tau is excluded since it corresponds to $\mu = C$.

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