Compatibility and attainability of matrices of correlation-based measures of concordance

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An example of compatibility and attainability

- Compatibility: Given a 3×3 matrix

$$P = \begin{pmatrix} 1 & -0.95 & 0.5 \\ -0.95 & 1 & -0.4 \\ 0.5 & -0.4 & 1 \end{pmatrix},$$

how to check whether P is a correlation matrix?

- Attainability: For a correlation matrix P, one can always find a r.v. $\boldsymbol{X}$ (for e.g., $\mathrm{N}_3(\mathbf{0}_3, P)$) s.t. $\rho(\boldsymbol{X}) = P$.

$\Rightarrow$ P is ρ-**compatible and** ρ-**attainable** by Normal distribution.

- What about matrices of pairwise Spearman's rho, Kendall's tau or other pairwise measures of concordance (MOC)?

Outline

1. **Preliminaries**

   Definitions of concepts, motivations, and main questions.

2. **Correlation-based Measures of Concordance**

   Axioms of MOC and characterization of correlation-based MOCs.

3. **Bounds of Compatible Sets**

   Upper and lower bounds of compatible set for correlation-based MOCs.

4. **Other Topics and Future Work**

   Other topics: attainability, extension to block matrices.
   Future work: Kendall's tau compatibility, non-continuous case...etc.

Compatibility problem

Definition 1.1 (κ-compatibility)

For a given $d \times d$ matrix R and an $\mathbb{R}$-valued functional κ on a space of bivariate random vectors, we call R κ-compatible if there exists a continuous d-random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ such that

$$\kappa_d(\boldsymbol{X}) := (\kappa(X_i, X_j))_{i,j=1,\ldots,d} = R.$$

Definition 1.2 (κ-compatible set)

A set of all κ-compatible matrices $\mathcal{K}_d$ is called a κ-compatible set, that is,

$$\mathcal{K}_d = \{R \in \mathcal{M}^{d\times d} : \exists \boldsymbol{X}\text{: a continuous } d\text{-r.v. s.t. } \kappa_d(\boldsymbol{X}) = R\}.$$

Motivations

As κ we consider measures of concordance (MOC), such as Spearman's rho ρ_S and Kendall's tau τ (we will see later).

Why MOC?

$\Rightarrow$ MOC can capture non-linear dependence while ρ cannot.

Why pairwise?

$\Rightarrow$ analog to correlation matrices; a simple extension from bivariate to higher dimensions; see also Embrechts et al. (2016).

Why compatibility?

$\Rightarrow$ entries of a pairwise MOC matrix are typically estimated (possibly from limited data) or exogenously determined by expert opinion in risk management practice.

Main questions

1. Can we define a class of MOCs whose compatibility is easy to study?

   $\Rightarrow$ We introduce a correlation-based transformed rank measures of concordance.

2. Can we characterize κ-compatible sets for some paticular κ?

   $\Rightarrow$ Positive answers for our proposed class, which includes **Spearman's rho, Blomqvist's beta and van der Waerden's coefficient** as special cases.

   $\Rightarrow$ For **Kendall's tau and Gini's gamma**, their characterizations are left open problems.

Scarsini's seven axioms

What functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ make κ_{g_1,g_2} a MOC?... where

$$\kappa_{g_1,g_2}(X,Y) = \rho(g_1(X), g_2(Y)), \quad \rho: \text{ correlation.}$$

Definition 2.1 (Axioms for MOC, Scarsini, 1984)

1. **Domain**: $\kappa(X,Y)$ is defined for any continuous random variables X, Y.
2. **Symmetry**: $\kappa(X,Y) = \kappa(Y,X)$.
3. **Coherence**: if $C_{X,Y} \preceq C_{X',Y'}$, then $\kappa(X,Y) \leq \kappa(X',Y')$.
4. **Range**: $-1 \leq \kappa(X,Y) \leq 1$.
5. **Independence**: if X and Y are independent, then $\kappa(X,Y) = 0$.
6. **Change of sign**: $\kappa(-X,Y) = -\kappa(X,Y)$.
7. **Continuity**: $\lim_{n\to\infty} \kappa(X_n, Y_n) = \kappa(X,Y)$ if $\lim_{n\to\infty} H_n = H$ pointwise for $(X_n, Y_n) \sim H_n$ and $(X,Y) \sim H$.

Necessary conditions for g_1 and g_2

1. **Rank-based**: $\kappa(X,Y)$ must depend only on the copula of (X,Y); for $(U,V) := (F_X(X), F_Y(Y)) \sim C_{X,Y}$, redefine
$$\kappa_{g_1,g_2}(X,Y) = \rho\big(g_1(U), g_2(V)\big) =: \kappa_{g_1,g_2}(C_{X,Y}).$$
2. **Monotonicity**: g_1 and g_2 must be both increasing or both decreasing.

Theorem 2.2 (Monotonicity of g_1 and g_2)

Let g_1 and g_2 be two continuous functions. If κ_{g_1,g_2} is a MOC, then
$$(g_1(x) - g_1(y))(g_2(x) - g_2(y)) \geq 0 \text{ for any } x > y \text{ in } [0,1]. \quad (1)$$

Proof: $0 \leq \kappa_{g_1,g_2}(\tilde{Q}_N) - \kappa_{g_1,g_2}(Q_N) \overset{N\to\infty}{\longrightarrow} (1)$ for certain checkerboard copulas s.t. $Q_N \preceq \tilde{Q}_N$, $Q_N(u,v) = \tilde{Q}_N(u,v)$ except at blocks including $(x,x), (x,y), (y,x)$ and (y,y).

Transformed rank correlations

- W.l.o.g., we can assume g_1 and g_2 are both increasing.
- Furthermore, assume they are left-continuous. Then they are quantile functions $g_1 = G_1^{-1}$ and $g_2 = G_2^{-1}$ for some cdfs G_1 and G_2.

Definition 2.3 ((G_1, G_2)–transformed rank correlations)

For two cdfs G_1 and G_2, (G_1, G_2)-transformed rank correlation coefficient is defined by

$$\kappa_{G_1,G_2}(U,V) = \rho\big(G_1^{-1}(U), G_2^{-1}(V)\big).$$

We call the pair (G_1, G_2) concordance inducing if κ_{G_1,G_2} is a MOC.

Examples of κ_{G_1,G_2}

1. Spearman's rho: Let $G_1 = G_2 = G$ for G being the cdf of the uniform distribution on $[0,1]$. Then κ_{G_1,G_2} is called the Spearman's rho ρ_S:

$$\rho_S(C) \propto \iint_{[0,1]^2} (C(u,v) - \Pi(u,v))\mathrm{d}u\mathrm{d}v.$$

2. Blomqvist's beta: Let $G_1 = G_2 = G$ for G being the cdf of $\mathrm{Bern}(1/2)$. Then κ_{G_1,G_2} yields the Blomqvist's beta β:

$$\beta(C) = 4C(1/2,1/2) - 1.$$

3. Van der Waerden's coefficient: Let $G_1 = G_2 = G$ for G being the cdf of $\mathrm{N}(0,1)$. Then κ_{G_1,G_2} is called the van der Waerden's ζ.

Example of Bernoulli G-functions

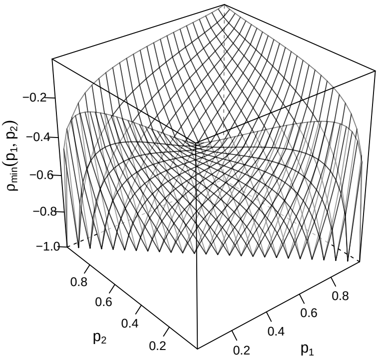

Figure: Plots of minimal (left, $(U,V) \sim W$) and maximal (right, $(U,V) \sim M$) correlation-based MOCs $\kappa_{G_1,G_2}(U,V)$ where G_j is the distribution function of $\mathrm{B}(1,p_j)$, $j=1,2$. The range axiom is violated except $(p_1,p_2)=(1/2,1/2)$.

Characterization of κ_{G_1,G_2}

Theorem 2.4 (Characterization of concordance-inducing G)

Let G_1 and G_2 be cdfs. The (G_1, G_2)-transformed rank correlation coefficient κ_{G_1,G_2} is a MOC **if and only if**

1. G_1 and G_2 are of the same type as G, where
2. G is a distribution function of a (i) non-degenerated (ii) radially symmetric distribution with (iii) finite second moment.

Proof: Key part: the correlation of $(X, Y) = (G_1^{-1}(U), G_2^{-1}(V))$ attain ± 1 at $C_{X,Y} = M,\ W$ (resp.) if and only if G_1 and G_2 are of the same type; see Embrechts et al. (2002).

Remark: If G_1, G_2 and G are all of the same type, then

$$\kappa_{G_1,G_2}(X_1, X_2) = \kappa_{G,G}(X_1, X_2) =: \kappa_G(X_1, X_2).$$

Properties of κ_G

Proposition 2.5 (Properties of κ_G)

1. **Uniqueness**: Let G and G' be two continuous concordance-inducing functions. If $\kappa_G(C) = \kappa_{G'}(C)$ for all 2-copulas, then G and G' are of the same type.
2. **Linearity**: For $n \in \mathbb{N}$, let $C_1, \ldots, C_n$ be 2-copulas and $\alpha_1, \ldots, \alpha_n \geq 0$ such that $\alpha_1 + \cdots + \alpha_n = 1$. Then

$$\kappa_G\left(\sum_{i=1}^{n} \alpha_i C_i\right) = \sum_{i=1}^{n} \alpha_i \kappa_G(C_i).$$

Limitations of κ_G

- Kendall's tau is a MOC defined by

$$\tau(C) = 4\int_{[0,1]^2} C(u,v)\,\mathrm{d}C(u,v) - 1,$$

but it is not a correlation based MOC since, in general

$$\tau(\alpha C + (1-\alpha)C') \neq \alpha\tau(C) + (1-\alpha)\tau(C'), \quad \alpha \in (0,1).$$

- κ_G measures quantify only concordance. It cannot measure the association among variables. For example,

$$\kappa_G\left(\frac{1}{2}(M+W)\right) = \frac{1}{2}(\kappa_G(M) + \kappa_G(W)) = 0$$
$$= \kappa_G(\Pi) \quad (\Pi\text{: independent copula}).$$

Bounds of the compatible set $\mathcal{K}_G$

Recall the notation of the κ_G-compatible set:

$\mathcal{K}_G = \{R : d \times d$ matrix : $\exists \boldsymbol{X}$: a continuous d-r.v. s.t. $\kappa_G(\boldsymbol{X}) = R\}$.

Proposition 3.1 (Bounds of $\mathcal{K}_G$)

For any concordance inducing G, $\mathcal{K}_G$ is convex and

$$\mathcal{P}_d^{\mathsf{B}}(1/2) \subseteq \mathcal{K}_G \subseteq \mathcal{P}_d,$$

where $\mathcal{P}_d$ is the set of all $d \times d$ correlation matrices, and

$$\mathcal{P}_d^{\mathsf{B}}(1/2) = \{\rho(\boldsymbol{B}) : B_j \sim \mathrm{Bern}(1/2),\ j = 1, \dots, d\}.$$

Proof: For $P = \rho(\boldsymbol{B}) \in \mathcal{P}_d^{\mathsf{B}}(1/2)$ and $U \sim \mathrm{U}(0,1)$ independent of $\boldsymbol{B}$, we have $\kappa_G(\boldsymbol{V}) = P$ for $\boldsymbol{V} = \boldsymbol{B}U + (\boldsymbol{1} - \boldsymbol{B})(1 - U)$: cont.

Attainability of the bounds

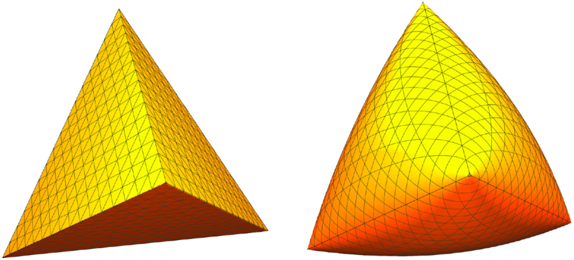

Figure: The set $\mathcal{P}_d^{\mathsf{B}}(1/2)$ (left, cut polytope) and $\mathcal{P}_d$ (right, elliptope) when $d = 3$. $d(d-1)/2 = 3$ off-diagonal entries are projected onto the Euclidean space and each vertex represents a matrix $P = \boldsymbol{c}\boldsymbol{c}^\top$ where $\boldsymbol{c} = (1,1,1), (1,-1,1), (1,1,-1)$ and $\boldsymbol{c} = (1,-1,-1)$ (Tropp, 2018).

Proposition 3.2 (Characterizations of some compatible sets)

1. **Normal variance mixture**: If $\sqrt{W}Z \sim G$ with $W \geq 0$, $\mathbb{E}W = 1$ and $Z \sim \mathrm{N}(0,1)$, then
$$\mathcal{K}_G = \mathcal{P}_d.$$
2. **Spearman's rho**: For the ρ_{S}-compatible set $\mathcal{S}_d$,
$$\mathcal{S}_d \begin{cases} = \mathcal{P}_d & d \leq 9, \\ \subset \mathcal{P}_d & d \geq 12. \end{cases}$$
3. **Blomqvist's beta**: For the β-compatible set $\mathcal{B}_d$, we have
$$\mathcal{B}_d = \mathcal{P}_d^{\mathsf{B}}(1/2) = \mathrm{conv}\{\boldsymbol{c}\boldsymbol{c}^\top : \boldsymbol{c} \in \{\pm 1\}^d\}.$$

Remark: (2) is shown in Devroye & Letac (2015) and Wang et al. (2018), and (3) is in Devroye & Letac (2015).

Other Topics

In the paper Hofert and Koike (2019) we also investigated...

- the attainability problem, that concerns whether, for a given $d \times d$ matrix R, we can construct a random vector $\boldsymbol{X}$ s.t. $\kappa_G(\boldsymbol{X}) = R$, and
- compatibility and attainability for block matrices and hierarchical matrices to solve the problem that checking compatibility and attainability is challenging for high-dimensional matrices.

Future work

- Compatibility for Kendall's τ: Is $\mathcal{T}_d = \mathcal{P}_d^{\mathsf{B}}(1/2)$?
- Compatibility for Gini's γ and generalized Blomqvist's β:

$$\gamma(C) = 4\int_{[0,1]^2} (M(u,v) + W(u,v))\,\mathrm{d}C(u,v) - 2.$$

- Comparison among MOCs, which is the best to be used?
- MOC for non-continuous margins: modified distributional transform (Rüschendorf, 2009) uniquely determines a MOC but it forms an interval due to arbitrariness of modification.
- Compatibility for measures of association, such as maximum mean discrepancy (MMD) s.t. $\mathrm{MMD}(C) = 0 \Leftrightarrow C = \Pi$.

Future work I: Kendall's tau compatibility

Conjecture: $\mathcal{T}_d = \mathcal{P}_d^B(1/2)$ for the Kendall's τ-compatible set.

- $\mathcal{T}_d \subseteq \mathcal{P}_d^B(1/2)$ is true for all $d \geq 2$.
- $\mathcal{T}_3 = \mathcal{P}_3^B(1/2)$ (Joe, 1996).
- Is $\mathcal{T}_d \supseteq \mathcal{P}_d^B(1/2)$ for all $d > 3$?
    - $\mathcal{T}_d$ may not be convex.
    - All the vertices of $\mathcal{P}_d^B(1/2)$ are attainable by τ.
    - Constructive approach?

Future work II: Gini's γ compatibility

Edwards et al. (2005) proposed the class of MOC

$$\kappa_\mu(C) \propto \iint C \mathrm{d}\mu,$$

where μ is a D_4-invariant measure on $[0,1]^2$, i.e.,

$$\mu(A) = \mu(\sigma(A)),\ A \in \mathfrak{B}(0,1)^2,$$

for σ: compositions of transpositions and partial reflections.

- κ_G corresponds to $\mu = \lambda_{G,G}$: pushforward Lebesgue measure by $G \otimes G$.
- Gini's gamma is a special case when $\mu = (M + W)/2$.

Future work III: Generalized Blomqvist's β

- Consider an easier case, for $p \in (0,1)$,

$$\mu_p = \delta_{p,p} + \delta_{p,1-p} + \delta_{1-p,p} + \delta_{1-p,1-p},$$

  which leads to the generalized Blomqvist's beta β_p.

- Its pairwise matrix admits the representation:

$$\beta_p(C) = \frac{1}{2^d} \sum_{\boldsymbol{i} \in \{0,1\}^d} \rho_{\boldsymbol{i}p+(\mathbf{1}-\boldsymbol{i})(1-p)}(C),$$

  where $\rho_{\boldsymbol{p}}(C)$ is a pairwise correlation matrix of a joint distribution with margins $\mathrm{Bern}(p_1), \dots, \mathrm{Bern}(p_d)$ and a copula C.

Thank you for your attention!

References: see Hofert and Koike (2019).

Website: https://uwaterloo.ca/scholar/tkoike/home

(The paper and this slide are also available here.)