Modality for Scenario Analysis and Maximum Likelihood Allocation

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Setup and problems

Notation

- $\boldsymbol{X} = (X_1, \ldots, X_d) \sim F_{\boldsymbol{X}}$: static loss random vector on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- $S = X_1 + \cdots + X_d$: total loss.
- $K \in \mathbb{R}$: total capital, typically $K = \varrho(S)$ for a risk measure ϱ, but not always: adjusted under regulation (Asimit et al., 2019) or even given exogenously (Laeven and Goovaerts, 2004 and Dhaene et al., 2012).

Problems

- Find an allocation $(K_1, \cdots, K_d)$ of K to d units.
- Test reliability of $(K_1, \cdots, K_d)$ = stress test of an allocation.

Existing allocation methods

Optimization

- Laeven and Goovaerts (2004) and Dhaene et al. (2012) considered

$$(K_1^*, \ldots, K_d^*) = \operatorname{argmin}\{L_{\boldsymbol{X}}(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{K}_d(K)\}$$

for some loss function $L_{\boldsymbol{X}}$ and a set of allocations

$$\mathcal{K}_d(K) = \{\boldsymbol{x} \in \mathbb{R}^d : x_1 + \cdots + x_d = K\}.$$

Euler method

- Find a confidence level $p \in (0,1)$ such that $K = \mathrm{VaR}_p(S)$ and apply the Euler princple, which leads to (what we call) the Euler allocation

$$K_j^* = \mathbb{E}[X_j \mid \{S = K\}] \quad j = 1, \ldots, d.$$

Soundness of risk allocations

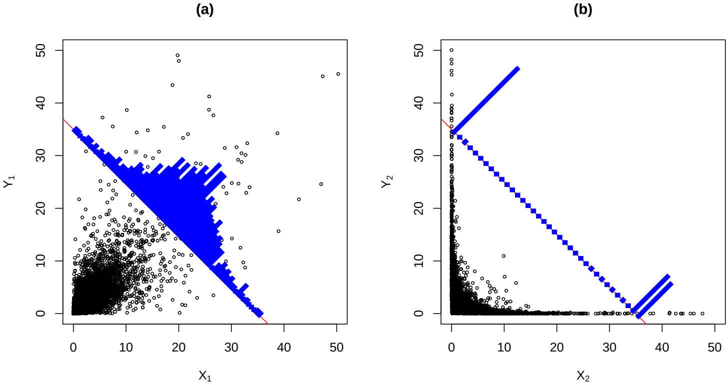

Figure: 1.1. $K = 35$, $(X_1, Y_1) \sim C^t_{\nu,\rho_1}(F, F)$ and $(X_2, Y_2) \sim C^t_{\nu,\rho_2}(F, F)$: exchangeable r.v.s, where $F = \mathrm{Par}(3, 5)$, $\nu = 5$, $\rho_1 = 0.8$ and $\rho_2 = -0.8$.

Stress test of risk allocations

- Breuer et al. (2009) requires stress scenarios to be severe and plausible.
- Consider a set of scenarios with $t > 0$ the level of plausibility:

$$L_t(\boldsymbol{X}) := \{\boldsymbol{x} \in \mathbb{R}^d : f_{\boldsymbol{X}}(\boldsymbol{x}) \geq t\},$$

  which is a level set of $\boldsymbol{X}$ (having a p.d.f. $f_{\boldsymbol{X}}$) at $t > 0$.
- Then the set of most severe scenarios K can cover is

$$\begin{aligned} L_t(\boldsymbol{X}) \cap \mathcal{K}_d(K) &= \{\boldsymbol{x} \in \mathbb{R}^d : f_{\boldsymbol{X}}(\boldsymbol{x}) \mathbf{1}_{\{\mathbf{1}_d^\top \boldsymbol{x} = K\}} \geq t\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^d : f_{\boldsymbol{X} | \{S = K\}}(\boldsymbol{x}) \geq t / f_S(K)\} \\ &= L_{t/f_S(K)}(\boldsymbol{X} \mid \{S = K\}). \end{aligned}$$

Related questions

Distributional properties of $\boldsymbol{X} \mid \{S = K\}$

- Detect uni/multi-modality of $\boldsymbol{X} \mid \{S = K\}$ from $\boldsymbol{X}$ to assess soundness of a risk allocation and simplicity of a scenario set?
- Unimodality, dependence and tail behavior of $\boldsymbol{X} \mid \{S = K\}$ are inherited from those of $\boldsymbol{X}$?

Mode of $\boldsymbol{X} \mid \{S = K\}$

- The most plausible and severe stress scenario K can cover.
- Searching for (local) modes of $\boldsymbol{X} \mid \{S = K\}$ can be beneficial to evaluate soundness of risk allocations.
- Desirable as a risk allocation?

Outline

1. **Preliminaries**

   Density and support.

2. **Properties of $\boldsymbol{X} \mid \{S = K\}$**

   Elliptical case, dependence, tail behavior and modality.

3. **Maximum likelihood allocation**

   Definition and properties.

4. **Numerical experiments**

   Simulation and empirical studies.

5. **Conclusion and future work**

   Tail dependence, measures of concordance and MCMC methods.

Density of $\boldsymbol{X} \mid \{S = K\}$

- We conventionally write
$$f_{\boldsymbol{X}|\{S=K\}}(\boldsymbol{x}) := \frac{f_{\boldsymbol{X}}(\boldsymbol{x})\mathbf{1}_{\{\mathbf{1}_d^\top \boldsymbol{x}=K\}}}{f_S(K)}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$
but $\boldsymbol{X}|\{S = K\}$ is degenerate and thus does not admit a density on $\mathbb{R}^d$.
- Instead, we work with $\boldsymbol{X}' \mid \{S = K\}$ where $d' = d - 1$ and $\boldsymbol{X}' = (X_1, \ldots, X_{d'})$ since it admits a density
$$f_{\boldsymbol{X}'|\{S=K\}}(\boldsymbol{x}') = \frac{f_{(\boldsymbol{X}',S)}(\boldsymbol{x}', K)}{f_S(K)} = \frac{f_{\boldsymbol{X}}(\boldsymbol{x}', K - \mathbf{1}_{d'}^\top \boldsymbol{x}')}{f_S(K)}, \quad \boldsymbol{x}' \in \mathbb{R}^{d'},$$
provided $\boldsymbol{X}$ and $(\boldsymbol{X}', S)$ have densities, and
$$X_d \mid \{S = K\} = K - (\mathbf{1}_{d'}^\top \boldsymbol{X}') \mid \{S = K\}.$$

Support of $\boldsymbol{X} \mid \{S = K\}$

Profit & loss: $\operatorname{supp} \boldsymbol{X} = \mathbb{R}^d$

- By $f_{\boldsymbol{X}'|\{S=K\}}(\boldsymbol{x}') = f_{\boldsymbol{X}}(\boldsymbol{x}', K - \mathbf{1}_{d'}^\top \boldsymbol{x}')/f_S(K)$, we have

$$\operatorname{supp}(\boldsymbol{X}' \mid \{S = K\}) = \mathbb{R}^{d'}.$$

Pure loss: $\operatorname{supp} \boldsymbol{X} = \mathbb{R}_+^d$

- $X_1, \dots, X_d$ cannot exceed K. Consequently, the support of $\boldsymbol{X}' \mid \{S = K\}$ forms a K-simplex:

$$\operatorname{supp}(\boldsymbol{X}' \mid \{S = K\}) = \{\boldsymbol{x}' \in \mathbb{R}_+^{d'} : \mathbf{1}_{d'}^\top \boldsymbol{x}' \leq K\}.$$

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Elliptical distributions

Definition 2.1 (Elliptical distribution)

A d-dimensional random vector $\boldsymbol{X}$ is said to have an elliptical distribution, denoted by $\boldsymbol{X} \sim \mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$, if its c.f. is

$$\phi_{\boldsymbol{X}}(\boldsymbol{t}) = \exp(i\boldsymbol{t}^\top \boldsymbol{\mu})\ \psi\left(\frac{1}{2}\boldsymbol{t}^\top \Sigma \boldsymbol{t}\right)$$

for $\boldsymbol{\mu} \in \mathbb{R}^d$, $\Sigma \in \mathcal{M}_+^{d\times d}$ and $\psi \in \Psi_d$. When $\boldsymbol{X} \sim \mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$ admits a density, it is of the form

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{c_d}{\sqrt{|\Sigma|}} g\left(\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}); d\right), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

for some normalizing constant $c_d > 0$ and a density generator $g(\cdot) = g(\cdot; d)$.

Elliptical case

Proposition 2.2 (Ellipticality of $\boldsymbol{X}' \mid \{S = K\}$)

If $\boldsymbol{X} \sim \mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$, then $\boldsymbol{X}' \mid \{S = K\} \sim \mathcal{E}_{d'}(\boldsymbol{\mu}_K, \Sigma_K, \psi_K)$ for some characteristic generator $\psi_K \in \Psi_{d'}$ and

$$\boldsymbol{\mu}_K = \boldsymbol{\mu}' + \frac{K - \mu_S}{\sigma_S^2}(\Sigma \mathbf{1}_d)' \quad \text{and} \quad \Sigma_K = \Sigma' - \frac{1}{\sigma_S^2}(\Sigma \mathbf{1}_d)'(\Sigma \mathbf{1}_d)'^{\top},$$

where Σ' is the principal submatrix of Σ deleting the dth row and column, $\mu_S = \mathbf{1}_d^{\top}\boldsymbol{\mu}$ and $\sigma_S^2 = \mathbf{1}_d^{\top}\Sigma\mathbf{1}_d$. Moreover, if $\boldsymbol{X}$ admits a density with density generator g, then so does $\boldsymbol{X}' \mid \{S = K\}$ with

$$g_K(t) = g(t + \Delta_K) \quad \text{where} \quad \Delta_K = \frac{1}{2}\left(\frac{K - \mu_S}{\sigma_S}\right)^2.$$

Example: Student t distributions

- A d-dimensional Student t distribution $t_\nu(\boldsymbol{\mu}, \Sigma)$ is an elliptical distribution $\mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$ with density generator

$$g(t; d) = \left(1 + \frac{t}{\nu}\right)^{-\frac{d+\nu}{2}}, \quad t \geq 0,$$

where $\nu \geq 1$ is the degrees of freedom parameter.

- By the previous proposition, we have that

$$\boldsymbol{X}' \mid \{S = K\} \sim t_{\nu+1}(\boldsymbol{\mu}_K, (\nu + \Delta_K)\Sigma_K/(\nu + 1))$$

since

$$g_K(t) \propto \left(1 + \frac{t}{\nu + \Delta_K}\right)^{-\frac{d+\nu}{2}} \propto \left(1 + \frac{\nu + 1}{\nu + \Delta_K} \frac{t}{\nu + 1}\right)^{-\frac{d'+\nu+1}{2}}.$$

Extremal positive dependent case

Proposition 2.3 ($\boldsymbol{X}' \mid \{S = K\}$ under comonotonicity)

Suppose $\boldsymbol{X}$ has continuous margins $F_1, \dots, F_d$ and is comonotone, i.e., $\boldsymbol{X} \stackrel{\text{d}}{=} (F_1^{-1}(U), \dots, F_d^{-1}(U))$ for some $U \sim \mathrm{U}(0,1)$. Then

$$\boldsymbol{X} \mid \{S = K\} = (F_1^{-1}(u^*), \dots, F_d^{-1}(u^*)) \quad \mathbb{P}\text{-a.s.},$$

where $u^* \in [0,1]$ is the unique solution to $\sum_{j=1}^d F_j^{-1}(u) = K$ as an equation of $u \in [0,1]$.

- An extremal case where positive dependence (comonotonicity) implies unimodality of $\boldsymbol{X} \mid \{S = K\}$ (taking on one point $(F_1^{-1}(u^*), \dots, F_d^{-1}(u^*))$ with probability 1).

Extremal negative dependent case: 1/2

- We construct $\boldsymbol{X}$ s.t. $\boldsymbol{X} \mid \{S = K\}$ is multimodal.
- For $X \geq 0$ having a c.d.f. F, suppose that $F_{X|\{X \leq K\}}$ admits a d-complete mix $\boldsymbol{Y} = (Y_1, \dots, Y_d)$ with center $K > 0$ (d-CM(K)), that is,

$$Y_j \sim F_{X|\{X \leq K\}}, \quad j = 1, \dots, d \quad \text{and} \quad \mathbf{1}_d^\top \boldsymbol{Y} = K \text{ a.s.}$$

- For $U \sim \mathrm{U}(0,1)$, $Z_1, \dots, Z_d \overset{\text{iid}}{\sim} F_{X|\{X > K\}}$ and $\boldsymbol{Y}$ being a d-CM(K) of $F_{X|\{X \leq K\}}$, define

$$\boldsymbol{X} = (X_1, \dots, X_d), \quad X_j = Y_j \mathbf{1}_{\{U \leq F(K)\}} + Z_j \mathbf{1}_{\{U > F(K)\}}$$

where $\boldsymbol{Y}$, U and $Z_1, \dots, Z_d$ are independent of each other.

Extremal negative dependent case: 2/2

- Then $X_j \sim F$ and $\{X_1 + \cdots + X_d = K\} = \{U \leq F(K)\}$ since

$$S := X_1 + \cdots + X_d = (\mathbf{1}_d^\top \boldsymbol{Y})\mathbf{1}_{\{U \leq F(K)\}} + (\mathbf{1}_d^\top \boldsymbol{Z})\mathbf{1}_{\{U > F(K)\}},$$

  $\mathbf{1}_d^\top \boldsymbol{Y} = K$ and $\mathbf{1}_d^\top \boldsymbol{Z} > K$ a.s.
- Consequently,

$$\boldsymbol{X} \mid \{S = K\} = \boldsymbol{X} \mid \{U \leq F(K)\} = \boldsymbol{Y} \ a.s.$$

- $\boldsymbol{X} \mid \{S = K\}$ is multimodal for example when $\boldsymbol{Y}$ is an equally weighted mixture of $\mathrm{Dir}(\alpha, \alpha, \beta)$, $\mathrm{Dir}(\alpha, \beta, \alpha)$ and $\mathrm{Dir}(\beta, \alpha, \alpha)$ distributions with $\alpha = 2$ and $\beta = 10$.

Dependence in elliptical case

- When $\boldsymbol{X} \sim \mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$, we have

$$\begin{aligned}\mathrm{Cov}[X_i, X_j \mid \{S = K\}] &= \mathrm{Cov}[X_i, X_j] - \frac{1}{\sigma_S^2}(\Sigma \mathbf{1}_d)_i (\Sigma \mathbf{1}_d)_j \\ &= \mathrm{Cov}[X_i, X_j] - \frac{\mathrm{Cov}[X_i, S]\,\mathrm{Cov}[X_j, S]}{\sigma_S^2} \\ &= \sigma_i \sigma_j \left(\rho_{X_i, X_j} - \rho_{X_i, S}\ \rho_{X_j, S}\right),\end{aligned}$$

  where $\sigma_j^2 = \mathrm{Var}(X_j)$ and ρ_{X_i, X_j} is the correlation coefficient of (X_i, X_j).

- The dependence structure of $\boldsymbol{X}' \mid \{S = K\}$ is typically described in terms of the dependence among X_j and S for $j = 1, \dots, d'$.

MTP2, MRR2 and TP2-order

Definition 2.4 (MTP2, MRR2 and TP2-order)

Suppose random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ have densities $f_{\boldsymbol{X}}$ and $f_{\boldsymbol{Y}}$, resp.

1. $\boldsymbol{X}$ is multivariate totally positively ordered of order 2 (MTP2) if

$$f_{\boldsymbol{X}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{y}) \leq f_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{y})f_{\boldsymbol{X}}(\boldsymbol{x} \vee \boldsymbol{y}), \quad \text{for all } \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^d.$$

2. $\boldsymbol{X}$ is said to be multivariate reverse rule of order 2 (MRR2) if

$$f_{\boldsymbol{X}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{y}) \geq f_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{y})f_{\boldsymbol{X}}(\boldsymbol{x} \vee \boldsymbol{y}), \quad \text{for all } \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^d.$$

3. $\boldsymbol{Y}$ is said to be larger than $\boldsymbol{X}$ in $TP2$-order, denoted as $\boldsymbol{X} \leq_{tp} \boldsymbol{Y}$ if

$$f_{\boldsymbol{X}}(\boldsymbol{x})f_{\boldsymbol{Y}}(\boldsymbol{y}) \leq f_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{y})f_{\boldsymbol{Y}}(\boldsymbol{x} \vee \boldsymbol{y}), \quad \text{for all } \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^d.$$

Dependence of $\boldsymbol{X} \mid \{S = K\}$

Proposition 2.5 (MTP2, MRR2 and TP2 order of $\boldsymbol{X}' \mid \{S = K\}$)

Suppose $(\boldsymbol{X}', S)$ and $(\boldsymbol{Y}', T)$ with $S = \mathbf{1}_d^\top \boldsymbol{X}$ and $T = \mathbf{1}_d^\top \boldsymbol{Y}$ have densities $f_{(\boldsymbol{X}', S)}$ and $f_{(\boldsymbol{Y}', T)}$, respectively.

1. If $(\boldsymbol{X}', S)$ is MTP2 (MRR2) then $\boldsymbol{X}' \mid \{S = K\}$ is MTP2 (MRR2).
2. If $(\boldsymbol{X}', S) \leq_{tp} (\boldsymbol{Y}', T)$ then $\boldsymbol{X}' \mid \{S = K\} \leq_{tp} \boldsymbol{Y}' \mid \{T = K\}$.

Implications:

- When $\boldsymbol{X}' \mid \{S = K\}$ is MTP2, then $\boldsymbol{X}' \mid \{S = K\}$ is positively associated, i.e., $\mathrm{Cov}[g(X_i), h(X_j) \mid \{S = K\}] \geq 0 \ \forall g,\ h : \mathbb{R} \to \mathbb{R} : \nearrow$.
- $\boldsymbol{X}' \mid \{S = K\} \leq_{tp} \boldsymbol{Y}' \mid \{T = K\} \Rightarrow \ \boldsymbol{X}' \mid \{S = K\} \leq_{st} \boldsymbol{Y} \mid \{T = K\}$, that is, $\mathbb{E}[h(\boldsymbol{X}') \mid \{S = K\}] \leq \mathbb{E}[h(\boldsymbol{Y}') \mid \{T = K\}]$ for all bounded and increasing functions $h : \mathbb{R}^{d'} \to \mathbb{R}$.

Regular and rapid variations

Definition 2.6 (Multivariate regular and rapid variations of a density)

Let $\boldsymbol{X}$ be a d-dimensional random vector $\boldsymbol{X}$ with a density $f_{\boldsymbol{X}}$.

1. $\boldsymbol{X}$ is called multivariate regularly varying with limit function $\lambda : \mathbb{R}^{2d} \to \mathbb{R}_+$ (at ∞ and on the first orthant), denoted by MRV(λ) if

$$\lim_{t\to\infty} \frac{f_{\boldsymbol{X}}(t\boldsymbol{y})}{f_{\boldsymbol{X}}(t\boldsymbol{x})} =: \lambda(\boldsymbol{x},\boldsymbol{y}) > 0 \quad \text{for any} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d_+,$$

   provided the limit function λ exists.

2. $\boldsymbol{X}$ is called multivariate rapidly varying (at ∞ and on the first orthant), denoted by MRV(∞) if,

$$\lim_{t\to\infty} \frac{f_{\boldsymbol{X}}(st\boldsymbol{x})}{f_{\boldsymbol{X}}(t\boldsymbol{x})} = \begin{cases} 0, & s > 1, \\ \infty, & 0 < s < 1, \end{cases} \quad \text{for any} \quad s > 0,\ \boldsymbol{x} \in \mathbb{R}^d_+.$$

Tail behavior of $X' \,|\{S = K\}$: 1/2

- We focus on the case where $\operatorname{supp}\{\boldsymbol{X}\} = \mathbb{R}^d$, and thus $\operatorname{supp}\{\boldsymbol{X}' \mid \{S = K\}\} = \mathbb{R}^{d'}$.
- There are $2^{d'}$ orthants to be considered. We consider tail behavior only in the first orthant $\{\boldsymbol{x}' \in \mathbb{R}^{d'} : x_1, \ldots, x_{d'} > 0\}$.
- We introduce the auxiliary random vector

$$\tilde{\boldsymbol{X}} = (\boldsymbol{X}', K - X_d),$$

which has margins $\tilde{F}_j = F_j$, $j = 1, \ldots, d'$ and $\tilde{F}_d(x_d) = \bar{F}_d(K - x_d)$, and the copula $\tilde{C}$ is the distribution function of $(U_1, \ldots, U_{d'}, 1 - U_d)$ where $\boldsymbol{U} \sim C$ is the copula of $\boldsymbol{X}$.

Tail behavior of $\boldsymbol{X}' \mid \{S = K\}$: 2/2

Proposition 2.7 (MRV of $\boldsymbol{X}' \mid \{S = K\}$)

1. Assume that $\tilde{\boldsymbol{X}} = (\boldsymbol{X}', K - X_d)$ is MRV($\tilde{\lambda}$). Then $\boldsymbol{X}' \mid \{S = K\}$ is MRV(λ') with limit function

$$\lambda'(\boldsymbol{x}', \boldsymbol{y}') = \tilde{\lambda}((\boldsymbol{x}', \mathbf{1}_{d'}^{\top}\boldsymbol{x}'), (\boldsymbol{y}', \mathbf{1}_{d'}^{\top}\boldsymbol{y}')), \quad \boldsymbol{x}', \boldsymbol{y}' \in \mathbb{R}_{+}^{d'}.$$

2. If $\tilde{\boldsymbol{X}}$ is MRV(∞), then $\boldsymbol{X}' \mid \{S = K\}$ is MRV(∞).

Note:

- See Li (2013), Li and Wu (2013), Li and Hua (2015) and Joe and Li (2019) for how to find the limit function of $\tilde{\boldsymbol{X}}$ given its joint distribution.

Tail behavior in elliptical case

Proposition 2.8 (MRV for elliptical distribution)

Assume $\boldsymbol{X} \sim \mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$ admits a density with density generator g continuous on $\mathbb{R}_+$.

1. If g is regularly varying in the sense that

$$\lim_{t\to\infty} g(tu)/g(ts) = \lambda_g(s,u), \quad s,u>0,$$

then $\boldsymbol{X}' \mid \{S = K\}$ is MRV(λ_K) with

$$\lambda_K(\boldsymbol{x}', \boldsymbol{y}') = \lambda_g({\boldsymbol{x}'}^\top \Sigma_K^{-1} \boldsymbol{x}',\ {\boldsymbol{y}'}^\top \Sigma_K^{-1} \boldsymbol{y}'), \quad \boldsymbol{x}',\ \boldsymbol{y}' \in \mathbb{R}^{d'}.$$

2. $\boldsymbol{X}' \mid \{S = K\}$ is MRV(∞) if g is rapidly varying in the sense that

$$\lim_{t\to\infty} \frac{g(st)}{g(t)} = \begin{cases} 0, & s > 1, \\ \infty, & 0 < s < 1. \end{cases}$$

Examples: Normal and Student t distributions

- **Normal distribution** has a rapidly varying density generator $g(t) = \exp(-t)$, and thus $\boldsymbol{X}' \mid \{S = K\}$ is MRV(∞).
- **Student t distribution** with dimension d and d.o.f. $\nu \geq 1$ has the regularly varying density generator with limit function

$$\lim_{t\to\infty} \frac{g(tu)}{g(ts)} = \left(\frac{u}{s}\right)^{-\frac{\nu+d}{2}}, \quad u, s > 0.$$

Therefore, $\boldsymbol{X}' \mid \{S = K\}$ is MRV(λ_K) with

$$\lim_{t\to\infty} \frac{f_{\boldsymbol{X}'|\{S=K\}}(t\boldsymbol{y}')}{f_{\boldsymbol{X}'|\{S=K\}}(t\boldsymbol{x}')} = \left(\frac{||\Sigma_K^{-\frac{1}{2}}\boldsymbol{y}'||}{||\Sigma_K^{-\frac{1}{2}}\boldsymbol{x}'||}\right)^{-(\nu+d)} =: \lambda_K(\boldsymbol{x}', \boldsymbol{y}'),$$

where $||\cdot||$ is an Euclidean norm on $\mathbb{R}^{d'}$.

Definition of unimodality

The level set of a bounded p.d.f. f on $\mathbb{R}^d$ is:

$$L_t(f) := \{\boldsymbol{x} \in \mathbb{R}^d : f(\boldsymbol{x}) \geq t\}, \quad t \in (0,\ \max\{f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}\}].$$

Definition 2.9 (Concepts of unimodality)

1. $M(f) = L_{t^*}(f)$, $t^* = \max_{\boldsymbol{x} \in \mathbb{R}^d} f(\boldsymbol{x})$ is the mode set of f.
2. If $L_{t^*}(f) = \{\boldsymbol{m}\}$ then we call $\boldsymbol{m} \in \mathbb{R}^d$ the mode of f.
3. Furthermore, f is said to be weakly unimodal if $L_t(f)$ is connected, star unimodal about the center $\boldsymbol{x}_0 \in \mathbb{R}^d$ if $L_t(f)$ is star-shaped $(*)$ about $\boldsymbol{x}_0$ and convex unimodal if $L_t(f)$ is convex, for all $0 < t \leq t^*$.

$(*)$ A set $A \subseteq \mathbb{R}^d$ is star-shaped about $\boldsymbol{x}_0 \in A$ if, for any $\boldsymbol{y} \in A$, the line segment from $\boldsymbol{x}_0$ to $\boldsymbol{y}$ is in A.

Unimodality of $\boldsymbol{X}' \mid \{S = K\}$

Note: By definition, convex unimodality implies star unimodality and star unimodality implies weak unimodality.

Proposition 2.10 (Unimodality of $\boldsymbol{X}' \mid \{S = K\}$)

1. Suppose $\boldsymbol{X} \sim \mathcal{E}_d(\boldsymbol{\mu}, \Sigma, \psi)$ admits a density with density generator g. If g is decreasing on $\mathbb{R}_+$, then $f_{\boldsymbol{X}'|\{S=K\}}$ is convex unimodal. Furthermore, if the equation $g(t) = \Delta_K$ of $t \in \mathbb{R}_+$ has a unique solution t^*_K, then $f_{\boldsymbol{X}'|\{S=K\}}$ has the mode $\boldsymbol{m} = \boldsymbol{\mu}_K$.
2. If $\boldsymbol{X}$ is convex unimodal, then $\boldsymbol{X}' \mid \{S = K\}$ is convex unimodal.

Remark: Unlike convex unimodality, neither weak unimodality nor star unimodality of $\boldsymbol{X}$ imply any unimodality of $\boldsymbol{X}' \mid \{S = K\}$.

Unimodality not inherited from $\boldsymbol{X}$

- A homothetic distribution is defined through its level set by

$$L_t(f_D) = r(t)D := \{s\boldsymbol{x} : 0 \leq s \leq r(t),\ \boldsymbol{x} \in D\},$$

  for some $r : \mathbb{R}_+ \to \mathbb{R}_+$ and $D \in \mathbb{R}^d$.
- Consider a homothetic distribution with $r(t) = \frac{1}{2\sqrt{3}} \exp(-t/2)$ and $D = ([-2,2] \times [-1,1]) \cup ([-1,1] \times [-2,2])$.
- r is $\downarrow$ and D is star-shaped around $(0,0)$, which implies star-unimodality of $\boldsymbol{X}$.
- For $t = -2\log(\sqrt{3}/3) \approx 1.098$, we have

$$r(t) = 1/6 \quad \text{and} \quad L_t(f_D) = D/6.$$

- For this t, $L_t(\boldsymbol{X}' \mid \{S = 1/3\}) = [0, 1/6] \cup [1/3, 1/2]$, which is neither star-shaped nor even connected.

Joint v.s. marginal unimodality

- **Marginal $\not\Rightarrow$ joint**: the following bivariate density

$$f(u,v) = \frac{9}{4}\mathbf{1}_{\{(u,v)\in\bigcup_{i=1}^{3}[\frac{i-1}{3},\frac{i}{3}]^2\}} + \frac{9}{4}\mathbf{1}_{\{(u,v)\in[\frac{1}{3},\frac{2}{3}]^2\}}, \quad (u,v)\in[0,1],$$

 has the convex unimodal marginal densities

$$f_1(u) = f_2(u) = \frac{3}{4}\mathbf{1}_{\{u\in[0,1]\}} + \frac{3}{4}\mathbf{1}_{\{u\in[\frac{1}{3},\frac{2}{3}]\}}, \quad u\in[0,1].$$

 However,

$$L_{9/4}(f) = [0,1/3]^2 \cup [1/3,2/3]^2 \cup [2/3,1]^2$$

 is neither convex nor star-shaped.

- **Joint $\not\Rightarrow$ marginal**: Example A.3. of Balkema and Nolde (2010)

Figures in examples

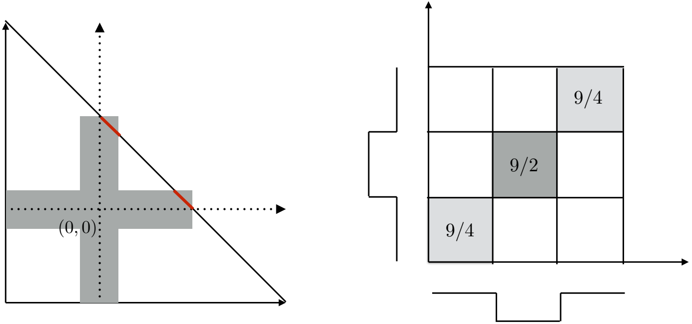

Figure: 2.11 Star unimodality of $\boldsymbol{X}' \mid \{S = K\}$ is not inherited from that of $\boldsymbol{X}$ (left), and joint unimodality does not imply marginal unimodality (right).

s-concave densities: definition

Definition 2.12 (s-concavity)

For $s \in \mathbb{R}$, a density f on $\mathbb{R}^d$ is called s-concave on a convex set $A \subseteq \mathbb{R}^d$ if

$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \geq M_s(f(\boldsymbol{x}), f(\boldsymbol{y}); \theta), \quad \boldsymbol{x}, \boldsymbol{y} \in A, \quad \theta \in (0,1),$$

where M_s is called the generalized mean defined by

$$M_s(a,b;\theta) = \begin{cases} \{\theta a^s + (1-\theta) b^s\}^{1/s}, & 0 < s < \infty \text{ or } (-\infty < s < 0 \text{ and } ab \neq 0), \\ 0, & -\infty < s < 0 \text{ and } ab = 0, \\ a^\theta b^{1-\theta}, & s = 0, \\ a \wedge b, & s = -\infty, \\ a \vee b, & s = +\infty, \end{cases}$$

for $s \in \mathbb{R}$, $a,\ b \geq 0$ and $\theta \in (0,1)$.

s-concave densities: properties and examples

- For $s = -\infty$, s-concavity is also known as quasi-concavity.
- 0-concavity is also known as log-concavity.
- The function $s \mapsto M_s(a, b; \theta)$ is increasing for fixed $(a, b; \theta)$.
- t-concavity implies s-concavity for $s < t$.
- Examples of s-concave densities: skew-normal distribution, Dirichlet with certain range of parameters and uniform distribution on a convex set in $\mathbb{R}^d$.

s-concave densities and convex unimodality

- A density f is convex unimodal iff it is $-\infty$-concave. Thus f is convex unimodal if it is s-concave for some $s \in \mathbb{R}$.
- $\boldsymbol{X}' \mid \{S = K\}$ has an s-concave density if $\boldsymbol{X}$ has.
- s-concavity is preserved under marginalization, convolution and weak-limit for certain ranges of $s \in \mathbb{R}$.
- Consequently, convex unimodality of $\boldsymbol{X}' \mid \{S = K\}$ can also be preserved under these operations if $f_{\boldsymbol{X}}$ is s-concave.

1. **Preliminaries**

   Density and support.

2. **Properties of $\boldsymbol{X} \mid \{S = K\}$**

   Elliptical case, dependence, tail behavior and modality.

3. **Maximum likelihood allocation**

   Definition and properties.

4. **Numerical experiments**

   Simulation and empirical studies.

5. **Conclusion and future work**

   Tail dependence, measures of concordance and MCMC methods.

Maximum likelihood allocation: set up

- Let $\mathcal{U}_d(K)$ be the set of all d-dim. r.v.s $\boldsymbol{X}$ such that
  (1) $\boldsymbol{X}$ and $(\boldsymbol{X}', S)$ admit p.d.f.s, and
  (2) $\boldsymbol{x} \mapsto f_{\boldsymbol{X}}(\boldsymbol{x})\mathbf{1}_{\{\boldsymbol{x}\in\mathcal{K}_d(K)\}}$ has a unique maximum.
- For $\boldsymbol{X} \in \mathcal{U}_d(K)$, $\boldsymbol{X}' \mid \{S = K\}$ admits a density $f_{\boldsymbol{X}'|\{S=K\}}$ having a unique maximum at its mode.
- We focus on the unique global maximizer of $f_{\boldsymbol{X}'|\{S=K\}}$ although $\mathcal{U}_d(K)$ contains multimodal random vectors in the sense that the level set $L_t(\boldsymbol{X}' \mid \{S = K\})$ is not connected for some $t > 0$ and the density $f_{\boldsymbol{X}'|\{S=K\}}$ has multiple local maximizers (we call them the local modes of $\boldsymbol{X}' \mid \{S = K\}$).

Maximum likelihood allocation: definition

Definition 3.1 (Maximum likelihood allocation)

For $K > 0$ and $\boldsymbol{X} \in \mathcal{U}_d(K)$, the maximum likelihood allocation (MLA) on a set $\mathcal{K} \subseteq \mathcal{K}_d(K)$ is defined by

$$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X};\mathcal{K}] = \operatorname{argmax}\{f_{\boldsymbol{X}}(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{K}\},$$

provided the function $\boldsymbol{x} \mapsto f_{\boldsymbol{X}}(\boldsymbol{x})\mathbf{1}_{\{\boldsymbol{x}\in\mathcal{K}\}}$ has a unique maximum. When $\mathcal{K} = \mathcal{K}_d(K)$, we call it the maximum likelihood allocation.

Note: MLA of K on $\mathcal{K}$ can be equivalently formulated as

$$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X};\mathcal{K}] = \operatorname{argmax}\{f_{\boldsymbol{X}'|\{S=K\}}(\boldsymbol{x}') : (\boldsymbol{x}', K - \mathbf{1}_{d'}^{\top}\boldsymbol{x}') \in \mathcal{K}\},$$

in terms of $\boldsymbol{X}' \mid \{S = K\}$.

Properties of MLA: 1/2

The following properties (1)–(4) are studied in Maume-Deschamps et al. (2016) for risk allocations derived from optimizations.

Proposition 3.2 (Properties of MLA: 1/2)

Suppose $K > 0$ and $\boldsymbol{X} \in \mathcal{U}_d(K)$.

(1) Translation invariance: For $\boldsymbol{c} \in \mathbb{R}^d$,

$$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X} + \boldsymbol{c};\ \mathcal{K}_d(K + \mathbf{1}_d^\top \boldsymbol{c})] = \boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}; \mathcal{K}_d(K)] + \boldsymbol{c}.$$

(2) Positive homogeneity: For $c > 0$,

$$\boldsymbol{K}_{\mathrm{M}}[c\boldsymbol{X}; \mathcal{K}_d(cK)] = c\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}; \mathcal{K}_d(K)].$$

Properties of MLA: 2/2

Proposition 3.3 (Properties of MLA: 2/2)

(3) Symmetry: For $(i,j) \in \{1,\dots,d\}$, $i \neq j$, let $\tilde{\boldsymbol{X}}$ be a d-dim random vector such that $\tilde{X}_j = X_i$, $\tilde{X}_i = X_j$ and $\tilde{X}_k = X_k$, $k \in \{1,\dots,d\}\backslash\{i,j\}$. If $\boldsymbol{X} \overset{\mathrm{d}}{=} \tilde{\boldsymbol{X}}$, then

$$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X};\mathcal{K}_d(K)]_i = \boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X};\mathcal{K}_d(K)]_j.$$

(4) Continuity: Suppose $\boldsymbol{X}_n,\ \boldsymbol{X} \in \mathcal{U}_d(K)$ have densities f_n and f for $n = 1,2,\dots,$ respectively. If f_n is uniformly continuous and bounded for $n = 1,2,\dots,$ and $\boldsymbol{X}_n \to \boldsymbol{X}$ weakly, then

$$\lim_{n\to\infty} \boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}_n;\mathcal{K}_d(K)] = \boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X};\mathcal{K}_d(K)].$$

Properties of MLA: degenerate case: 1/2

- Consider the case

$$X_j = \begin{cases} c_j \in \mathbb{R}, & j \in I \subseteq \{1,\dots,d\}, \\ \boldsymbol{X}_{-I} := (X_j, j \in \{1,\dots,d\}\backslash I), & \text{admitting a density } f_{\boldsymbol{X}_{-I}}. \end{cases}$$

- Since, for $\boldsymbol{c} = (c_j; j \in I) \in \mathbb{R}^{|I|}$,

$$(\boldsymbol{X}_I, \boldsymbol{X}_{-I}) \mid \{S = K\} \overset{\mathrm{d}}{=} (\boldsymbol{c},\ \boldsymbol{X}_{-I} \mid \{\mathbf{1}_{|-I|}^\top \boldsymbol{X}_{-I} = K - \mathbf{1}_{|I|}^\top \boldsymbol{c}\}),$$

  any realization $\boldsymbol{x}$ of $\boldsymbol{X} \mid \{S = K\}$ satisfies $\boldsymbol{x}_I = \boldsymbol{c}$ and its likelihood is quantified through $f_{\boldsymbol{X}_{-I} \mid \{\mathbf{1}_{|-I|}^\top \boldsymbol{X}_{-I} = K - \mathbf{1}_{|I|}^\top \boldsymbol{c}\}}(\boldsymbol{x}_{-I})$.
- Thus, we naturally extend the definition of MLA to such a random vector $\boldsymbol{X}$ by

$$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}; \mathcal{K}_d(K)]_I = \boldsymbol{c}, \quad \boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}; \mathcal{K}_d(K)]_{-I} = \boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}_{-I}; \mathcal{K}_{|-I|}(K - \mathbf{1}_{|I|}^\top \boldsymbol{c})].$$

Properties of MLA: degenerate case: 2/2

Following the extended definition of MLA, the following properties hold.

- Riskless asset:

  Sure loss $X_j = c_j$ for $c_j \in \mathbb{R}$ is covered by the amount of allocated capital c_j.

- Allocation under comonotonicity:

  Suppose $\boldsymbol{X}$ is a comonotone random vector with continuous margins $F_1, \dots, F_d$. Then

$$\boldsymbol{K}_{\mathrm{M}}(\boldsymbol{X}; \mathcal{K}_d(K)) = (F_1^{-1}(u^*), \dots, F_d^{-1}(u^*)),$$

  where $u^* \in [0, 1]$ is the unique solution to $\sum_{j=1}^{d} F_j^{-1}(u) = K$.

Suitability of MLA as an allocation

We compare MLA with Euler allocation $\mathbb{E}[\boldsymbol{X} \mid \{S = K\}]$.

- (+) Both of Euler and MLA possess properties naturally required as a risk allocation (TI, PH, RA).
- (+) Euler and MLA coincide when $\boldsymbol{X}$ is elliptically distributed.
- (+) Searching for the modes of $\boldsymbol{X}' \mid \{S = K\}$ is beneficial to evaluate the soundness of risk allocations and design more flexible allocations.
- (±) MLA is robust to severe but little plausible scenarios.
- (−) Estimating modes becomes more difficult than estimating a mean as d gets larger.

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Heuristic for simulating $\boldsymbol{X} \mid \{S = K\}$

Monte Carlo (MC) simulation

- The distribution of $\boldsymbol{X} \mid \{S = K\}$ is often intractable.
- Instead, simulate $\boldsymbol{X}$ and extract samples falling in $\{S = K\}$.
- However, $\mathbb{P}(S = K) = 0$ when S admits a density. Thus replace $\{S = K\}$ with $\{K-\delta < S < K+\delta\}$ for a small $\delta > 0$.
- The extracted samples are then standardized via $KX_j / \sum_{j=1}^{d} X_j$ so that they sum up to K.
- If data from $\boldsymbol{X}$ is available, then we regard the extracted and standardized samples as pseudo samples from $\boldsymbol{X} \mid \{S = K\}$

Empirical study: setting

- **Data**: We consider two portfolios (a) $\boldsymbol{X}_t^{\text{pos}} = (X_{t,1}, X_{t,2}, X_{t,3})$ and (b) $\boldsymbol{X}_t^{\text{neg}} = (X_{t,1}, -X_{t,2}, X_{t,3})$ for daily log-returns of FTSE $X_{t,1}$, S&P 500 $X_{t,2}$ and Dow Jones Index (DJI) $X_{t,3}$ from January 2, 1990 to March 25, 2004 ($T = 3712$ log-returns).
- **Goal**: Allocate the capital $K = 1$ based on the conditional loss distribution at time $T + 1$ given $\mathcal{F}_T$.
- **Model**: GARCH(1,1) model with empirical copula $\hat{C}$ and skew-t innovations.
- **Estimation**: Based on the pseudo samples (sample size: (a) 354 and (b) 558), estimate Euler and MLA. The function kms of the R package ks was used to estimate the modes.

Empirical study: plots

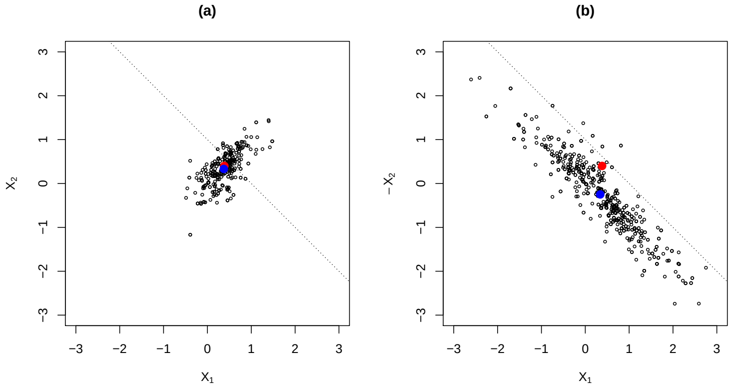


Figure: 4.1 Scatter plots (black dots) of the first two components of the pseudo samples from $\boldsymbol{X} \mid \{S = K\}$, where $\delta = 0.3$ and $K = 1$.

Empirical study: table

Table: 4.2 Bootstrap estimates and estimated standard errors of the Euler allocation and MLA. The subsample size is $N = 3712$ and the bootstrap sample size is $B = 100$.

	Estimator			Standard error		
	X_1	X_2	X_3	X_1	X_2	X_3
$\mathbb{E}[\boldsymbol{X}^{\text{pos}} \mid \{S = K\}]$	0.378	0.338	0.285	0.019	0.022	0.038
$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}^{\text{pos}}; \mathcal{K}_d(K)]$	0.367	0.365	0.268	0.019	0.024	0.041
$\mathbb{E}[\boldsymbol{X}^{\text{neg}} \mid \{S = K\}]$	0.345	−0.248	0.903	0.037	0.039	0.015
$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}^{\text{neg}}; \mathcal{K}_d(K)]$	0.371	−0.280	0.909	0.040	0.039	0.013

Simulation study: model description

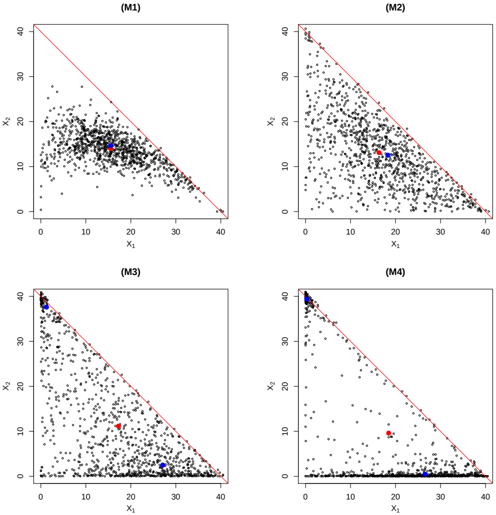
- We consider four models, referred to as (M1), (M2), (M3) and (M4), resp, with $d = 3$ and having the same margins $X_1 \sim \mathrm{Par}(2.5, 5)$, $X_2 \sim \mathrm{Par}(2.75, 5)$ and $X_3 \sim \mathrm{Par}(3, 5)$ but different t copulas with d.o.f. $\nu = 5$ and dispersion matrices

$$P_1 = \begin{pmatrix} 1 & 0.8 & 0.5 \\ 0.8 & 1 & 0.8 \\ 0.5 & 0.8 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & -0.5 & 0.5 \\ -0.5 & 1 & -0.5 \\ 0.5 & -0.5 & 1 \end{pmatrix}.$$

- $K = 40$ and $\delta = 1$.

Simulation study: plots



Simulation study: tables: 1/2

	Estimator			Standard error		
	X_1	X_2	X_3	X_1	X_2	X_3
(M1) Pareto + t copula: strong positive dependence						
$\mathbb{E}[\boldsymbol{X} \mid \{S = K\}]$	15.549	13.889	10.562	0.336	0.157	0.288
$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}; \mathcal{K}_d(K)]$	15.849	14.434	9.718	0.482	0.213	0.356
(M2) Pareto + t copula: positive dependence						
$\mathbb{E}[\boldsymbol{X} \mid \{S = K\}]$	16.228	13.042	10.562	0.399	0.355	0.288
$\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X}; \mathcal{K}_d(K)]$	17.689	12.481	9.830	0.759	0.663	0.475

Simulation study: tables: 2/2

	Estimator			Standard error		
	X_1	X_2	X_3	X_1	X_2	X_3
(M3) Pareto $+$ t copula: independence						
$\mathbb{E}[\boldsymbol{X} \mid \{S = K\}]$	17.479	11.368	10.562	0.517	0.530	0.288
$\boldsymbol{K}_{\mathrm{M},1}[\boldsymbol{X}; \mathcal{K}_d(K)]$	25.678	3.107	11.215	1.185	0.278	1.205
$\boldsymbol{K}_{\mathrm{M},2}[\boldsymbol{X}; \mathcal{K}_d(K)]$	2.639	35.275	2.086	0.973	1.306	0.424
(M4) Pareto $+$ t copula: negative dependence						
$\mathbb{E}[\boldsymbol{X} \mid \{S = K\}]$	19.062	9.272	10.562	0.556	0.614	0.288
$\boldsymbol{K}_{\mathrm{M},1}[\boldsymbol{X}; \mathcal{K}_d(K)]$	28.353	0.684	10.962	2.125	1.646	2.154
$\boldsymbol{K}_{\mathrm{M},2}[\boldsymbol{X}; \mathcal{K}_d(K)]$	0.710	38.385	0.905	1.719	3.537	2.705

Exact Simulation of $\boldsymbol{X}' \mid \{S = K\}$ with MCMC

By repeating (1)–(2) a Markov chain is constructed such that each of $\boldsymbol{X}'_1, \boldsymbol{X}'_2, \dots$ has a density $f_{\boldsymbol{X}'|\{S=K\}}$.

(1) From the current state $\boldsymbol{X}'_n$, simulate a candidate $\boldsymbol{Y}'_n$ from the proposal density $q(\boldsymbol{X}'_n, \cdot)$.

(2) Accept the candidate, i.e., $\boldsymbol{X}'_{n+1} = \boldsymbol{Y}'_n$, with the acceptance probability $\alpha(\boldsymbol{X}'_n, \boldsymbol{Y}'_n)$:

$$\alpha(\boldsymbol{x}', \boldsymbol{y}') = 1 \wedge \frac{q(\boldsymbol{x}', \boldsymbol{y}') f_{\boldsymbol{X}}(\boldsymbol{y}', K - \mathbf{1}_{d'}^{\top}\boldsymbol{y}')}{q(\boldsymbol{y}', \boldsymbol{x}') f_{\boldsymbol{X}}(\boldsymbol{x}', K - \mathbf{1}_{d'}^{\top}\boldsymbol{x}')},$$

and otherwise reject, i.e., $\boldsymbol{X}'_{n+1} = \boldsymbol{X}'_n$.

Performance of MCMC methods

An appropriate choice of q is important depending on distributional properties of $\boldsymbol{X}' \mid \{S = K\}$.

- Support: a candidate outside of $\operatorname{supp}(\boldsymbol{X}' \mid \{S = K\})$ is immediately rejected.
- Tail-heaviness: most standard MCMC methods such as random walk MH, independent MH, Gibbs samplers and the Hamiltonian Monte Carlo method cannot guarantee the theoretical convergence when $\boldsymbol{X}' \mid \{S = K\}$ is heavy-tailed.
- Multimodality: the chain needs to traverse from one mode to another to explore the entire support of $\boldsymbol{X}' \mid \{S = K\}$.

Core-compatible allocations

We compute the Euler allocation and MLA on the (atomic) core:

$$\mathcal{K}_d^{\mathsf{C}}(K;r) = \{\boldsymbol{x} \in \mathbb{R}^d : \mathbf{1}_d^\top \boldsymbol{x} = K,\ \boldsymbol{\lambda}^\top \boldsymbol{x} \le r(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \in \{0,1\}^d\}.$$

- $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ is a participation profile where $\lambda_j = 1/0$ represents the presence/absence of the jth entity.
- $r : \{0,1\}^d \to \mathbb{R}$ is called a participation profile function typically determined as $r(\boldsymbol{\lambda}) = \varrho(\boldsymbol{\lambda}^\top \boldsymbol{X})$.
- We call an element of $\mathcal{K}_d^{\mathsf{C}}(K;r)$ a core allocation.
- **Interpretation**: under the core allocation $\boldsymbol{x} \in \mathcal{K}_d^{\mathsf{C}}(K;r)$, any subportfolio $(\lambda_1 X_1, \ldots, \lambda_d X_d)$ gains benefit of capital reduction from the stand-alone capital $r(\boldsymbol{\lambda})$ to $\boldsymbol{\lambda}^\top \boldsymbol{x}$.

Core-compatible MLA: setting

- **Goal**: Calculate the core-compatible versions of Euler allocation $\mathbb{E}[\boldsymbol{X} \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K;r)\}]$, MLA $\boldsymbol{K}_{\mathrm{M}}[\boldsymbol{X};\mathcal{K}_d^{\mathsf{C}}(K;r)]$ and local modes of $f_{\boldsymbol{X}|\{\boldsymbol{X}\in\mathcal{K}_d^{\mathsf{C}}(K;r)\}}$ (if they exist).

- **Method**: We utilize an MCMC method, especially the Hamiltonian Monte Carlo (HMC) method with reflection to directly simulate $f_{\boldsymbol{X}'|\{\boldsymbol{X}\in\mathcal{K}_d^{\mathsf{C}}(K;r)\}}$, because

$$\operatorname{supp}\{\boldsymbol{X}' \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K;r)\}\} = \bigcap_{\boldsymbol{\lambda}\in\{0,1\}^d} \{\boldsymbol{x}' \in \mathbb{R}^{d'} : \boldsymbol{\lambda}^\top(\boldsymbol{x}', K - \mathbf{1}_{d'}^\top \boldsymbol{x}') \le r(\boldsymbol{\lambda})\}.$$

- In HMC, a candidate is proposed according to the Hamiltonian dynamics, and the chain reflects at the boundaries.

Core-compatible MLA: model description

- Let $\boldsymbol{X} \sim t_\nu(\mathbf{0}_d, P)$ with $d = 3$, $\nu = 5$ and $P = (\rho_{ij})$ being a correlation matrix with $\rho_{12} = \rho_{23} = 1/3$ and $\rho_{13} = 2/3$.
- For $p = 0.99$, we set $r(\boldsymbol{\lambda}) = \mathrm{VaR}_p(\boldsymbol{\lambda}^\top \boldsymbol{X})$ for $\boldsymbol{\lambda} \in \{0,1\}^3$ and $K = r(\mathbf{1}_3)$.
- For $\delta = 0.001$, we first generate $N_{\text{MC}} = 10^6$ samples from $\boldsymbol{X}$ and estimate K and $(r(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \{0,1\}^3)$.
- Samples of $\boldsymbol{X} \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K; r)\}$ are extracted as pseudo MC samples.
- We conduct an MCMC simulation to generate $N_{\text{MCMC}} = 10^4$ samples directly from $\boldsymbol{X} \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K; r)\}$.
- Hyperparameters of the HMC method are estimated based on the 189 MC samples.

Core-compatible MLA: plots

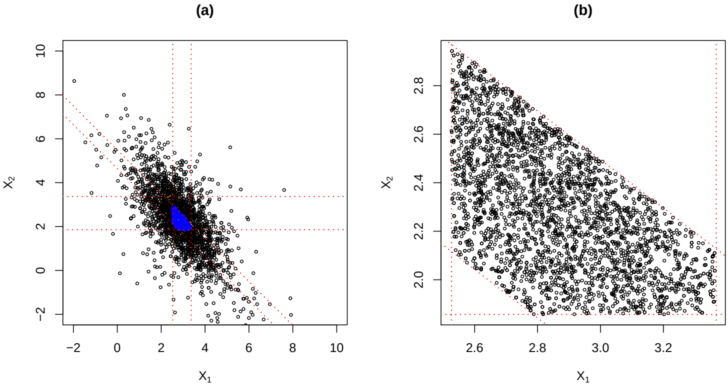


Figure: 4.3 (a) The first two components of the MC samples (black) from $\boldsymbol{X}$ and the extracted samples (blue) falling in $\mathcal{K}_d^{\mathsf{C}}(K;r)$. (b) The first 3000 MCMC samples of $\boldsymbol{X}' \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K;r)\}$.

Core-compatible MLA: table

Table: 4.4 MC and MCMC estimates and standard errors of the Euler and maximum likelihood allocations on $\mathcal{K}_d(K)$ and those on $\mathcal{K}_d^{\mathsf{C}}(K;r)$.

	Estimator			Standard error		
	X_1	X_2	X_3	X_1	X_2	X_3
$\hat{\mathbb{E}}^{\mathrm{MC}}[\boldsymbol{X} \mid \{\boldsymbol{X} \in \mathcal{K}_d(K)\}]$	2.865	2.310	2.846	0.026	0.034	0.026
$\hat{\boldsymbol{K}}_{\mathrm{M}}^{\mathrm{MC}}[\boldsymbol{X};\mathcal{K}_d(K)]$	2.861	2.366	2.793	–	–	–
$\hat{\mathbb{E}}^{\mathrm{MC}}[\boldsymbol{X} \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K;r)\}]$	2.852	2.267	2.903	0.016	0.019	0.016
$\hat{\boldsymbol{K}}_{\mathrm{M}}^{\mathrm{MC}}[\boldsymbol{X};\mathcal{K}_d^{\mathsf{C}}(K;r)]$	2.838	2.262	2.920	–	–	–
$\hat{\mathbb{E}}^{\mathrm{MCMC}}[\boldsymbol{X} \mid \{\boldsymbol{X} \in \mathcal{K}_d^{\mathsf{C}}(K;r)\}]$	2.876	2.269	2.877	0.002	0.003	0.002
$\hat{\boldsymbol{K}}_{\mathrm{M}}^{\mathrm{MCMC}}[\boldsymbol{X};\mathcal{K}_d^{\mathsf{C}}(K;r)]$	2.866	2.283	2.871	–	–	–

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Conclusion

- Studying $\boldsymbol{X}' \,|\{S = K\}$, especially its modality is motivated from scenario analysis and assessing soundness of risk allocations.
- Dependence, tail behavior and modality of $\boldsymbol{X}' \mid \{S = K\}$ are inherited from those of $\boldsymbol{X}$.
- Dependence of $\boldsymbol{X}$ is important for modality of $\boldsymbol{X}' \,|\{S = K\}$.
- The mode of $\boldsymbol{X}' \,|\{S = K\}$ (MLA) can be used as a risk allocation method.
- Searching for modes of $\boldsymbol{X}' \,|\{S = K\}$ is beneficial to designing more flexible allocations.

Future work

- Further theoretical investigation of the relationship between negative dependence of $\boldsymbol{X}$ and multimodality of $\boldsymbol{X} \mid \{S = K\}$.
- Study the copulas, tail dependence and measures of concordance of $\boldsymbol{X} \mid \{S = K\}$ especially without assuming the existence of a density.
- More detailed analysis of efficient simulation approaches of $\boldsymbol{X} \mid \{S = K\}$ with MCMC and possibly other methods.

Thank you for your attention!

References: see Koike and Hofert (2020+).
Available at: https://arxiv.org/abs/2005.02950

Website: https://uwaterloo.ca/scholar/tkoike/home

(The paper and this slide are also available here.)