Estimation and Comparison of Correlation-based Measures of Concordance

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MOCs and related questions

- Pearson's linear correlation coefficient ρ does not possess desirable properties for measuring dependence (Embrechts et al., 2002).
- Alternatively, measures of concordance (MOCs) are widely used to quantify dependence in terms of a single number.
- **Examples**: Spearman's rho ρ_{S}, Blomqvist's beta β and Kendall's tau τ.

  (Q1) **Why are they popular?**

  (Q2) **How to compare MOCs? Which one is best to use?**

Why are ρ_S, β and τ popular?

(A1) They often admit explicit forms for elliptical and Archimedean copulas.

Ex: $\beta(C_\rho^{\mathsf{Ga}}) = \tau(C_\rho^{\mathsf{Ga}}) = \frac{2}{\pi}\arcsin(\rho)$ for $\rho \in [-1, 1]$.

(A2) Because of (A1), they can be used to estimate parameters of these copulas by method-of-moment-like estimators.

Ex: Estimate τ by $\hat{\tau}$ from data, and find $\rho \in [-1, 1]$ such that $\hat{\tau} = \frac{2}{\pi}\arcsin(\rho)$.

However, benefit of these features is limited in practice since these copulas may not always be realistic.

Interpretability as transformed correlation: 1/2

ρ_{S}, β and τ admit the forms:

$$\rho_{\mathsf{S}}(C) = 12\mathbb{E}[UV] - 3 = \rho(U, V),$$

$$\beta(C) = 4C(1/2, 1/2) - 1 = \rho(\mathbf{1}_{\{U>1/2\}}, \mathbf{1}_{\{V>1/2\}}),$$

$$\tau(C) = 4\int_{(0,1)^2} C(u, v)\ \mathrm{d}C(u, v) - 1 = \rho(\mathbf{1}_{\{U>\tilde{U}\}}, \mathbf{1}_{\{V>\tilde{V}\}}),$$

where (U, V) and $(\tilde{U}, \tilde{V})$ are independent copies from C. So they are popular partly because...

(A3) They are easy to interpret and explain!

Interpretability as transformed correlation: 2/2

This interpretability still holds for (g_1, g_2)-transformed rank correlation coefficients

$$\kappa_{g_1,g_2}(C) = \rho(g_1(U), g_2(V)) \quad \text{for some} \quad g_1,\ g_2 : [0,1] \to \mathbb{R}.$$

Ex: $g_1 = g_2 = G^{-1}$ with G being...

- $\mathrm{Bern}(1/2) \Rightarrow$ Blomqvist's beta / median correlation coefficient
- $\mathrm{Unif}(0,1) \Rightarrow$ Spearman's rho
- $\mathrm{N}(0,1) \Rightarrow$ van der Waerden's coefficient / normal score correlation

We answer which g_1 and g_2 to use in terms of ease of estimation / asymptotic variance.

Literature review

Comparing MOCs in terms of

- estimation and robustness (by numerical experiments); De Winter et al. (2016).
- influence function; Croux and Dehon (2010), Boudt et al. (2012), Borroni and Cifarelli (2017) and Raymaekers and Rousseeuw (2019).
- power in tests of independence; Bhuchongkul (1964), Behnen (1971), Behnen (1972), Luigi Conti and Nikitin(1999), Rodel and Kossler (2004) and Genest and Verret (2005).
- tractability; Schmid and Schmidt (2007).

Outline

1. **Preliminaries (cont'd)**
    - Copulas, MOCs and transformed rank correlations.
2. **Comparison of asymptotic variances**
    - Canonical estimators, optimal shifts, theoretical results for Fréchet copulas and optimality of Blomqvist's beta.
3. **Comparison with Kendall's tau**
    - Asymptotic variance for Kendall's tau, its optimality, standardization by sample size.
4. **Simulation study**
    - Investigation of asymptotic variances for various copulas and MOCs.

Notation on copulas

- $\mathcal{C}_2$: the set of all bivariate copulas.
- $C \preceq C'$: $C' \in \mathcal{C}_2$ is more concordant than $C \in \mathcal{C}_2$ if $C(u,v) \leq C'(u,v)$ for all $(u,v) \in [0,1]^2$.
- $\Pi(u,v) = uv$: independence copula, $M(u,v) = \min(u,v)$: comonotonic copula and $W(u,v) = \max(u+v-1,0)$: countermonotonic copula such that $W \preceq C \preceq M$, $\forall C \in \mathcal{C}_2$.
- For $\kappa : \mathcal{C}_2 \to \mathbb{R}$, we identify $\kappa(C)$ for $C \in \mathcal{C}_2$ with $\kappa(U,V)$ for a random vector $(U,V) \sim C$.
- $\bar{C}(u,v) = \mathbb{P}(U > u, V > v)$ for $(u,v) \in [0,1]$ and $(U,V) \sim C$: the survival function of C.

Axioms of MOCs

Definition 1.1 (Axioms for measures of concordance)

A map $\kappa : \mathcal{C}_2 \to \mathbb{R}$ is called a measure of concordance if it satisfies the followings axioms.

1. Domain: $\kappa(C)$ is defined for any $C \in \mathcal{C}_2$.
2. Symmetry: $\kappa(V, U) = \kappa(U, V)$ for any $(U, V) \sim C \in \mathcal{C}_2$.
3. Monotonicity: If $C \preceq C'$ for $C, C' \in \mathcal{C}_2$, then $\kappa(C) \leq \kappa(C')$.
4. Range: $-1 \leq \kappa(C) \leq 1$ and $\kappa(C) \pm 1$ are attainable.
5. Independence: $\kappa(\Pi) = 0$ for the independence copula $\Pi \in \mathcal{C}_2$.
6. Change of sign: $\kappa(U, 1 - V) = -\kappa(U, V)$.
7. Continuity: If $C_n \to C$ pointwise for $C_n,\ C \in \mathcal{C}_2$, $n \in \mathbb{N}$, then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.

Transformed rank correlation coefficients

- Consider a class of maps on $\mathcal{C}_2$ written as

$$\kappa_{g_1,g_2}(U,V) = \rho(g_1(U), g_2(V)) \quad \text{for} \quad g_1,\ g_2 : [0,1] \to \mathbb{R}.$$

- Hofert and Koike (2019) showed that κ_{g_1,g_2} is MOC only if g_1 and g_2 are monotone with each other (w.l.o.g., increasing).
- Assuming left-continuity, g_1 and g_2 are quantiles $g_1 = G_1^{-1}$ and $g_2 = G_2^{-1}$ of some cdfs $G_1,\ G_2 : \mathbb{R} \to [0,1]$.
- Define the set of concordance-inducing functions by

$$\mathcal{G} = \{G : \mathbb{R} \to [0,1] : \text{ nondegenerate radially symmetric cdfs with finite second moment.}\}.$$

Properties of κ

Proposition 1.2 (Basic properties of $\kappa_{G_1^{-1},G_2^{-1}}$)

1. For cdfs G_1 and G_2, $\kappa_{G_1^{-1},G_2^{-1}}$ is a measure of concordance i.f.f. G_1, G_2 are of the same type with some $G \in \mathcal{G}$.
2. $\kappa_{G_1^{-1},G_2^{-1}} = \kappa_{G^{-1},G^{-1}} =: \kappa_G$ (call it the G-transformed rank correlation coefficient).
3. κ_G is invariant under location-scale transforms of G, that is, $\kappa_{G_{\mu,\sigma}}(C) = \kappa_G(C)$ where $G_{\mu,\sigma}(x) = G\left(\frac{x-\mu}{\sigma}\right)$, $\mu \in \mathbb{R}$, $\sigma > 0$.
4. κ_G is linear, that is, $\kappa_G(\alpha_1 C_1 + \alpha_2 C_2) = \alpha_1 \kappa_G(C_1) + \alpha_2 \kappa_G(C_2)$ for $C_1,\ C_2 \in \mathcal{C}_2$ and $\alpha_1,\ \alpha_2 \geq 0$ s.t. $\alpha_1 + \alpha_2 = 1$.

Remark: Kendall's tau is not included in this class since it is not linear.

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Canonical estimator of κ_G

- Assume that $\mathbb{E}_G[X] = 0$ and $\mathrm{Var}_G(X) = 1$.
- Define a canonical estimator of κ_G

$$\hat{\kappa}_G = \frac{1}{n} \sum_{i=1}^{n} G^{-1}(U_i) G^{-1}(V_i),$$

where $(U_1, V_1), \ldots, (U_n, V_n) \overset{\text{iid}}{\sim} C$.

- Then the CLT $\sqrt{n}\,\{\hat{\kappa}_G - \kappa_G(C)\} \xrightarrow{d} \mathrm{N}(0, \sigma_G^2(C))$ holds, where

$$\sigma_G^2(C) = \mathrm{Var}(G^{-1}(U) G^{-1}(V))$$

and $G \in \mathcal{G}_4$ with

$$\mathcal{G}_4 = \{G \in \mathcal{G} : \mathbb{E}_G[X] = 0,\ \mathrm{Var}_G(X) = 1 \text{ and } \mathbb{E}_G[X^4] < \infty\}.$$

Optimal asymptotic variances: 1/2

- For $\mathcal{H} \subseteq \mathcal{G}_4$ and $\mathcal{D} \subseteq \mathcal{C}_2$, consider

$$\underline{\sigma}^2_G(\mathcal{D}) = \inf_{C\in\mathcal{D}}\sigma^2_G(C), \quad \overline{\sigma}^2_G(\mathcal{D}) = \sup_{C\in\mathcal{D}}\sigma^2_G(C),$$

  and the optimal best and worst asymptotic variances and their attainers defined by

$$\underline{\sigma}^2_*(\mathcal{H},\mathcal{D}) = \inf_{G\in\mathcal{H}} \underline{\sigma}^2_G(\mathcal{D}), \quad \underline{G}_*(\mathcal{H},\mathcal{D}) = \operatorname*{arginf}_{G\in\mathcal{H}} \underline{\sigma}^2_G(\mathcal{D}),$$
$$\overline{\sigma}^2_*(\mathcal{H},\mathcal{D}) = \inf_{G\in\mathcal{H}} \overline{\sigma}^2_G(\mathcal{D}), \quad \overline{G}_*(\mathcal{H},\mathcal{D}) = \operatorname*{arginf}_{G\in\mathcal{H}} \overline{\sigma}^2_G(\mathcal{D}),$$

  respectively.

- Typically $\mathcal{H} = \mathcal{G}_4$ but can be limited to continuous functions or those with bounded supports.

Optimal asymptotic variances: 2/2

- **Interpretation**: $\underline{G}_*(\mathcal{H},\mathcal{D})$ and $\overline{G}_*(\mathcal{H},\mathcal{D})$ are the best choices of G to accurately estimate κ_G if one believes that $\mathcal{D}$ is the set of copulas which one wants to quantify and compare in terms of their concordance.

- Does there exist $G \in \underline{G}_*(\mathcal{H},\mathcal{D}) \cap \overline{G}_*(\mathcal{H},\mathcal{D})$?

- **Reflection invariance**: By radial symmetry of $G \in \mathcal{G}_4$, $C \mapsto \sigma_G^2(C)$ is reflection invariant in the sense that

$$\begin{aligned}\sigma_G^2(U,V) = \sigma_G^2(1-U,V) &= \sigma_G^2(U,1-V)\\ &= \sigma_G^2(1-U,1-V).\end{aligned}$$

Optimal location shifts of G: 1/3

- κ_G is location-scale invariant but its canonical estimator is not.
- Let $G_0 \in \mathcal{G}_4$ be s.t. $\mathbb{E}_{G_0}[X] = 0$ and $\mathrm{Var}_{G_0}(X) = 1$, and let $G_{\mu,\sigma}(x) = G_0\left(\frac{x-\mu}{\sigma}\right)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$.
- For known μ and σ, the canonical estimator of $\kappa_{G_{\mu,\sigma}}$ is

$$\hat{\kappa}_{G_{\mu,\sigma}} = \frac{1}{n}\sum_{i=1}^{n} \frac{G_{\mu,\sigma}^{-1}(U_i)G_{\mu,\sigma}^{-1}(V_i)}{\sigma^2} - \left(\frac{\mu}{\sigma}\right)^2.$$

- By the CLT, it has the asymptotic variance

$$\sigma^2_{G_{\mu,\sigma}}(C) = \mathrm{Var}\left(\frac{G_{\mu,\sigma}^{-1}(U)G_{\mu,\sigma}^{-1}(V)}{\sigma^2}\right).$$

Optimal location shifts of G: 2/3

- W.l.o.g, one can take $\sigma = 1$ since $G^{-1}_{\mu,\sigma}(u) = G^{-1}_{\mu/\sigma,1}(u)$.
- Consider $G_\mu(x) = G_0(x-\mu)$ where $\mu \in \mathbb{R}$. Then

$$\sigma^2_{G_\mu}(C) = \mathrm{Var}(G_\mu^{-1}(U)G_\mu^{-1}(V)) = \mathrm{Var}(X_0Y_0 + \mu(X_0+Y_0))$$

where $X_0 = G_0^{-1}(U)$, $Y_0 = G_0^{-1}(V)$ and $(U,V) \sim C$.

- Provided $C \neq W$, $\sigma^2_{G_\mu}(C)$ is minimized when

$$\mu = \mu_* = \mu_*(G_0, C) = -\frac{\mathrm{Cov}(X_0Y_0, X_0+Y_0)}{\mathrm{Var}(X_0+Y_0)}.$$

- We call μ_* an optimal shift of $G_0 \in \mathcal{G}_4$ under $C \in \mathcal{C}_2$.

Optimal location shifts of G: 3/3

Proposition 2.1 (Sufficient condition for $\mu_* = 0$)

For $C \in \mathcal{C}_2$ and $G_0 \in \mathcal{G}_4$ with mean zero and variance one, $\mu_*(G_0, C) = 0$ holds if C is radially symmetric, that is,

$$(U, V) \overset{\mathrm{d}}{=} (1 - U, 1 - V) \quad \text{for } (U, V) \sim C.$$

- $\mu_* = 0$ if C is M, W, Π, a Gaussian copula, t copula or their mixtures.
- $\mu_* \neq 0$ if C is a Clayton or Gumbel copula. Nevertheless, we will see that, even in this case, $\sigma^2_{G_{\mu_*}}(C)$ and $\sigma^2_{G_0}(C)$ are very close.
- In the following we focus on the case $\mu = 0$.

Asymptotic variances for fundamental copulas

Proposition 2.2 (Asymptotic variances for fundamental copulas)

1. Suppose $\mathcal{D} = \{\Pi\}$. Then, for any $\mathcal{H} \subseteq \mathcal{G}_4$,

$$\underline{\sigma}_*^2(\mathcal{H}, \{\Pi\}) = \overline{\sigma}_*^2(\mathcal{H}, \{\Pi\}) = 1,$$
$$\underline{G}_*(\mathcal{H}, \{\Pi\}) = \overline{G}_*(\mathcal{H}, \{\Pi\}) = \mathcal{H}.$$

2. Suppose $\mathcal{D} = \{M\}, \{W\}$ or $\{M, W\}$. Then, for $\mathcal{H} \subseteq \mathcal{G}_4$,

$$\underline{\sigma}_*^2(\mathcal{H}, \mathcal{D}) = \overline{\sigma}_*^2(\mathcal{H}, \mathcal{D}) = \text{inf}_{G \in \mathcal{H}} \mathrm{Var}_G(X^2),$$
$$\underline{G}_*(\mathcal{H}, \mathcal{D}) = \overline{G}_*(\mathcal{H}, \mathcal{D}) = \text{arginf}_{G \in \mathcal{H}} \mathrm{Var}_G(X^2).$$

Proof: Use $\sigma_G^2(C) = \mathrm{Var}(XY) = \mathrm{Cov}(X^2, Y^2) + 1 - \mathrm{Cov}(X, Y)^2$.

Ordering MOCs for fundamental copulas

Let $\mathcal{H}_{\mathrm{N}}$, $\mathcal{H}_{\mathrm{Unif}}$ and $\mathcal{H}_{\mathrm{Bern}}$ be singletons of normal, uniform and Bernoulli distributions with mean zero and variance one, respectively.

Proposition 2.3 (Orders of MOCs for fundamental copulas)

For $\mathcal{D}_{\mathsf{F}} = \{\Pi, M, W\}$, it holds that

$$\underline{\sigma}_*^2(\mathcal{H}_{\mathrm{Bern}}, \{M, W\}) < \underline{\sigma}_*^2(\mathcal{H}_{\mathrm{Unif}}, \{M, W\}) < \underline{\sigma}_*^2(\mathcal{H}_{\mathrm{N}}, \{M, W\}),$$
$$\overline{\sigma}_*^2(\mathcal{H}_{\mathrm{Bern}}, \{M, W\}) < \overline{\sigma}_*^2(\mathcal{H}_{\mathrm{Unif}}, \{M, W\}) < \overline{\sigma}_*^2(\mathcal{H}_{\mathrm{N}}, \{M, W\}).$$
$$\underline{\sigma}_*^2(\mathcal{H}_{\mathrm{Bern}}, \mathcal{D}_{\mathsf{F}}) < \underline{\sigma}_*^2(\mathcal{H}_{\mathrm{Unif}}, \mathcal{D}_{\mathsf{F}}) < \underline{\sigma}_*^2(\mathcal{H}_{\mathrm{N}}, \mathcal{D}_{\mathsf{F}}),$$
$$\overline{\sigma}_*^2(\mathcal{H}_{\mathrm{Bern}}, \mathcal{D}_{\mathsf{F}}) = \overline{\sigma}_*^2(\mathcal{H}_{\mathrm{Unif}}, \mathcal{D}_{\mathsf{F}}) < \overline{\sigma}_*^2(\mathcal{H}_{\mathrm{N}}, \mathcal{D}_{\mathsf{F}}).$$

Proof: $\mathrm{Var}_{G_{\mathrm{N}}}(X^2) = 2$, $\mathrm{Var}_{G_{\mathrm{Unif}}}(X^2) = 0.8$ and $\mathrm{Var}_{G_{\mathrm{Bern}}}(X^2) = 0$.

Fréchet copulas

- A bivariate Fréchet copula is defined by

$$C_{\boldsymbol{p}}^{\mathrm{F}} = p_M M + p_\Pi \Pi + p_W W, \quad \boldsymbol{p} = (p_M, p_\Pi, p_W) \in \Delta_3,$$

where $\Delta_3 = \{(p_1, p_2, p_3) \in \mathbb{R}_+^3 : p_1 + p_2 + p_3 = 1\}$.

- Let $\mathcal{C}^{\mathrm{F}} = \{C_{\boldsymbol{p}}^{\mathrm{F}} : \boldsymbol{p} \in \Delta_3\}$.
- Fréchet copulas can be applied in insurance and finance, and for approximating bivariate copulas (Yang et. al, 2006).
- $\kappa_G(C_{\boldsymbol{p}}^{\mathrm{F}})$ can take any value in $[-1, 1]$ for any $G \in \mathcal{G}$;

$$\begin{aligned}\kappa_G(C_{\boldsymbol{p}}^{\mathrm{F}}) &= p_M \kappa_G(M) + p_\Pi \kappa_G(\Pi) + p_W \kappa_G(W) \\ &= p_M - p_W \in [-1, 1].\end{aligned}$$

Asymptotic variances for Fréchet copulas

Proposition 2.4 (Asymptotic variances for Fréchet copulas)

For $G \in \mathcal{G}_4$, we have that

$$\overline{\sigma}_G^2(\mathcal{C}^{\mathrm{F}}) = 1 + \mathrm{Var}_G(X^2) \quad \text{and} \quad \underline{\sigma}_G^2(\mathcal{C}^{\mathrm{F}}) = 1 \wedge \mathrm{Var}_G(X^2)$$

with the maximum and minimum attained, respectively, by

$$C_{\mathsf{max}} = \begin{cases} \frac{M+W}{2} & \text{if } \mathrm{Var}_G(X^2) > 0, \\ p\frac{M+W}{2} + (1-p)\Pi \ \text{ for any } p \in [0,1] & \text{if } \mathrm{Var}_G(X^2) = 0, \end{cases}$$

$$C_{\mathsf{min}} = \begin{cases} M, W & \text{if } 0 \le \mathrm{Var}_G(X^2) < 1, \\ M, W, \Pi & \text{if } \mathrm{Var}_G(X^2) = 1, \\ \Pi & \text{if } 1 < \mathrm{Var}_G(X^2). \end{cases}$$

Optimal asymptotic variances for Fréchet copulas

Corollary 2.5 (Optimal asymptotic variances for Fréchet copulas)

For $\mathcal{H} \subseteq \mathcal{G}_4$ and $\mathcal{D} = \mathcal{C}^{\mathrm{F}}$, it holds that

$$\overline{\sigma}_*^2(\mathcal{H}, \mathcal{C}^{\mathrm{F}}) = 1 + \inf_{G \in \mathcal{H}} \mathrm{Var}_G(X^2),$$

$$\underline{\sigma}_*^2(\mathcal{H}, \mathcal{C}^{\mathrm{F}}) = 1 \wedge \inf_{G \in \mathcal{H}} \mathrm{Var}_G(X^2),$$

$$\overline{G}_*(\mathcal{H}, \mathcal{C}^{\mathrm{F}}) = \operatorname*{arginf}_{G \in \mathcal{H}} \mathrm{Var}_G(X^2),$$

$$\underline{G}_*(\mathcal{H}, \mathcal{C}^{\mathrm{F}}) = \begin{cases} \mathrm{arginf}_{G \in \mathcal{H}} \mathrm{Var}_G(X^2), & \text{if } \inf_{G \in \mathcal{H}} \mathrm{Var}_G(X^2) < 1, \\ \mathcal{H}, & \text{if } \inf_{G \in \mathcal{H}} \mathrm{Var}_G(X^2) \geq 1. \end{cases}$$

Remark: Again $\mathrm{Var}_G(X^2)$ determines the order. The upper bound increases but the lower bound is unchanged.

Proof of Prop 2.4 and Cor 2.5

- For $C^{\mathrm{F}}_{\boldsymbol{p}} \in \mathcal{C}^{\mathrm{F}}$ with $\boldsymbol{p} = (p_M, p_\Pi, p_W) \in \Delta_3$ and $v = \mathrm{Var}_G(X^2)$,

$$\sigma_G^2(C^{\mathrm{F}}_{\boldsymbol{p}}) = (p_M + p_W)v + 1 - (p_M - p_W)^2 =: f(p_M, p_W).$$

- Taking $(p_M, p_W) = (p - r, r)$ where $0 \leq r \leq p \leq 1$,

$$f(p - r, r) = -4\left(r - \frac{p}{2}\right)^2 + pv + 1 \quad \text{(parabolic cylinder)}.$$

$$\Rightarrow \text{ optimize over } 0 \leq r \leq p \leq 1.$$

<u>**Remark**</u>: (Restrictions of $\mathcal{C}^{\mathrm{F}}$) One can consider $\mathcal{D} = \mathcal{C}^{\mathrm{F}}_{\underline{k},\overline{k}}(G)$ where

$$\mathcal{C}^{\mathrm{F}}_{\underline{k},\overline{k}}(G) = \{C \in \mathcal{C}^{\mathrm{F}} : \underline{k} \leq \kappa_G(C) \leq \overline{k}\}, \quad -1 \leq \underline{k} \leq \overline{k} \leq 1.$$

Then the problem reduces to optimizing $f(p_M, p_W)$ subject to $0 \leq p_M, p_W$, $p_M + p_W \leq 1$ and $\underline{k} \leq p_M - p_W \leq \overline{k}$.

Asymptotic variance for Blomqvist's beta

Let $p(C) = C(1/2, 1/2) + \bar{C}(1/2, 1/2)$ for $C \in \mathcal{C}_2$. C is called balanced if $p(C) = 1/2$, imbalanced if $p(C) \neq 1/2$, totally positively imbalanced (TPI) if $p(C) = 1$ and totally negatively imbalanced (TNI) if $p(C) = 0$.

Proposition 2.6 (Asymptotic variance for Blomqvist's beta)

For any $\mathcal{D} \subseteq \mathcal{C}_2$, we have that

$$0 \leq \underline{\sigma}^2_*(\mathcal{H}_{\text{Bern}}, \mathcal{D}) \leq \overline{\sigma}^2_*(\mathcal{H}_{\text{Bern}}, \mathcal{D}) = 1.$$

The upper bound is attained i.f.f. $\mathcal{D}$ contains a balanced copula, and the lower bound is attained i.f.f. $\mathcal{D}$ contains a TPI or TNI copula.

Proof: By calculation, $\sigma^2_{G_{\text{Bern}}}(C) = 4p(C)(1 - p(C))$.

Optimality of Blomqvist's beta

Corollary 2.7 (Optimality of Blomqvist's beta)

Consider $\mathcal{D} \subseteq \mathcal{C}_2$ and $\mathcal{H}_{\mathrm{Bern}} \subseteq \mathcal{H}$ for $\mathcal{H} \subseteq \mathcal{G}_4$. If $\Pi \in \mathcal{D}$, then

$$\overline{\sigma}_*^2(\mathcal{H}, \mathcal{D}) = 1 \text{ and } \mathcal{H}_{\mathrm{Bern}} \subseteq \overline{G}_*(\mathcal{H}, \mathcal{D}).$$

If $\mathcal{D}$ includes at least one TPI or TNI copula, then

$$\underline{\sigma}_*^2(\mathcal{H}, \mathcal{D}) = 0 \text{ and } \mathcal{H}_{\mathrm{Bern}} \subseteq \underline{G}_*(\mathcal{H}, \mathcal{D}).$$

Proof: For any $G \in \mathcal{G}_4$, $1 = \sigma_G^2(\Pi) \leq \sup_{C \in \mathcal{D}} \sigma_G^2(C)$ for $\overline{\sigma}_*^2(\mathcal{H}, \mathcal{D})$, and $0 \leq \inf_{C \in \mathcal{D}} \sigma_G^2(C)$ for $\underline{\sigma}_*^2(\mathcal{H}, \mathcal{D})$.

Remark: Blomqvist's beta attains the optimum when $\mathcal{D}$ is, for e.g., $\mathcal{C}_2$, $\mathcal{C}_2^{\succeq} = \{C \in \mathcal{C}_2 : C \succeq \Pi\}$ or $\mathcal{C}_2^{\preceq} = \{C \in \mathcal{C}_2 : C \preceq \Pi\}$.

Non-uniqueness of the optimality of β

$G \in \underline{G}_*(\mathcal{H}, \mathcal{D}) \Leftrightarrow$ there exists $C \in \mathcal{D}$ s.t. $\sigma^2_G(C) = 0$

$\Leftrightarrow G^{-1}(U)G^{-1}(V) \overset{\text{a.s.}}{=} {}^\exists a \in \mathbb{R}$ and $(U, V) \sim {}^\exists C \in \mathcal{D}$.

Proposition 2.8 (Necessary conditions on $G \in \underline{G}_*(\mathcal{H}, \mathcal{D})$)

Let $G \in \underline{G}_*(\mathcal{H}, \mathcal{D})$. Then $C \in \mathcal{D}$ and $a \in \mathbb{R}$ above satisfy the followings.

(C1) If $\mathbb{P}(X = 0) > 0$ for $X \sim G$, then $a = 0$ and $\mathbb{P}(X = 0) \geq 1/2$.

(C2) If $\mathbb{P}(X = 0) = 0$, then $a \neq 0$ and the copula C is either TPI or TNI with $0 < a \leq 1$ if C is TPI and $-1 \leq a < 0$ if C is TNI. Moreover, the distribution function $G_+(x) = 2G(x) - 1$, $x > 0$ satisfies $\mathbb{E}_{G_+}[Z] \geq |a|^{1/2}$, $Z \sim G_+$, and

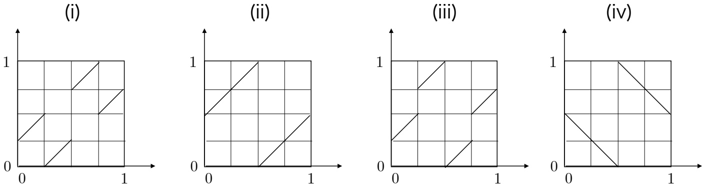
$$G_+(x) = 1 - G_+\left(|a|/x-\right), \quad x > 0.$$

Examples of $G \in \underline{G}_*(\mathcal{H}, \mathcal{D})$

Consider $X_+ \sim G_+$ where

(C1) $X_+ = \begin{cases} 0 & \text{w.p. } 1/2, \\ U \sim \mathrm{Unif}(0, \sqrt{6}) & \text{w.p. } 1/2, \end{cases}$ (C2) $X_+ = \begin{cases} \sqrt{1 - \sqrt{2}/2} & \text{w.p. } 1/2, \\ 1/\sqrt{2 - \sqrt{2}} & \text{w.p. } 1/2, \end{cases}$

and four shuffles of M copulas



Then (C1) + (i), (ii), (iii) and (C2) + (iv) attain $\sigma_G^2(C) = 0$.

Discussion on $G \in \overline{G}_*(\mathcal{H}, \mathcal{D})$

- Much less is known for $G \in \overline{G}_*(\mathcal{H}, \mathcal{D})$ compared to $\underline{G}_*(\mathcal{H}, \mathcal{D})$.
- Assuming that $\Pi \in \mathcal{D}$ and $\mathcal{H}_{\mathrm{Bern}} \subseteq \mathcal{H}$, we have that

$$G \in \overline{G}_*(\mathcal{H}, \mathcal{D}) \quad \Leftrightarrow \quad \sigma_G^2(C) \leq 1 \quad \text{for all } C \in \mathcal{D}$$
$$\Leftrightarrow \quad \mathrm{Cov}(X^2, Y^2) \leq \mathrm{Cov}(X, Y)^2 \ (= \kappa_G(C)^2).$$

- If M or W is in $\mathcal{D}$, then

$$\mathrm{Var}_G(X^2) \leq 1$$

is a sufficient condition for $G \in \overline{G}_*(\mathcal{H}, \mathcal{D})$.

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   - Canonical estimators, optimal shifts, theoretical results for Fréchet copulas and optimality of Blomqvist's beta.
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   - Asymptotic variance for Kendall's tau, its optimality, standardization by sample size.
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   - Investigation of asymptotic variances for various copulas and MOCs.

Canonical estimator of τ

- For $(U, V),\ (\tilde{U}, \tilde{V}) \overset{\text{iid}}{\sim} C$, Kendall's tau admits

$$\tau(C) = \rho(g(U, \tilde{U}), g(V, \tilde{V})) \quad \text{where} \quad g(l, m) = \begin{cases} 1 & \text{if } l \leq m, \\ -1 & \text{if } l > m. \end{cases}$$

- The canonical estimator of $\tau(C)$ is defined by

$$\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} g(U_i, \tilde{U}_i) g(V_i, \tilde{V}_i) \quad \text{where} \quad (U_i, V_i),\ (\tilde{U}_i, \tilde{V}_i),\ i = 1, \ldots, n, \overset{\text{iid}}{\sim} C.$$

- By the CLT, $\hat{\tau}$ satisfies the asymptotic normality

$$\sqrt{n}\,\{\hat{\tau} - \tau(C)\} \xrightarrow{d} \mathrm{N}(0, \sigma_\tau^2(C)) \quad \text{where} \quad \sigma_\tau^2(C) = \mathrm{Var}(g(U, \tilde{U}) g(V, \tilde{V})).$$

Asymptotic variances of τ

Proposition 3.1 (Asymptotic variances of τ)

1. For all $C \in \mathcal{C}_2$, it holds that $0 \leq \sigma_\tau^2(C) \leq 1$.
2. For a given $C \in \mathcal{C}_2$, $\sigma_\tau^2(C) = 1$ i.f.f. $\tau(C) = 0$, which holds, for example, when $C = \Pi$ or $C = (M + W)/2$. More generally, $\sigma_\tau^2(C) = 1$ if C satisfies $(U, 1 - V) \overset{\text{d}}{=} (U, V)$ or $(1 - U, V) \overset{\text{d}}{=} (U, V)$ for $(U, V) \sim C$.
3. For a given $C \in \mathcal{C}_2$, $\sigma_\tau^2(C) = 0$ i.f.f. $\tau(C) = 1$ or -1, i.e., $C = M$ or W, resp.

Proof: $\sigma_\tau^2(C) = 4p_\tau(C)(1 - p_\tau(C))$ where $p_\tau(C) = \frac{1}{2}(\tau(C) + 1)$.

Remark: Attainers of $\sigma_\tau^2(C) = 0$ are characterized but those of $\sigma_\tau^2(C) = 1$ are not.

Optimality of Kendall's tau

For $\mathcal{D} \subseteq C_2$, define

$$\underline{\sigma}_\tau^2(\mathcal{D}) = \inf_{C\in\mathcal{D}}\sigma_\tau^2(C) \quad \text{and} \quad \overline{\sigma}_\tau^2(\mathcal{D}) = \sup_{C\in\mathcal{D}}\sigma_\tau^2(C).$$

Proposition 3.2 (Optimality of τ)

1. Suppose $\mathcal{H}_{\mathrm{Bern}} \subseteq \mathcal{H}$ and $\Pi \in \mathcal{D}$. Then $\overline{\sigma}_\tau^2(\mathcal{D}) = \overline{\sigma}_*^2(\mathcal{H},\mathcal{D}) = 1$.
2. Suppose $\mathcal{H}_{\mathrm{Bern}} \subseteq \mathcal{H}$, and $M \in \mathcal{D}$ or $W \in \mathcal{D}$. Then $\underline{\sigma}_\tau^2(\mathcal{D}) = \underline{\sigma}_*^2(\mathcal{H},\mathcal{D}) = 0$.

Considering the drawback of β that it depends only on the local value $C(1/2,1/2)$, Kendall's tau can be a good alternative of β.

Characterization of the attainers of $\overline{\sigma}_\tau^2(\mathcal{C}^{\mathsf{F}})$

Characterization of attainers of $\overline{\sigma}_\tau^2(\mathcal{D})$ is in general not known but is known when $\mathcal{D} = \mathcal{C}^{\mathsf{F}}$.

Proposition 3.3 (Characterization of copulas attaining $\overline{\sigma}_\tau^2(\mathcal{C}^{\mathsf{F}})$)

A Fréchet copula $C = C^{\mathsf{F}}_{(p_M, p_\Pi, p_W)} \in \mathcal{C}^{\mathsf{F}}$ attains $\overline{\sigma}_\tau^2(\mathcal{C}^{\mathsf{F}}) = 1$ i.f.f. $p_M = p_W \in [0, 1/2]$. Equivalently,

$$C = p\frac{M+W}{2} + (1-p)\Pi, \quad p \in [0,1].$$

Proof: Solve $\tau(\mathcal{C}^{\mathsf{F}}_{(p_M, p_\Pi, p_W)}) = (p_M - p_W)(p_M + p_W + 2)/3 = 0$.

Standardization by sample size

- $\hat{\tau}$ requires twice more samples than $\hat{\kappa}_G$ for $G \in \mathcal{G}_4$.
- If only n i.i.d. samples are given, then

$$\mathrm{Var}(\hat{\kappa}_G) = \frac{\sigma_G^2(C)}{n} \quad \text{and} \quad \mathrm{Var}(\hat{\tau}) = \frac{\sigma_\tau^2(C)}{n/2} = \frac{2\sigma_\tau^2(C)}{n}.$$

  Thus $\sigma_G^2(C)$ should be compared with $\sigma_\tau^{2\star}(C) = 2\sigma_\tau^2(C)$.
- Optimality of τ in terms of the worst asymptotic variance is valid with this modification since

$$\underline{\sigma}_\tau^{2\star}(\mathcal{D}) = 2\underline{\sigma}_\tau^2(\mathcal{D}) = 0 = \underline{\sigma}_*^2(\mathcal{G}_4, \mathcal{D}).$$

- That of the best asymptotic variance becomes invalid since

$$\overline{\sigma}_\tau^{2\star}(\mathcal{D}) = 2\overline{\sigma}_\tau^2(\mathcal{D}) = 2 > 1 = \overline{\sigma}_*^2(\mathcal{G}_4, \mathcal{D}).$$

1. **Preliminaries (cont'd)**
   - Copulas, MOCs and transformed rank correlations.
2. **Comparison of asymptotic variances**
   - Canonical estimators, optimal shifts, theoretical results for Fréchet copulas and optimality of Blomqvist's beta.
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Simulation study

1. Set
$$\rho = -0.99 + 1.98\frac{k}{49}, \quad k = 0, 1, \dots, 49,$$
$\nu = 5$ and $\theta = 2\rho/(1-\rho)$ (so that $\tau(C_\theta^{\mathrm{Cl}}) = \rho$) in $C = C_\rho^{\mathrm{Ga}}$ (Gauss), $C_{\rho,\nu}^t$ (t) and C_θ^{Cl} (Clayton).
2. For each C, simulate $(U_1, V_1), \dots, (U_n, V_n) \overset{\text{iid}}{\sim} C$, $n = 10^5$.
3. Estimate $\sigma_G^2(C)$ and $\sigma_\tau^2(C)$ by the sample variances of $G^{-1}(U_i)G^{-1}(V_i)$, $i = 1, \dots, n$, and of $g(U_i, U_{i+n/2})g(V_i, V_{i+n/2})$, $i = 1, \dots, n/2$, where G is a standardized, and optimally shifted Bernoulli, uniform, normal, $t(10)$ and Beta$(0.5, 0.5)$ distribution function.

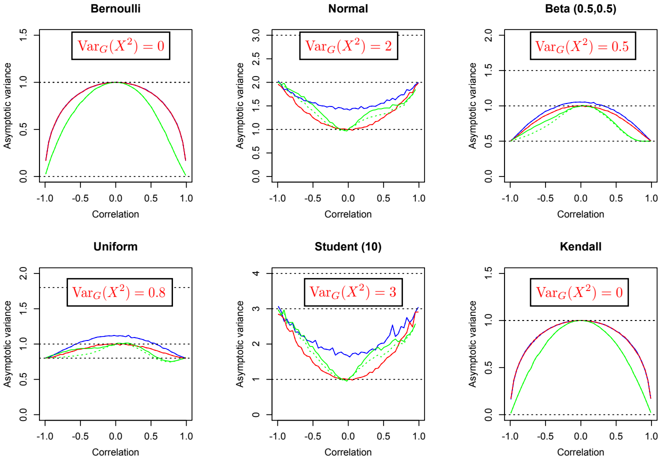

Figure: Standardized (solid) and optimally shifted (dotted) $\sigma_G^2(C)$ for $C = C_\rho^{\mathsf{Ga}}$, $C_{\rho,\nu}^t$ and C_θ^{Cl}. The black dotted lines are $y = 1$, $\mathrm{Var}_G(X^2)$ and $\mathrm{Var}_G(X^2) + 1$.

Discussion: 1/2

- **Shapes of the curves**: The curves of $\sigma_G^2(C)$ against $\rho \in [-1, 1]$ are symmetric around $\rho = 0$, convex when $\mathrm{Var}_G(X^2) > 1$ and concave when $\mathrm{Var}_G(X^2) < 1$.
- **Upper and lower bounds**: For all the cases of G, the upper and lower bounds are $1 \vee \mathrm{Var}_G(X^2)$ and $1 \wedge \mathrm{Var}_G(X^2)$, resp. The upper bound $1 + \mathrm{Var}_G(X^2)$ is not attained except for the case where $\mathrm{Var}_G(X^2) = 0$ (Blomqvist and Kendall)
- **Choices of G**: Smaller $\mathrm{Var}_G(X^2)$ is more preferable. Normal (van der Waerden) is better than t and Beta$(0.5, 0.5)$ outperforms uniform (Spearman).

Discussion: 2/2

- **Blomqvist's beta and Kendall's tau**: The curves seem to coincide for all choices of C. Some theoretical results are known:

$$\beta(C_\rho^{\mathsf{Ga}}) = \tau(C_\rho^{\mathsf{Ga}}) = \beta(C_{\rho,\nu}^t) = \tau(C_{\rho,\nu}^t) = \frac{2}{\pi}\arcsin(\rho),$$

$$\sigma^2_{G_{\mathrm{Bern}}}(C) = 1 - \beta^2(C) \quad \text{(Schmid and Schmidt, 2007).}$$

- **Strength of dependence and model of copula**: The strength of dependence affects $\sigma_G^2(C)$. Among the models of C with the same strength of dependence, differences of $\sigma_G^2(C)$ are small.

- **Effect of optimal shifts**: As theoretically shown, $\sigma_G^2(C)$ is not reduced by the optimal shift of G when $C = C_\rho^{\mathsf{Ga}}$ or $C_{\rho,\nu}^t$. Even when the copula is C_θ^{Cl}, only a small reduction was observed.

Concluding remarks

- We proposed a framework for comparing transformed rank correlations in terms of the asymptotic variances $\sigma_G^2(C)$ of their canonical estimators.
- Blomqvist's beta β was shown to be optimal.
- Kendall's tau is also optimal if not standardized by sample size.
- The curve of $\sigma_G^2(C)$ against the strength of dependence is typically symmetric convex or concave.
- Smaller $\mathrm{Var}_G(X^2)$ is more preferable.
- Normal (van der Waerden's coefficient) is better than t. Beta $(0.5, 0.5)$ is more preferable than uniform (Spearman's rho).

Future work

- Comparison of κ_G to Gini's gamma and other MOCs.
- Parametric classes of G (e.g., Beta(α, α)).
- Optimal G and $\sigma_G^2(C)$ for more practical choices of $\mathcal{D}$ (e.g., $\mathcal{D}_\epsilon(C^*) = \{C \in \mathcal{C}_2 : d(C, C^*) \leq \epsilon\}$).
- Comparison of multivariate MOCs and matrices of pairwise MOCs.
- Comparison based on pseudo-samples (when the margins are unknown and estimated non-parametrically).
- Effect of optimal shifts.

Thank you for your attention!

References: see Koike and Hofert (2020+).
Available at: https://arxiv.org/abs/2006.13975

Website: https://uwaterloo.ca/scholar/tkoike/home

(The paper and these slides are also available here.)

Appendix I: Alternative estimators of Kendall's tau

- Consider

$$\hat{\tau}^{\maltese} = \frac{1}{n}\sum_{i=1}^{n} g(U_i, U_{i+1})g(V_i, V_{i+1}), \quad (U_i, V_i) \overset{\text{iid}}{\sim} C.$$

- The Markov chain CLT holds with the asymptotic variance

$$\begin{aligned}\sigma_{\tau}^{2\maltese}(C) &= \mathrm{Var}(g(U_1, U_2)g(V_1, V_2)) \\ &\quad + \mathrm{Cov}(g(U_1, U_2)g(V_1, V_2), g(U_2, U_3)g(V_2, V_3)).\end{aligned}$$

- $\sigma_{\tau}^{2\maltese}(C)$ can be directly compared to $\sigma_{G}^{2}(C)$ as $\frac{n+1}{n} \to 1$.
- $\underline{\sigma}_{\tau}^{2\maltese}(\mathcal{D}) = 0 = \underline{\sigma}_{*}^{2}(\mathcal{G}_4, \mathcal{D})$ if $M \in \mathcal{D}$ or $W \in \mathcal{D}$.
- $\overline{\sigma}_{\tau}^{2\maltese}(C) \leq \overline{\sigma}_{\tau}^{2\star}(\mathcal{D}) = 2$ so $\hat{\tau}^{\maltese}$ is not worse than $\hat{\tau}$.

Appendix II: A class of discrete G functions

- Let $G_{m,\boldsymbol{z},\boldsymbol{p}} \in \mathcal{G}$ take $-z_m, \dots, -z_1, 0, z_1, \dots, z_m \in \mathbb{R}$ with probs $p_m, \dots, p_1, p_0, p_1, \dots, p_m \in \mathbb{R}_+$ s.t. $\sum_{i=1}^m p_i z_i^2 = 1/2$.
- Then $\kappa_{G_{m,\boldsymbol{z},\boldsymbol{p}}}(C)$ and $\sigma^2_{G_{m,\boldsymbol{z},\boldsymbol{p}}}(C)$ admits

$$\kappa_{G_{m,\boldsymbol{z},\boldsymbol{p}}}(C) = \sum_{(i,j)\in\{-m,\dots,m\}} z_i z_j V_C(I_i \times I_j),$$

$$\sigma^2_{G_{m,\boldsymbol{z},\boldsymbol{p}}}(C) = \sum_{(i,j)\in\{-m,\dots,m\}} z_i^2 z_j^2 V_C(I_i \times I_j) - \left(\sum_{(i,j)\in\{-m,\dots,m\}} z_i z_j V_C(I_i \times I_j)\right)^2,$$

where $\quad z_{-i} = -z_i, \quad p_+ = p_1 + \cdots + p_m,$

$$I_{-i} = [p_+ - \sum_{j=1}^{i} p_j, p_+ - \sum_{j=1}^{i-1} p_j], \quad I_0 = [p_+,\ p_+ + p_0],$$

$$I_i = [p_+ + p_0 + \sum_{j=1}^{i-1} p_j,\ p_+ + p_0 + \sum_{j=1}^{i} p_j], \quad i = 1, \dots, m.$$

Appendix III: Joint distribution of (X^2, Y^2)

- By radial symmetry of G,

$$G^{[2]}(x) = \mathbb{P}(X^2 \leq x) = 2G(\sqrt{x}) - 1, \quad x \geq 0.$$

- Suppose G is continuous. Then the copula of (X^2, Y^2) is

$$C^{[2]}(u,v) = \sum_{\varphi \in \{\iota, \nu_1, \nu_2, \nu_1 \circ \nu_2\}} \bar{C}_\varphi\left(\frac{1}{2}, \frac{1}{2}\right) C_{\varphi,(1/2,1/2)}\left(\frac{u+1}{2}, \frac{v+1}{2}\right),$$

where $\nu_1(C)(u,v) = v - C(1-u, v), \quad \nu_2(C)(u,v) = u - C(u, 1-v),$
$C_{\varphi,(1/2,1/2)}(u,v) = \mathbb{P}\left(U_\varphi \leq u,\ V_\varphi \leq v \mid U_\varphi > 1/2,\ V_\varphi > 1/2\right),$
$(U_\varphi, V_\varphi) \sim C_\varphi = \varphi(C).$

- **Examples**: $M^{[2]} = W^{[2]} = M$ and $\Pi^{[2]} = \Pi$.

Appendix IV: Linearity of $C \mapsto \sigma_G^2(C)$

Suppose $G \in \mathcal{G}_4$ is continuous. For $p \in [0,1]$ and $C, C' \in \mathcal{C}_2$,

$$\tilde{C}_p^{[2]} = pC^{[2]} + (1-p){C'}^{[2]} \quad \text{where} \quad \tilde{C}_p = pC + (1-p)C'.$$

This yields the following proposition.

Proposition (Linearity of $\sigma_G^2(C)$)

For any $G \in \mathcal{G}_4$ and $k \in [-1,1]$, the map $C \mapsto \sigma_G^2(C)$ is linear on $\mathcal{C}_G(k) = \{C \in \mathcal{C}_2 : \kappa_G(C) = k\}$, that is,

$$\sigma_G^2(pC + (1-p)C') = p\sigma_G^2(C) + (1-p)\sigma_G^2(C')$$

for $p \in [0,1]$ and $C, C' \in \mathcal{C}_2$ s.t. $\kappa_G(C) = \kappa_G(C') = k$.