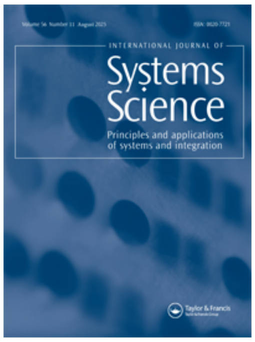

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Static output feedback strategy for mean-field social control with nonlinear stochastic dynamics

Hiroaki Mukaidani[a], Hua Xu[b] and Weihua Zhuang[c]

[a]Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima, Japan; [b]Graduate School of Business Sciences, The University of Tsukuba, Bunkyo-ku, Tokyo, Japan; [c]Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, ON, Canada

ABSTRACT
A mean-field social control problem for uncertain nonlinear stochastic systems is investigated by using a robust static output feedback (SOF) strategy. First, the problem in the single decision maker case is investigated in terms of guaranteed cost control approaches to derive suboptimal conditions at the supremum of the cost function. The Karush-Kuhn-Tucker (KKT) condition is used to derive the necessary conditions which are expressed as a large stochastic combined matrix equation (SCME). Second, the preliminary results in the single decision maker case are used to study the Pareto optimal strategy in a cooperative game. As our main contribution, we derive the high-order centralised strategies and the low-order decentralised strategies, respectively, for the cooperative game. In order to avoid the difficulty of higher-order dimensional problem related to SCMEs, a new reduced-order decomposition numerical scheme by means of Newton's method is developed. The computation for designing the proposed strategy set can be performed in low dimension, even when the number of decision makers approachs to infinity. Moreover, the degradation of the cost function is rigorously evaluated by comparing the centralised strategy set with the proposed strategy set. Finally, several numerical experiments are conducted to demonstrate the usefulness and effectiveness of the proposed strategy set.

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Mean-field social control; multiple decision makers; reduced-order technique; computational algorithm; uncertain nonlinear stochastic system

1. Introduction

1.1. Motivation and challenges

In control and dynamic game problems, it is difficult to design state feedback strategies when all states cannot always be measured. In such cases, observer-based state estimation is effective, and static output feedback (SOF) strategies are also known to be practical (Kučera & De Souza, 1995). Many researchers have widely adopted algorithms based on SOF strategies and have achieved the expected results. Game-theoretic approaches to various optimisation problems have also attracted significant attention (see, for example, Gadewadikar et al. (2006), Movric et al. (2015) and Zhu et al. (2015) and the references therein). Several researchers have extensively studied dynamic mean-field game problems (Caines et al., 2017; Huang, Caines et al., 2007). In Moon and Başar (2017), zero-sum mean-field difference game problem was studied for general stochastic systems. In Moon and Başar (2018), the mean-field Stackerberg dynamic game with an open-loop control law was studied. In Ma and Huang (2020), the asymptotic analysis problem was addressed for linear quadratic (LQ) Nash games with a major player and N minor players. In Xu and Zhang (2020), the finite horizon ε-Nash mean-field LQ game for stochastic systems with many decision makers was addressed. Moreover, the infinite horizon Pareto-based strategies were studied to solve the optimisation problem of the supremum of the cost function via linear matrix inequality (LMI) techniques (Y. Lin et al., 2018). Furthermore, necessary and sufficient conditions for the existence of the Pareto solution for infinite horizon nonlinear mean field probability systems associated with cooperative dynamic games were established (Y. Lin, 2020). The explicit conditions for feedback saddle point equilibria in LQ stochastic zero-sum dynamic games for mean-field stochastic systems were derived (Moon, 2020). However, all of the above studies are based on state feedback strategies.

CONTACT Hiroaki Mukaidani mukaida@hiroshima-u.ac.jp Graduate School of Advanced Science and Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan

On the other hand, dynamic game problems based on SOF strategies have become a popular research topic (Engwerda & Weeren, 2008; Movric et al., 2015; Zhu et al., 2015). The disturbance attenuation problem based on saddle-point equilibrium with cooperative distributed SOF in multi-agent systems has been studied (Movric et al., 2015). Necessary and sufficient conditions for the existence of an SOF Nash equilibrium have been established (Engwerda & Weeren, 2008). Meanwhile, a new dynamic output feedback (DOF) control strategy was presented to solve the linear dynamic zero-sum game problem (Rizvi & Lin, 2020). However, it should be noted that the implementation of DOF strategies is generally complicated and costly. As with stabilisation by the usual LQ optimal control, when considering dynamic game problems, the SOF strategy is particularly effective when full-state measurements are not available or are too prohibitively expensive to obtain (Mukaidani, Sagara et al., 2024). Given the limited number of output measurements that can be used to design a controller that stabilises the stochastic system and achieves the desired control objective, its active use in solving dynamic game problems should be considered; however, it is important to note that SOF has limitations compared to full-state feedback control because it uses only a portion of the total state measurements. In particular, with respect to finding feedback gains, the development of numerical algorithms remains a challenging problem that has not yet been fully solved, despite the existence of several useful algorithms based on nonlinear optimisation approaches (Ebihara et al., 2015; Sadabadi & Peaucelle, 2016). However, in scenarios where complete state information is not accessible, SOF remains a reliable approach. Therefore, the application of SOF to dynamic game problems is very interesting. For these reasons, a robust SOF strategy for incentive Stackelberg games with a large number of decision makers for mean-field stochastic systems has recently been solved (Mukaidani, Irie et al., 2022). In particular, if the player size is sufficiently large, a decentralised strategy that relies only on local information can be implemented instead of a centralised strategy. Furthermore, the cost degradation between the centralised SOF-incentive Stackelberg strategy and the approximate SOF-incentive Stackelberg strategy was also determined. However, when the number of decision makers in a mean-field system becomes very large, the curse of dimensionality makes computation in polynomial time physically impossible.

1.2. Main contributions

In recent years, many useful and reliable results have been obtained for practical applications of SOF control. For example, a composite controller consisting of SOF and nonlinear compensation was constructed as a new control mechanism. Reliable SOF control of time-varying delay systems with sensor failures was addressed in Shen, Lim et al. (2017). Mismatched quantised H_∞ output feedback control of fuzzy Markov jump systems via a dynamic guaranteed cost triggering scheme was investigated in Shen, Gu et al. (2024). In this paper, we study the implementation of SOF in mean-field stochastic systems with the goal of designing actual SOF strategies. In particular, we addresse the SOF problem when the number of players becomes very large. The existence of strategies and the asymptotic structure of the solution are clarified when the population size, N, goes to infinity. In existing works (Moon & Başar, 2017, 2018), the asymptotic properties of the cost function has been well studied, but little attention has been paid to strategy design when N goes to infinity. In this work, we show that the theory of weakly coupled systems can be used to implement a robust strategy (30) even when the number of players is very large, as in SIR infected systems (Armbruster & Beck, 2017).

In this paper, a robust SOF strategy for mean-field games of nonlinear stochastic systems with a large number of players is dealt with as an extension of the existing result in Kikuchi et al. (2022). To the best of our knowledge, the mean-field games for uncertain nonlinear stochastic systems in terms of the SOF strategy have not been studied so far. It is supposed that the nonlinear part of the original stochastic system includes the un-modelled dynamics and it is constrained in terms of a norm-bounded function. Nonlinear large-scale stochastic systems with higher-order interconnections based on the SOF strategies have been solved by LMI optimisation (Mukaidani, 2009), but this is the first time to consider mean-field stochastic games with a many decision makers.

The main contributions of the present study are as follows: First, we introduce some essential preliminary results related to SOF Pareto strategy in terms of GCC methods (Moheimani & Petersen, 1996). In general, LMI techniques are powerful tools when considering GCC problems, but are undesirable in mean-field stochastic systems due to their high dimensionality and computational complexity. Moreover, it is well known that implementing SOF control is very difficult, both theoretically and numerically. This is because the design of SOF control leads to a non-convex optimisation problem, since the output feedback matrix must satisfy both stability and performance criteria. Unlike state feedback, where linear matrix inequality (LMI) techniques can provide convex conditions for controller design, designing a SOF strategy often results in a bilinear matrix inequality (BMI) that is much more difficult to solve optimally. Most BMI approaches (e.g. iterative LMI approaches) do not guarantee global optimality, and their convergence depends on characteristics of the system. Currently, solving the non-convex BMI problems that arise in SOF relies on sophisticated optimisation algorithms that are computationally expensive and do not guarantee convergence to a global solution. Algorithms such as the branch and bound algorithm, alternating projection method, matrix valued sum-of-squares (SOS) techniques or gradient-based approaches may be able to produce viable solutions, but are often stuck in local minima and the computational load increasess rapidly with system size (Gadewadikar et al., 2006). As a result, SOF control remains a challenging area, both theoretically and in terms of practical, scalable implementation. To deal with these difficulties, in this study, we introduce a centralised diagonal output feedback control problem specific to mean-field systems and transforms it into the problem of finding individual SOF feedback gains. After setting the existence condition for the upper bound of the cost function, we use the Karush–Kuhn–Tucker (KKT) condition to solve the associated optimisation problem. In addition to the optimisation conditions, numerical algorithms are discussed in detail. It is theoretically pointed out that the existing algorithms in Gadewadikar et al. (2006); Kučera and De Souza (1995) do not require the selection of an initial gain, but do not guarantee the optimum of the cost bound. Furthermore, it was shown that the optimum is not guaranteed in simple examples. In contrast, the proposed composite algorithm based on Newton's method and the existing sequential algorithm does not require the selection of initial gains, and the optimum is guaranteed. Second, large-scale stochastic coupled-matrix equations (SCMEs) are derived and their solution is used to obtain a centralised strategy set. Nevertheless, we need to avoid the important issue of a large number of the decision makers. In this sequel, we apply a decomposition technique to compute the exact strategy set. The results show that each strategy set can be computed with the same dimension as that of each player. Third, a reduced-order SOF strategy using the feature of the uniform feature of the mean-field system is proposed. The mean-field game theory has been studied by using two approaches, the direct approach (Huang & Yang, 2021b, 2021a; Huang & Zhou, 2020; Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020), and the fixed point approach (see Caines et al., 2017; Huang, Caines et al., 2007, and the references therein for details). In this study, the direct approach based on the weakly coupled system theory is applied (Gajic et al., 2008). We first derive a set of SCMEs by using the preliminary optimisation result for designing a centralised SOF strategy. It should be noted that the approximation approach for the mean-field systems are not fully investigated in the existing results (Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020). On the contrary, our approach has a limitation that requires a mean field measurement. It should be noted that this can be designed in a decentralised manner compared to the case of using the existing direct approach (Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020). Although our result is similar to those in Huang and Zhou (2020), Huang and Yang (2021b) and Huang and Yang (2021a), the direct decomposition technique is considered to establish the existence conditions for the SCMEs. Moreover, an infinite horizon case with the SOF strategy set is investigated for the first time.

The remainder of this paper is organised as follows. In Section 2, we present some of the preliminary results to be used. Section 3 defines the problem of mean-field social control for uncertain nonlinear stochastic systems, along with a review of existing algorithms for solving the SOF problem. In Section 4, we propose a decentralised scheme to reduce the complexity due to the large number of agents. In addition, we present several numerical algorithms and their convergence results. In Section 5, we establish an approximate strategy set that ignores small parameter $\varepsilon = 1/N$ and derive the cost degradation. In Section 6, we present numerical examples to demonstrate the effectiveness of the proposed method. Section 7 concludes this study.

Notation: The following definitions are used through out the paper for simplification and mathematical notation: The symbol $\#\mathbb{A}$ stands for the number of elements in the set $\mathbb{A}$; The symbol $\mathbb{E}[\cdot]$ stands for the mathematical expectation, and The symbol **Tr** stands for trace of a matrix; The symbol block diag stands for the block diagonal matrix; The symbol $\mathbf{vec}M$ stands for an ordered stack of the columns of M; The symbol $\otimes$ stands for Kronecker product; The symbol $\mathbb{U}_{pm}$ stands for a commutation matrix in Kronecker matrix sense such that $\mathbb{U}_{pm}\text{vec}M^\top = \text{vec}M,\ M \in \mathbb{R}^{m\times p}$; The symbol $\|\cdot\|$ stands for the Euclidean norm of a real vector; I_n denotes the identity matrix of size n; $\boldsymbol{J}_n$ denotes the all-ones matrix of size $\mathbb{R}^{n\times n}$.

2. Preliminary results

First, we discuss some important preliminary results before discussing the main results. Consider the following nonlinear stochastic system with norm-bounded uncertainty:

$$\mathrm{d}x(t) = \{Ax(t) + Bu(t) + Gf(t, x(t))\}\,\mathrm{d}t + \sum_{k=1}^{N} A_{kp}x(t)\mathrm{d}w_k(t), \tag{1a}$$

$$y(t) = C_c x(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measurement output, $f(t, x(t)) \in \mathbb{R}^n$ is the unknown nonlinear vector function, and $w_i(t) \in \mathbb{R}$, $i = 1, \ldots, N$ is a standard Wiener process with a filtered probability space. It is supposed that the stochastic processes, $w_i(t)$, $i = 1, \ldots, N$, are mutually independent. A, A_{ip}, $i = 1, \ldots, N$, B, G, and C, are known constant coefficient matrices with appropriate dimensions.

The following assumption is introduced for the norm-bounded uncertainty $f(t, x(t))$:

Assumption 2.1: There exists known constant matrix W for all $x(t) \in \mathbb{R}^n$, such that

$$\|f(t, x(t))\| \le \|Wx(t)\|. \tag{2}$$

It should be noted that $f(t, x(t))$ of Assumption 2.1 is a piecewise-continuous nonlinear function in $x(t)$. Furthermore, $f(t, x(t))$ is assumed to be uncertain, but bounded by a quadratic inequality (Zecevic et al., 2004). Therefore, since a dominating linear part, known to the designer, has already been separated such that $Ax(t) + Bu(t)$ is as described in Equation (1a), the problem considered still faces difficulty in establishing the mean-square stable nature of the stochastic system.

Remark 2.1: Consider the following uncertain linear stochastic system, instead of stochastic system (1a):

$$\begin{aligned}
\mathrm{d}x(t) &= [(A + G\eta(t)H)x(t) + Bu(t)]\ \mathrm{d}t + \sum_{k=1}^{N} A_{kp}x(t)\,\mathrm{d}w_k(t) \\
&= [Ax(t) + Bu(t) + G\eta(t)Hx(t)]\ \mathrm{d}t + \sum_{k=1}^{N} A_{kp}x(t)\,\mathrm{d}w_k(t)
\end{aligned}$$

$$\eta^\top(t)\eta(t) \le I_s, \quad \forall\, t \in [0, \infty), \tag{3}$$

where H is the given constant matrix with appropriate dimension. Then, $f(t, x(t)) = \eta(t)Hx(t)$ and $W^\top W = H^\top H$ in (2) is satisfied.

On the other hand, the case of $G = 0$ is the same as that for mean-field linear stochastic systems. As a result, by using the existing method presented in Mukaidani, Sagara et al. (2024), such a problem can be solved. Although the idea based on the SOF strategy is the same as that in the present study, Mukaidani, Sagara et al. (2024) considered the Nash equilibrium strategy, which is well-known in non-cooperative games. Therefore, the results obtained in this study are significantly different.

Based on the control design, the following cost function is defined:

$$J(x(0), u(t)) = \mathbb{E}\left[\int_0^\infty \left\{x^\top(t)Qx(t) + u^\top(t)Ru(t)\right\}\,\mathrm{d}t\right], \tag{4}$$

where $Q = Q^\top > 0$ and $R = R^\top > 0$.

Because deterministic uncertainty (2) is considered here, the exact value of the cost function cannot be computed. Therefore, we consider optimising the upper bound of the cost function as in the case of cost-guaranteed control (Moheimani & Petersen, 1996).

Definition 2.1: Consider the nonlinear stochastic system with norm-bounded uncertainty (1a). If there exist a control law $u^*(t)$ and a positive scalar $J^*(x(0), u^*(t))$ such that, for all admissible uncertainties (2), the closed-loop system is mean-square stable and the closed-loop value of the cost function (4) satisfies $J(x(0), u(t)) \leq J^*(x(0), u^*(t))$, then $J^*(x(0), u^*(t))$ is said to be a supremum of the cost function and $u^*(t)$ is said to be a guaranteed cost control law.

If the state variables are not available, then the following SOF control law is considered instead:

$$\begin{aligned} u(t) &= F_c y(t) = F_c C_c x(t) \\ &= \text{block diag}\begin{pmatrix} F_1 & \cdots & F_N \end{pmatrix} C_c x(t) \\ &= \sum_{k=1}^{N} \xi_k^1 F_k \xi_k^{2\top} C_c x(t), \end{aligned} \tag{5}$$

where

$$\xi_k^1 = \begin{bmatrix} 0 \\ \vdots \\ I_{m_k} \\ \vdots \\ 0 \end{bmatrix}, \quad m = \sum_{k=1}^{N} m_k, \quad \xi_k^2 = \begin{bmatrix} 0 \\ \vdots \\ I_{p_k} \\ \vdots \\ 0 \end{bmatrix},$$

$$p = \sum_{k=1}^{N} p_k.$$

In this case, the problem considered is to find the SOF strategy set (5) such that the closed-loop uncertain nonlinear stochastic system is mean-square stable and the supremum on the quadratic cost function in (4) should be minimised.

Consider $\mathbb{E}[x(0)x^\top(0)] = \Lambda \neq I_n$ for initial condition $x(0)$. When dealing with SOF problems, in order to eliminate dependence on initial values, $\mathbb{E}[x(0)x^\top(0)] = I_n$ has been assumed (Moheimani & Petersen, 1996). However, this assumption implies that the expectation of the initial values of each other's players is 0. In general, such an assumptions is idealised and conservative. Therefore, in order to treat them more generally, we make the assumption that $\mathbb{E}[x(0)x^\top(0)] = \Lambda \neq I_n$. It should be noted that this assumption significantly changes the results of the optimisation problem with the bound of the cost function. In particular, the structure of Lagrange multiplier S is clearly different from the conventional results to be described later.

The following important preliminary has already been derived in Kikuchi et al. (2022).

Lemma 2.1. *Consider the stochastic nonlinear system in (1a). Assume there exist the matrices $P = P^\top > 0$ and $S = S^\top > 0$, and positive scale constant parameter λ, such that the SCMEs (6a) with $C_c S C_c^\top > 0$ hold:*

$$\begin{aligned} \Sigma_1(F_c, P, \lambda) &= P(A + BF_cC_c) + (A + BF_cC_c)^\top P \\ &\quad + \sum_{k=1}^{N} A_{kp}^\top P A_{kp} \\ &\quad + \lambda P \Omega P + \frac{1}{\lambda} W^\top W + Q \\ &\quad + C_c^\top F_c^\top R F_c C_c = 0, \end{aligned} \tag{6a}$$

$$\begin{aligned} \Sigma_2(S, F_c, \lambda) &= S(A + BF_cC_c + \lambda \Omega P)^\top \\ &\quad + (A + BF_cC_c + \lambda \Omega P) S \\ &\quad + \sum_{k=1}^{N} A_{kp} S A_{kp}^\top + \Lambda = 0, \end{aligned} \tag{6b}$$

$$\begin{aligned} F_i &= -R^{-1} \xi_i^{1T} B^\top P S C_c^\top \xi_i^2 \\ &\quad \left(\xi_i^{2\top} C_c S C_c^\top \xi_i^2 \right)^{-1}, \end{aligned} \tag{6c}$$

$$\lambda = \sqrt{\frac{\mathbf{Tr}[SW^\top W]}{\mathbf{Tr}[SP\Omega P]}}, \tag{6d}$$

where $\Omega = GG^\top$. If the following three assumptions hold:

(i) $\left[A + BF_cC_c + \lambda \Omega P,\ A_{1p},\ \ldots,\ A_{Np} \,\middle|\, \sqrt{\Lambda} \right]$ *is exactly observable,*
(ii) $(A + BF_cC_c + \lambda \Omega P,\ A_{1p},\ \ldots,\ A_{Np})$ *is stable,*
(iii) $\text{rank}\left(\frac{\partial}{\partial F_c} \Sigma_1^{ij}(\check{F}_c, \check{P}, \check{\lambda}) \right)_{j \in \mathbb{J}_0} = \#\mathbb{J}_0$, *where*

$$\mathbb{J}_0 = \left\{ i, j \in \{1, \ldots, n\},\ i \leq j \,\middle|\, \Sigma_1^{ij}(\check{F}_c, \check{P}, \check{\lambda}) = 0 \right\}, \tag{7a}$$

$$\Sigma_1(F_c, P, \lambda) = \begin{bmatrix} \Sigma_1^{11}(F_c, P, \lambda) & \cdots & \Sigma_1^{1n}(F_c, P, \lambda) \\ \vdots & \ddots & \vdots \\ \Sigma_1^{1n}(F_c, P, \lambda) & \cdots & \Sigma_1^{nn}(F_c, P, \lambda) \end{bmatrix} \tag{7b}$$

then, the SOF strategy is designed by

$$u^*(t) = F_c^* y(t) = F_c^* C_c x(t), \tag{8}$$

where the SOF gain $F_c = F_c^*$ can be computed by solving SCMEs (6a).

Furthermore, for the supremum of cost function (4), *the follow inequality is derived:*

$$J(x(0), u(t)) \leq J^*(x(0), u^*(t)) = \mathbf{Tr}\,[\Lambda P]. \quad (9)$$

Proof: The proof can be found in Kikuchi et al. (2022) for details. ■

It should be noted that Condition (iii) is well known as the singularity condition (Göllmann et al., 2009) or KKT condition; that is, the necessary conditions are applicable only if the assumption of the regularity of $(F_c, P, \lambda) = (F_c^*, P^*, \lambda^*)$ at a local optimal point is satisfied. This condition is the same as Condition (iii).

On the other hand, for more information on stable and exactly observable, reference (Y. Lin et al., 2018) is helpful to the readers. Note also that when Conditions (i) and (ii) hold true, asymptotically mean square stable (AMSS) nature of the stochastic system is guaranteed because $P > 0$ is satisfied if P exists.

The derivation, under the assumption that nonlinear term $f(t, x(t))$ is norm bounded uncertainty, is standard. However, the derivation of output feedback gain F_c in a diagonal matrix form is novel. Each element of the output feedback gain can be computed in a small dimension by using this novel concept.

To design SOF gain F_c in (8), the SCMEs in (6a) should be solved. It is well-known that it is difficult to solve the SOF control problem and there exist many feasible algorithms that have been developed in Kučera and De Souza (1995) and Gadewadikar et al. (2006). It should be noted that these algorithms have the feature that does not require the initial stabilising gain. When the algorithm via the core idea is applied, the following algorithm can be utilised:

$$\begin{aligned} 0 &= P^{(\ell+1)}A + A^\top P^{(\ell+1)} \\ &\quad + \sum_{k=1}^{N} A_{kp}^\top P^{(\ell+1)} A_{kp} \\ &\quad + P^{(\ell+1)} \left(\lambda^{(\ell)}\Omega - U\right) P^{(\ell+1)} \\ &\quad + \frac{1}{\lambda^{(\ell)}} W^\top W \\ &\quad + Q + L^{(\ell)T} R^{-1} L^{(\ell)}, \end{aligned} \quad (10a)$$

$$\begin{aligned} 0 &= S^{(\ell+1)} \left[A + (\lambda^{(\ell)}\Omega - U)P^{(\ell+1)} \right. \\ &\quad \left. + BR^{-1}L^{(\ell)}\right]^\top \\ &\quad + \left[A + (\lambda^{(\ell)}\Omega - U)P^{(\ell+1)} \right. \\ &\quad \left. + BR^{-1}L^{(\ell)}\right] S^{(\ell+1)} \\ &\quad + \sum_{k=1}^{N} A_{kp} S^{(\ell+1)} A_{kp}^\top + \Lambda, \end{aligned} \quad (10b)$$

$$F_c^{(\ell+1)} = -R^{-1}B^\top P^{(\ell+1)} S^{(\ell+1)} C_c^\top \left(C_c S^{(\ell+1)} C_c^\top\right)^{-1}, \quad (10c)$$

$$L^{(\ell+1)} = RF_c^{(\ell+1)} C_c + B^\top P^{(\ell+1)}, \quad (10d)$$

$$L^{(0)} = 0, \quad \lambda^{(0)} = 1, \quad \ell = 0, 1, 2, \ldots. \quad (10e)$$

where $U = BR^{-1}B^\top$.

It should be noted that it Algorithm (10a) can be easily conceived of by using the key properties presented in the literatures (Gadewadikar et al., 2006; Kučera & De Souza, 1995), in which a combined algorithm for the Lyapunov equation that is independent of the initial condition was proposed, and this part is related to Algorithm (10a). On the other hand, note also that, different from the previous results (Gadewadikar et al., 2006; Kučera & De Souza, 1995), the iterative algorithm in (10a) contains the information of Lagrange multiplier S. In particular, without this information, the optimality may not be achieved. The same feature can be observed for the existing algorithms (Gadewadikar et al., 2006; Kučera & De Souza, 1995). In fact, without the information of Lagrange multiplier S, it is confirmed that the minimisation cannot be achieved by using a simple numerical example in practice as presented in the section of numerical results. Finally, although Algorithm (10a) does not need the initial stabilising gain $F_c^{(0)}$, the convergence is not guaranteed owing to the absence of a rigorous proof.

3. Problem statement

In this section, the mean-field social control problem based on the Pareto suboptimal strategy is studied using the results obtained in the previous section. Consider the following nonlinear stochastic system with a sufficiently large N:

$$\mathrm{d}x_i(t) = \left\{Ax_i(t) + Bu_i(t) + Dx^{(N)}(t)\right.$$

$$+ Gf(t, x_i(t))\Big\}\,\mathrm{d}t$$

$$+ A_p x_i(t)\mathrm{d}w_i(t), \tag{11a}$$

$$y_i(t) = Cx_i(t),\ i = 1, \ldots, N, \tag{11b}$$

$$x^{(N)}(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t) \in \mathbb{R}^n. \tag{11c}$$

where for the ith subsystem, $x_i(t) \in \mathbb{R}^n$ is the state vector, $u_i(t) \in \mathbb{R}^m$ is the control input, $y_i(t) \in \mathbb{R}^p$ is the measurement output, $f(t, x(t)) \in \mathbb{R}^n$ is the unknown nonlinear vector functions, and $x^{(N)}(t) \in \mathbb{R}^n$ is the mean-field term (Moon & Başar, 2017, 2018; Wang et al., 2021a; Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020). Here N stands for number of the decision maker, which is assumed to be a sufficiently large given constant. It is well known that mean field game theory is the study on strategic decision making by small interacting agents in a very large population. For example, it focuses on large-population dynamic games involving nonlinear stochastic dynamical systems such as the SIR infection systems (Armbruster & Beck, 2017). In such a case, the population size can be represented as N which means sufficiently large. Without loss of generality, we assume that the population size, N, in the SIR system is very large.

Without loss of generality, it is assumed that initial states $x_i(0)$, $i = 1, \ldots, N$, are mutually independent and have the same mathematical expectation such that $\mathbb{E}[x_i(0)x_i^{\top}(0)] = \Lambda_d$. The considered mean-field stochastic system based on the existing models with mean-field term $x^{(N)}(t)$ was intensively studied in Moon and Başar (2017), Wang, Zhang and Zhang (2020), Wang and Zhang (2021) and Wang et al. (2021a). It can be applied to many practical plants such as residential power storage control, communication networks, and stochastic growth models (Caines et al., 2017).

It should be noted that, although the uniform stabilisation and social optimality for mean-field Pareto suboptimal strategies have been investigated for the systems with a many decision makers (Wang et al., 2021a; Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020), there are no known nonlinear dynamics. That is, the term, $f(t, x_i(t))$ in (11a), is unknown bounded nonlinear function representing any disturbance signals (Wang et al., 2021a; Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020). Therefore, this study deals with not only disturbance signals, but also uncertainties more generally. Furthermore, the SOF strategies have not been studied so far in the previous works.

It is assumed that the nonlinear part with unmodeled dynamics can be described as follows.

Assumption 3.1: There exists known constant matrix W for all $x_i(t) \in \mathbb{R}^n$, such that

$$\|f(t, x_i(t))\| \leq \|Wx_i(t)\|. \tag{12}$$

It should be noted that, in Wang et al. (2021b), unknown disturbances reflecting the impact of an external environment on each agent were taken into account. Furthermore, the uncertainty of the model appears in both the drift and diffusion terms of each agent (Liang et al., 2022). On the other hand, we investigate deterministic uncertainty expressed as norm-bounded. This is the crucial difference.

Furthermore, the essence of Assumptions 2.1 and 3.1 is the same. However, Assumption 2.1 is an assumption for general nonlinear stochastic systems based on existing results, whereas Assumption 3.1 is an assumption for the system studied here where each subsystem is uniform.

It is known that the model of quadrotor craft (Lin et al., 2021) can be represented by a known coupling of bounded periodic functions, without loss of generality, as Assumption 3.1 holds true. In this study, the following mean-field SOF strategy set is introduced:

$$u_i(t) = F_i y^{(N)}(t), \quad i = 1, \ldots, N, \tag{13}$$

where the mead-field measurement is given as follows:

$$y^{(N)}(t) = \frac{1}{N}\sum_{k=1}^{N} y_k(t) \in \mathbb{R}^p.$$

The cost functional which is referred to as the social cost in Wang, Zhang and Zhang (2020), Wang and Zhang (2021) and Wang et al. (2021a) is given by

$$J_{\mathrm{soc}}(\boldsymbol{x}(0), \boldsymbol{u}) = \sum_{k=1}^{N} J_k(x_k(0), u_k), \tag{14}$$

where

$$J_i(x_i(0), u_i)$$
$$= \mathbb{E}\Bigg[\int_0^{\infty} \Big\{\Big(x_i(t) - \Phi x^{(N)}(t)\Big)^{\top}$$

$$Q\left(x_i(t) - \Phi x^{(N)}(t)\right)$$

$$+ u_i^\top(t) R u_i(t)\Big\} \, \mathrm{d}t\Bigg],$$

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} \in \mathbb{R}^{nN}, \quad \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix} \in \mathbb{R}^{mN},$$

and Φ denotes the cost parameters.

Note that for each decision maker, the coefficient matrix of (11a) is the same as compared to (1a). Furthermore, it is assumed that the mean-field stochastic system has uniform structure and appropriate dimensions. This means that the coefficient matrices for the decision makers are the same.

The following centralised stochastic system is introduced instead of (11a):

$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t) &= [(A_c + \varepsilon \boldsymbol{J}_N \otimes D)\,\boldsymbol{x}(t) + B_c \boldsymbol{u}(t) \\ &\quad + G_c \boldsymbol{f}_c(t, \boldsymbol{x}(t))]\,\mathrm{d}t \\ &\quad + \sum_{k=1}^{N} A_{pk} \boldsymbol{x}(t)\,\mathrm{d}w_k(t), \qquad (15a)\end{aligned}$$

$$\begin{aligned}J_{\mathrm{soc}}(\boldsymbol{u}(t), \boldsymbol{x}(0)) &= \mathbb{E}\left[\int_0^\infty \left\{\boldsymbol{x}^\top(t) Q_c \boldsymbol{x}(t)\right.\right. \\ &\quad \left.\left. + \boldsymbol{u}^\top(t) R_c \boldsymbol{u}(t)\right\}\mathrm{d}t\right], \qquad (15b)\end{aligned}$$

$$\boldsymbol{u}(t) = F_c \boldsymbol{y}^{(N)}(t) = \varepsilon F_c C_c \boldsymbol{x}(t), \qquad (15c)$$

where

$$\boldsymbol{y}^{(N)}(t) = \varepsilon C_c \boldsymbol{x}(t) = \begin{bmatrix} y^{(N)}(t) \\ \vdots \\ y^{(N)}(t) \end{bmatrix},$$

$$\boldsymbol{f}_c(t, \boldsymbol{x}(t)) = \begin{bmatrix} f(t, x_1) \\ \vdots \\ f(t, x_N) \end{bmatrix},$$

$$A_c = I_N \otimes A,\ B_c = I_N \otimes B,\ G_c = I_N \otimes G,$$

$$C_c = J_N \otimes C,$$

$$A_{pi} = \textbf{block diag}\begin{pmatrix} 0 & \cdots & A_p & \cdots & 0 \end{pmatrix},$$

$$\varepsilon = \frac{1}{N}, \quad Q_c = \sum_{k=1}^{N} Q_k, \quad R_c = I_N \otimes R,$$

$$Q_i = \begin{bmatrix} q_1 & \cdots & q_{2i} & \cdots & q_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ q_{2i}^\top & \cdots & q_{3i} & \cdots & q_{2i} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ q_1 & \cdots & q_{2i}^\top & \cdots & q_1 \end{bmatrix},$$

$$q_1 = \varepsilon^2 \Phi^\top Q \Phi, \quad q_{2i} = q_1 - \varepsilon \Phi^\top Q,$$

$$q_{3i} = (I_n - \varepsilon \Phi^\top) Q (I_n - \varepsilon \Phi),$$

$$\begin{aligned}F_c &= \sum_{k=1}^{N} \delta_k^1 F_k \delta_k^{2\top} \\ &= \textbf{block diag}\begin{pmatrix} F_1 & \cdots & F_N \end{pmatrix},\end{aligned}$$

$$\delta_k^1 = \begin{bmatrix} 0 & \cdots & I_m & \cdots & 0 \end{bmatrix}^\top,$$

$$\delta_k^2 = \begin{bmatrix} 0 & \cdots & I_p & \cdots & 0 \end{bmatrix}^\top.$$

By centralising inequality (12) under Assumption 3.1, the following structure holds:

$$\|\boldsymbol{f}_c(t, \boldsymbol{x}(t))\| \le \|W_c \boldsymbol{x}(t)\|, \qquad (16)$$

where $W_c = I_N \otimes W$. In Kikuchi et al. (2022) the gain F_c corresponding to (15c) is assumed to be a diagonal form in the design of the decentralised SOF strategy. Therefore, the minimal supremum of (15b) calculated using the diagonal matrix of F_c is higher than the one in the case of a centralised matrix F_c of (15c), because in the optimisation scheme, one loses the off-diagnosed information of the feedback gain matrix F_c. On the contrary, it is well known that if the mean-field stochastic game with large decision maker game problem is considered, the centralised strategy strongly relies on the global information instead of the local information. From the global information perspective, the diagonal form under the local static information is assumed in Kikuchi et al. (2022), and the resulting analysis is not appropriate. In this study, the above-mentioned concerns are addressed. Finally, the following problem can be stated.

Problem 3.1: Design the SOF Pareto suboptimal strategy set $\boldsymbol{u}(t)$ that minimises the supremum of the cost functional in (15b), subject to (15a).

4. Full-order strategy

The full-order strategy set based on the centralised scheme is derived. Based on the previous section, the optimisation problem is stated:

$$J_{\text{soc}}(\boldsymbol{x}(0), \boldsymbol{u}(t)) \le J^*_{\text{soc}}(\boldsymbol{x}(0), \boldsymbol{u}^*(t)) = \mathbf{Tr}\,[\Lambda_N P_c], \tag{17}$$

where $A_{\text{CL}c} = A_c + \varepsilon \boldsymbol{J}_N \otimes D + \varepsilon B_c F_c C_c$,

$$\begin{aligned}
\Sigma_{1c}(P_c, F_c, \lambda_c) &= P_c A_{\text{CL}c} + A_{\text{CL}c}^\top P_c \\
&\quad + \sum_{k=1}^{N} A_{kp}^\top P_c A_{kp} + \lambda_c P_c \Omega_c P_c \\
&\quad + \frac{1}{\lambda_c} W_c^\top W_c + Q_c \\
&\quad + \varepsilon^2 C_c^\top F_c^\top R_c F_c C_c, \\
C_c^\top F_c^\top R_c F_c C_c &= \sum_{k=1}^{N} \bar{C}^\top F_k^\top R F_k \bar{C}, \\
B_c F_c C_c &= \sum_{k=1}^{N} B_k F_k \bar{C}, \\
B_i &= \begin{bmatrix} 0 & \cdots & B^\top & \cdots & 0 \end{bmatrix}^\top, \\
\bar{C} &= \begin{bmatrix} C & \cdots & C \end{bmatrix}, \\
\Omega_c &= G_c G_c^\top, \\
\mathbb{E}\left[\boldsymbol{x}(0)\boldsymbol{x}^\top(0)\right] &= \Lambda_N = \boldsymbol{J}_N \otimes \Lambda_o + (\Lambda_d - \Lambda_o) I_{nN}, \\
\mathbb{E}[x_i^\top(0) x_i(0)] &= \Lambda_d, \quad i = 1, \ldots, N, \\
\mathbb{E}[x_i^\top(0) x_j(0)] &= \Lambda_o, \quad i \ne j, \quad i, j = 1, \ldots, N.
\end{aligned}$$

As mentioned in Subsection 1.2 on the main contribution of this work, conventionally, when dealing with SOF problems, in order to eliminate dependence on initial values, $\mathbb{E}[\boldsymbol{x}(0)\boldsymbol{x}^T(0)] = I_{nN}$ was assumed (Moheimani & Petersen, 1996). However, this assumption implies that the expectation of initial values 0 of independent players. In general, such an assumption is idealised and conservative. Hence, we introduce the above assumption, which can handle more general cases.

Using the same technique as in Lemma 2.1, the necessary conditions for the optimisation problem regarding the right-hand side of (17) are derived by:

$$0 = \Sigma_{1c}(P_c, F_c, \lambda_c), \tag{18a}$$

$$0 = \Sigma_{2c}(S_c, F_c, \lambda_c), \tag{18b}$$

$$\begin{aligned}
F^* &= F_1 = \cdots = F_N \\
&= -(\varepsilon R)^{-1} B_i^\top P_c S_c \bar{C}^\top (\bar{C} S_c \bar{C}^\top)^{-1},
\end{aligned} \tag{18c}$$

$$\lambda_c = \sqrt{\frac{\mathbf{Tr}\left[S_c W_c^\top W_c\right]}{\mathbf{Tr}[S_c P_c \Omega_c P_c]}}, \tag{18d}$$

where the matrix S_c denotes the Lagrange multiplier and $\bar{C} S_c \bar{C}^\top > 0$,

$$\begin{aligned}
&\Sigma_{2c}(S_c, F_c, \lambda_c) \\
&= S_c (A_{\text{CL}c} + \lambda_c \Omega_c P_c)^\top + (A_{\text{CL}c} + \lambda_c \Omega_c P_c) S_c \\
&\quad + \sum_{k=1}^{N} A_{kp} S_c A_{kp}^\top + \Lambda_N.
\end{aligned}$$

Then, the full-order SOF strategy set is established from:

$$\boldsymbol{u}^*(t) = F_c^* \boldsymbol{y}^{(N)}(t) = \varepsilon F_c^* C_c \boldsymbol{x}(t) = \varepsilon \sum_{k=1}^{N} \delta_k^1 F^* \bar{C} \boldsymbol{x}(t). \tag{19}$$

In this case, the following equality holds:

$$J_{\text{soc}}(\boldsymbol{x}(0), \boldsymbol{u}^*(t)) = \min_{P_c, S_c, F_c, \lambda_c} \mathbf{Tr}\,[\Lambda_N P_c]. \tag{20}$$

To obtain a full-order strategy set, the SCME in (18a) must be solved. However, when the number of decision makers N goes to infinity, it is not possible to calculate the feasible solution. Therefore, using the uniform features in mean-field stochastic system, a reduced order decomposition algorithm is established.

4.1. Decomposition technique

In this subsection, a decomposition technique is investigated. The important feature of the considered technique is that the solution set can be calculated with the dimension of each subsystem and low SCMEs. Using the weakly-coupled systems theory (Gajic et al., 2008), the solution set P_c, S_c, and F_c are assumed to be the following structure, and coefficient matrix Q_c is defined:

$$F_c = I_N \otimes F,$$

$$P_c = \begin{bmatrix} P_d & \varepsilon P_o & \varepsilon P_o & \cdots & \varepsilon P_o \\ \varepsilon P_o & P_d & \varepsilon P_o & \cdots & \varepsilon P_o \\ \varepsilon P_o & \varepsilon P_o & P_d & \cdots & \varepsilon P_o \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_o & \varepsilon P_o & \varepsilon P_o & \cdots & P_d \end{bmatrix},$$

$$S_c = \begin{bmatrix} S_d & S_o & S_o & \cdots & S_o \\ S_o & S_d & S_o & \cdots & S_o \\ S_o & S_o & S_d & \cdots & S_o \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_o & S_o & S_o & \cdots & S_d \end{bmatrix},$$

$$Q_c = \begin{bmatrix} Q_d & \varepsilon Q_o & \varepsilon Q_o & \cdots & \varepsilon Q_o \\ \varepsilon Q_o & Q_d & \varepsilon Q_o & \cdots & \varepsilon Q_o \\ \varepsilon Q_o & \varepsilon Q_o & Q_d & \cdots & \varepsilon Q_o \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_o & \varepsilon Q_o & \varepsilon Q_o & \cdots & Q_d \end{bmatrix},$$

$$\varepsilon = \frac{1}{N}, \quad \mu = \frac{1}{N-1}. \tag{21}$$

It should be noted that the structure of Lagrange multiplier S_c is different from P_c under the assumption $\mathbb{E}[x(0)x^\top(0)] = \Lambda \neq I_n$. Substituting (21) into (18a), and performing a tedious algebraic calculation results in the following decentralised SCMEs:

$$\begin{aligned} 0 &= P_d[A + \varepsilon(D + BFC)] \\ &\quad + [A + \varepsilon(D + BFC)]^\top P_d \\ &\quad + A_p^\top P_d A_p + \varepsilon(1-\varepsilon) \\ &\qquad [P_o(D + BFC) + (D + BFC)^\top P_o] \\ &\quad + \frac{1}{\lambda_c} W^\top W + \lambda_c P_d \Omega P_d + \varepsilon(1-\varepsilon)\lambda_c P_o \Omega P_o \\ &\quad + Q_d + \varepsilon C^\top F^\top RFC = \mathbb{F}_1(F, P_d, P_o, \lambda_c, \varepsilon), \end{aligned} \tag{22a}$$

$$\begin{aligned} 0 &= P_d(D + BFC) + (D + BFC)^\top P_d \\ &\quad + P_o[A + (1-\varepsilon)(D + BFC) + \lambda_c \Omega P_d] \\ &\quad + [A + (1-\varepsilon)(D + BFC) + \lambda_c \Omega P_d]^\top P_o \\ &\quad + (1-2\varepsilon)\lambda_c P_o \Omega P_o + Q_o + C^\top F^\top RFC \\ &= \mathbb{F}_2(F, P_d, P_o, \lambda_c, \varepsilon), \end{aligned} \tag{22b}$$

$$\begin{aligned} 0 &= S_d(A + \varepsilon(D + BFC) + \lambda_c \Omega P_d)^\top \\ &\quad + (A + \varepsilon(D + BFC) + \lambda_c \Omega P_d) S_d \\ &\quad + A_p S_d A_p^\top + (1-\varepsilon) \\ &\qquad [S_o(D + BFC)^\top + (D + BFC) S_o] \\ &\quad + (1-\varepsilon)\lambda_c(S_o P_o \Omega + \Omega P_o S_o) + \Lambda_d \\ &= \mathbb{F}_3(F, P_d, P_o, S_d, S_o, \lambda_c, \varepsilon), \end{aligned} \tag{22c}$$

$$\begin{aligned} 0 &= \varepsilon[S_d(D + BFC)^\top + (D + BFC) S_d] \\ &\quad + S_o[A + (1-\varepsilon)(D + BFC) \\ &\quad + (1-2\varepsilon)\lambda_c \Omega P_o + \lambda_c \Omega P_d]^\top \\ &\quad + [A + (1-\varepsilon)(D + BFC) \\ &\quad + (1-2\varepsilon)\lambda_c \Omega P_o + \lambda_c \Omega P_d] S_o \\ &\quad + \lambda_c \varepsilon(S_d P_o \Omega + \Omega P_o S_d) + \Lambda_o \\ &= \mathbb{F}_4(F, P_d, P_o, S_d, S_o, \lambda_c, \varepsilon), \end{aligned} \tag{22d}$$

$$\begin{aligned} F &= -R^{-1}B^\top [\mu P_d S_d + P_d S_o \\ &\quad + \varepsilon P_o S_d + (1-\varepsilon) P_o S_o] \\ &\quad \times C^\top(\mu C S_d C^\top + C S_o C^\top)^{-1}, \end{aligned} \tag{22e}$$

$$\lambda_c = \sqrt{\frac{\mathbf{Tr}[S_d W^\top W]}{\mathbf{Tr}[\Xi]}}, \tag{22f}$$

where $\Xi = [S_d P_d + (1-\varepsilon) S_o P_o]\Omega P_d + (1-\varepsilon)[\varepsilon S_d P_o + S_o P_d + (1-2\varepsilon) S_o P_o]\Omega P_o$.

It should be noted that the following relation holds for (18a):

$$\begin{aligned} 0 &= \Sigma_{1c}(P_c, F_c, \lambda_c) \\ &= \begin{bmatrix} \mathbb{F}_1(F, P_d, P_o, \lambda_c, \varepsilon) & \cdots & \mathbb{F}_2(F, P_d, P_o, \lambda_c, \varepsilon) \\ \vdots & \ddots & \vdots \\ \mathbb{F}_2(F, P_d, P_o, \lambda_c, \varepsilon) & \cdots & \mathbb{F}_1(F, P_d, P_o, \lambda_c, \varepsilon) \end{bmatrix}. \end{aligned}$$

Moreover, other Equations (18a) have same feature. Note that, using the uniform feature, $F = F_1 = F_2 = \cdots = F_N$ holds. It should be noted that, instead of solving the large-scale SCMEs in (18a) with $\mathbb{R}^{nN \times nN}$ order, where N is sufficiently large, solving very low-dimensional reduced-order SCMEs in (22a) with $\mathbb{R}^{n \times n}$ order is required.

4.2. Asymptotic properties of a solution

In this subsection, the asymptotic properties of a solution set for the SCMEs in (22a) are investigated. The novel idea comes from the weakly coupled systems approach (Gajic et al., 2008). Letting $\varepsilon = 0$, the following decomposition SCMEs are derived:

$$\begin{aligned} 0 &= \bar{P}_d A + A^\top \bar{P}_d + A_p^\top \bar{P}_d A_p \\ &\quad + \frac{1}{\bar{\lambda}} W^\top W + \bar{\lambda} \bar{P}_d \Omega \bar{P}_d + Q_d, \end{aligned} \tag{23a}$$

$$\begin{aligned} 0 &= \bar{P}_o(A + D + B\bar{F}C + \bar{\lambda}\Omega \bar{P}_d) \\ &\quad + (A + D + B\bar{F}C + \bar{\lambda}\Omega \bar{P}_d)^\top \bar{P}_o \\ &\quad + \bar{P}_d(D + B\bar{F}C) + (D + B\bar{F}C)^\top \bar{P}_d \\ &\quad + \bar{\lambda} \bar{P}_o \Omega \bar{P}_o + Q_o + C^\top \bar{F}^\top R \bar{F} C, \end{aligned} \tag{23b}$$

$$
\begin{aligned}
0 &= \bar{S}_d(A+\bar{\lambda}\Omega\bar{P}_d)^\top + (A+\bar{\lambda}\Omega\bar{P}_d)\bar{S}_d + A_p\bar{S}_dA_p^\top \\
&\quad + \bar{\lambda}(\bar{S}_o\bar{P}_o\Omega + \Omega\bar{P}_o\bar{S}_o) \\
&\quad + \bar{S}_o(D+B\bar{F}C)^\top + (D+B\bar{F}C)\bar{S}_o + \Lambda_d, \qquad (23c)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \bar{S}_o[A+D+B\bar{F}C+\bar{\lambda}\Omega(\bar{P}_o+\bar{P}_d)]^\top \\
&\quad + [A+D+B\bar{F}C+\bar{\lambda}\Omega(\bar{P}_o+\bar{P}_d)]\bar{S}_o + \Lambda_o, \qquad (23d)
\end{aligned}
$$

$$
\bar{F} = -R^{-1}B^\top(\bar{P}_d+\bar{P}_o)\bar{S}_oC^\top(C\bar{S}_oC^\top)^{-1}, \qquad (23e)
$$

$$
\bar{\lambda} = \sqrt{\frac{\mathbf{Tr}\,[\bar{S}_dW^\top W]}{\mathbf{Tr}\,[\bar{\Xi}]}}, \qquad (23f)
$$

where $\bar{\Xi} = (\bar{S}_d\bar{P}_d + \bar{S}_o\bar{P}_o)\Omega\bar{P}_d + (\bar{S}_o\bar{P}_d + \bar{S}_o\bar{P}_o)\Omega\bar{P}_o, C\bar{S}_dC^\top >$ $0\bar{\Xi} = (\bar{S}_d\bar{P}_d + \bar{S}_o\bar{P}_o)\Omega\bar{P}_d + (\bar{S}_o\bar{P}_d + \bar{S}_o\bar{P}_o)\Omega\bar{P}_o, C\bar{S}_dC^\top >$ 0 is assumed.

Theorem 4.1. *Suppose that parameter independent reduced-order SCMEs in (23a) have solution set $(\bar{F}, \bar{P}_d, \bar{P}_o, \bar{S}_d, \bar{S}_o)$ and $\bar{\lambda} > 0$. Suppose that*

$$
\Delta(\bar{z}, 0) \neq 0, \quad C\bar{S}_dC^\top > 0, \qquad (24)
$$

where

$$
\Delta(z,\varepsilon) = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & 0 & 0 & \Delta_{15} & \Delta_{16} \\
\Delta_{21} & \Delta_{22} & 0 & 0 & \Delta_{25} & \Delta_{26} \\
\Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} & \Delta_{35} & \Delta_{36} \\
\Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} & \Delta_{45} & \Delta_{46} \\
\Delta_{51} & \Delta_{52} & \Delta_{53} & \Delta_{54} & \Delta_{55} & 0 \\
\Delta_{61} & \Delta_{62} & \Delta_{63} & \Delta_{64} & 0 & \Delta_{66}
\end{bmatrix},
$$

$$
z = \begin{bmatrix}
\mathbf{vec}\,[P_d] \\ \mathbf{vec}\,[P_o] \\ \mathbf{vec}\,[S_d] \\ \mathbf{vec}\,[S_o] \\ \mathbf{vec}\,[F] \\ \lambda_c
\end{bmatrix},
$$

$$
\begin{aligned}
\Delta_{11} &= I_n \otimes [A+\varepsilon(D+BFC)+\lambda_c\Omega P_d]^\top \\
&\quad + [A+\varepsilon(D+BFC)+\lambda_c\Omega P_d]^\top \otimes I_n \\
&\quad + A_p^\top \otimes A_p^\top, \\
\Delta_{12} &= \varepsilon(1-\varepsilon)\Delta_{21}, \\
\Delta_{15} &= C^\top \otimes \left[(1-\varepsilon)P_oB + P_dB + C^\top F^\top R\right] \\
&\quad + [(1-\varepsilon)P_oB + P_dB \\
&\quad + C^\top F^\top R] \otimes C^\top]\mathbb{U}_{pm}, \\
\Delta_{16} &= \mathbf{vec}\left[-\lambda_c^{-2}WW^\top + P_d\Omega P_d\right. \\
&\quad \left. + \varepsilon(1-\varepsilon)P_o\Omega P_o\right], \\
\Delta_{21} &= I_n \otimes \left[(D+BFC)^\top + \lambda_cP_o\Omega\right] \\
&\quad + \left[(D+BFC)^\top + \lambda_cP_o\Omega\right] \otimes I_n, \\
\Delta_{22} &= I_n \otimes [A+(1-\varepsilon)(D+BFC)+\lambda_c\Omega P_d \\
&\quad + \lambda_c(1-2\varepsilon)\Omega P_o]^\top \\
&\quad + [A+(1-\varepsilon)(D+BFC)+\lambda_c\Omega P_d \\
&\quad + \lambda_c(1-2\varepsilon)\Omega P_o]^\top \otimes I_n, \\
\Delta_{25} &= C^\top \otimes \left[P_dB + (1-\epsilon)P_oB + C^\top F^\top R\right] \\
&\quad + \left[P_dB + (1-\epsilon)P_oB + C^\top F^\top R\right] \\
&\quad \otimes C^\top]\mathbb{U}_{pm}, \\
\Delta_{26} &= \mathbf{vec}\,[P_d\Omega P_o + P_o\Omega P_d + (1-2\varepsilon)P_o\Omega P_o], \\
\Delta_{31} &= \lambda_c[S_d \otimes \Omega + \Omega \otimes S_d], \\
\Delta_{32} &= (1-\varepsilon)\Delta_{41}, \quad \Delta_{33} = \Delta_{11}^\top, \\
\Delta_{34} &= (1-\varepsilon)[I_n \otimes (D+BFC+\lambda_c\Omega P_o) \\
&\quad + (D+BFC+\lambda_c\Omega P_o) \otimes I_n], \\
\Delta_{35} &= \varepsilon[(S_dC^\top) \otimes B + [B \otimes (S_dC^\top)]\mathbb{U}_{pm}] \\
&\quad + (1-\varepsilon)[(S_oC^\top) \otimes B \\
&\quad + [B \otimes (S_oC^\top)]\mathbb{U}_{pm}], \\
\Delta_{36} &= \mathbf{vec}[S_dP_d\Omega + \Omega P_dS_d \\
&\quad + (1-\varepsilon)(S_oP_o\Omega + \Omega P_oS_o)], \\
\Delta_{41} &= \lambda_c[S_o \otimes \Omega + \Omega \otimes S_o], \\
\Delta_{42} &= (1-2\varepsilon)\Delta_{41} + \varepsilon\Delta_{31}, \\
\Delta_{43} &= \varepsilon[I_n \otimes (D+BFC+\lambda_c\Omega P_o) \\
&\quad + (D+BFC+\lambda_c\Omega P_o) \otimes I_n], \\
\Delta_{44} &= \Delta_{22}^\top, \quad \Delta_{45} = \Delta_{35}, \\
\Delta_{46} &= \mathbf{vec}\,[(1-2\varepsilon)(S_oP_o\Omega + \Omega P_oS_o) \\
&\quad + S_oP_d\Omega + \Omega P_dS_o \\
&\quad + \varepsilon(S_dP_o\Omega + \Omega P_oS_d)], \\
\Delta_{51} &= [C(\mu S_d + S_o)] \otimes B^\top, \\
\Delta_{52} &= [(1-\varepsilon)CS_o + \varepsilon CS_d] \otimes B^\top, \\
\Delta_{53} &= \mu C \otimes (RFC + B^\top P_d) + C \otimes (B^\top P_o), \\
\Delta_{54} &= C \otimes [RFC + B^\top P_d + (1-\varepsilon)B^\top P_o], \\
\Delta_{55} &= (\mu CS_dC^\top + CS_oC^\top) \otimes R,
\end{aligned}
$$

$$\Delta_{61} = \lambda_c^2 \mathbf{vec}\,[S_d P_d \Omega + \Omega P_d S_d + (1-\varepsilon)(S_o P_o \Omega + \Omega S_o P_o)]\,,$$

$$\Delta_{62} = \lambda_c^2 (1-\varepsilon)\mathbf{vec}\,[S_o P_d \Omega + \varepsilon S_d P_o \Omega + (1-2\varepsilon) S_o P_o \Omega + \Omega[\varepsilon S_d P_o + S_o P_d + (1-2\varepsilon) S_o P_o]^{\top}\Big],$$

$$\Delta_{63} = \mathbf{vec}\left[\lambda_c^2\{P_d \Omega P_d + \varepsilon(1-\varepsilon) P_o \Omega P_o\} - WW^{\top}\right],$$

$$\Delta_{64} = (1-\varepsilon)\mathbf{vec}\left[\lambda_c^2 P_d \Omega P_o + P_o \Omega\{P_d + (1-2\varepsilon) P_o\}\right],$$

$$\Delta_{66} = 2\lambda_c \mathbf{Tr}[\Xi].$$

The structures of the reduced-order SCMEs in (22a) *can be described as* (25).

$$\|F - \bar{F}\| = O(\varepsilon), \quad \|P_d - \bar{P}_d\| = O(\varepsilon), \quad \|P_o - \bar{P}_o\| = O(\varepsilon), \quad \|S_d - \bar{S}_d\| = O(\varepsilon), \quad \|S_o - \bar{S}_o\| = O(\varepsilon), \quad |\lambda_c - \bar{\lambda}| = O(\varepsilon). \tag{25}$$

Proof: The proof can be found in Appendix 1. ■

4.3. Numerical algorithm via Newton's method

The decomposition numerical method by means of Newton's method is discussed because it is hard to solve the full-order SCMEs in (18a) due to the restricted computational work space. The detailed algorithm is given as follows:

$$\boldsymbol{z}^{(\ell+1)} = \boldsymbol{z}^{(\ell)} - [\Delta(\boldsymbol{z}^{(\ell)}, \varepsilon)]^{-1} \boldsymbol{F}\left(\boldsymbol{z}^{(\ell)}\right), \quad \ell = 0, 1, \ldots, \tag{26}$$

where

$$\boldsymbol{z}^{(\ell)} = \begin{bmatrix} \mathbf{vec}\left[P_d^{(\ell)}\right] \\ \mathbf{vec}\left[P_o^{(\ell)}\right] \\ \mathbf{vec}\left[S_d^{(\ell)}\right] \\ \mathbf{vec}\left[S_o^{(\ell)}\right] \\ \mathbf{vec}\left[F^{(\ell)}\right] \\ \lambda^{(\ell)} \end{bmatrix}, \quad \boldsymbol{z}^{(0)} = \begin{bmatrix} \mathbf{vec}[\bar{P}_d] \\ \mathbf{vec}[\bar{P}_o] \\ \mathbf{vec}[\bar{S}_d] \\ \mathbf{vec}[\bar{S}_o] \\ \mathbf{vec}[\bar{F}] \\ \bar{\lambda} \end{bmatrix},$$

$$\boldsymbol{F}(\boldsymbol{z}) = \begin{bmatrix} \mathbf{vec}\,[\mathbb{F}_1(F, P_d, P_o, \lambda_c, \varepsilon)] \\ \mathbf{vec}\,[\mathbb{F}_2(F, P_d, P_o, \lambda_c, \varepsilon)] \\ \mathbf{vec}\,[\mathbb{F}_3(F, P_d, P_o, S_d, S_o, \lambda_c, \varepsilon)] \\ \mathbf{vec}\,[\mathbb{F}_4(F, P_d, P_o, S_d, S_o, \lambda_c, \varepsilon)] \\ \mathbf{vec}\,[\mathbb{F}_5(F, P_d, P_o, S_d, S_o, \varepsilon)] \\ \mathbf{vec}\,[\mathbb{F}_6(P_d, P_o, S_d, S_o, \lambda_c, \varepsilon)] \end{bmatrix},$$

$$\mathbb{F}_5(F, P_d, P_o, S_d, S_o, \varepsilon) = RF(\mu C S_d C^{\top} + C S_o C^{\top}) + B^{\top}[\mu P_d S_d + P_d S_o + \varepsilon P_o S_d + (1-\varepsilon) P_o S_o]\, C^{\top},$$

$$\mathbb{F}_6(P_d, P_o, S_d, S_o, \lambda_c, \varepsilon) = \lambda_c^2 \mathbf{Tr}[\Xi] - \mathbf{Tr}[S_d W^{\top} W].$$

It should be noted that $\mathbb{F}_i(\cdot)$, $i = 1, \ldots, 4$ are defined on the right-hand side of (22a)–(22d). It should also be noted that, to attain easy derivation, equations $\mathbb{F}_5$ and $\mathbb{F}_6$ are defined instead of (22e) and (22f). The important feature of the reduced-order decomposition Newton's method is given in the following theorem.

Theorem 4.2. *Suppose that the assumption in Theorem* 4.1 *holds and there exists the feasible solution set of reduced-order zeroth SCMEs in* (23a). *Then, there exists a small parameter, σ, such that for all $\varepsilon \in (0, \sigma)$, Newton's method* (26) *attains the following quadratic convergence property:*

$$\left\|\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^*\right\| = O\left(\varepsilon^{2^\ell}\right), \quad \ell = 0, 1, 2, \ldots, \tag{27}$$

where $\boldsymbol{z}^*$ is the required solution set of (22a).

Proof: The proof can be found in Appendix 2. ■

Remark 4.1: As is well known, if the initial conditions are sufficiently close to the set of required solutions, the Newton's method achieves quadratic convergence. On the other hand, there may be a limit cycle. As a result, the iteration enters in an infinite loop and does not converge. In this study, instead of addressing this limitation, it is preferable that if the sequence of numbers is infinitely repeated, there is no solution to the SCMEs (18a) and it cannot be stabilised in the sense of a sufficient condition.

It is shown that, even though it is difficult to compute the solution of (18a) due to memory constraints, it can be computed by applying the new decomposition algorithm for the SCMEs in (22a). Furthermore, it is expected that the computational time can be significantly improved as the reduction in the number of dimensions for the large-scale SCMEs.

4.4. Evaluation of costs

For the previous subsection, the high-order solution set can be computed in terms of Newton's method. Considering this solution set, the following strategy set can be used:

$$\boldsymbol{u}^{(\ell)}(t) = F_c^{(\ell)}\boldsymbol{y}^{(N)}(t) = \varepsilon F_c^{(\ell)}C_c\boldsymbol{x}(t), \quad \ell = 1, 2, \ldots. \tag{28}$$

where $F_c^{(\ell)} = \textbf{block diag}\left(F^{(\ell)} \quad \cdots \quad F^{(\ell)}\right)$, $u_i^{(\ell)}(t) = F^{(\ell)}y^{(n)}(t)$. It should be noted that $F_1 = F_2 = \cdots = F_N$ holds because the uniform structure is assumed from the mean-field stochastic system. The following result shows the degradation rate of the cost between the centralised strategy (19) and proposed strategy (28).

Theorem 4.3. *For the high-order strategy set* (28), *the following relation holds:*

$$\phi(\ell) = \left|\frac{J_{\text{soc}}\left(\boldsymbol{x}(0), \boldsymbol{u}^*(t)\right)}{N} - \frac{J_{\text{soc}}\left(\boldsymbol{x}(0), \boldsymbol{u}^{(\ell)}(t)\right)}{N}\right| = O\left(\varepsilon^{2^\ell}\right). \tag{29}$$

Proof: The proof can be found in Appendix 3. ■

The obtained strategy set in (28) can be calculated accurately if the iteration is large enough. However, if the number of the decision makers is sufficiently large, designing a strategy that does not depend on N is very useful and practical. In the next section, we consider strategic design that does not depend on N.

5. Decentralized strategy set

Since the distributed strategy has been well-known to be very useful and practical in terms of the cost reduction, there are many reliable results. In this paper, we consider the decentralised strategy design based on the results of the previous section. Specifically, the reduced-order parameter-independent strategy set is considered to avoid the high-order dimensional problem. Adopt the following parameter-independent strategy set:

$$\tilde{\boldsymbol{u}}(t) = \bar{F}_c\boldsymbol{y}(t) = \bar{F}_cC_c\boldsymbol{x}(t) = I_N \otimes (\bar{F}C)\boldsymbol{x}(t), \tag{30}$$

where $\bar{F}$ can be obtained by (23e).

The following corollary shows the feature for using the approximate strategy set in (30).

Corollary 5.1. *Considering the approximate strategy set in* (30), *the following relation holds:*

$$\phi(0) = \left|\frac{J_{\text{soc}}(\boldsymbol{x}(0), \boldsymbol{u}^*(t))}{N} - \frac{J_{\text{soc}}(\boldsymbol{x}(0), \tilde{\boldsymbol{u}}(t))}{N}\right| = O(\varepsilon). \tag{31}$$

Proof: The proof can be found in Appendix 4. ■

It should be noted that most previous studies (see, e.g. Moon & Başar, 2018; Wang et al., 2021a; Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020 and references therein) are $O(\sqrt{\varepsilon}) = O(1/\sqrt{N}) = O(\sqrt{\varepsilon})$, not $O(\varepsilon) = O(1/N) = O(\varepsilon)$. While these proof methods are limited to showing $O(\sqrt{\varepsilon}) = O(1/\sqrt{N})$, it is worth pointing out that this study shows a stricter evaluation by using the weak couple system theory. In the next section of numerical examples, evaluation formula (31) is actually calculated and confirmed to be correct. To the best of our knowledge, there has never been an example of actually calculating the cost function for a sufficiently large N value.

Remark 5.1: The major differences between this present study and the existing results (Huang & Yang, 2021b) are as follows. The results of previous study were derived by analysing a low-dimensional system of ordinary differential equations obtained by applying the rescaling method to high-dimensional centralised solutions. As a result, when N goes to infinity, the mean-field limiting behaviour is derived from the centralised solution with a new perspective by adopting a concept called asymptotic solvability. Consequently, the suboptimality has been proved. In contrast, the newly proposed method uses the weakly-coupled system theory (Gajic et al., 2008) to assume the asymptotic structure of solution set (21) of the centralised SCMEs (18a) with $\varepsilon = 1/N$. It is worth noting that, after assuming $\varepsilon = 0$, one can derive by direct calculation the reduced dimensional SCMEs independent of the parameter ε. Finally, the asymptotic structure of the solution of the reduced dimensional SCMEs (22a) is used to prove the deterioration of the cost function for the first time. Another difference is that the previous result deals with the case of a finite time horizon, whereas this study deals with the case of an infinite time horizon. The difficulty associated with the infinite time horizon lies in guaranteeing the stochastic stability of the mean-field stochastic system.

Remark 5.2: In (13), the SOF strategy set is $u_i(t) = F_i y^{(N)}(t)$ based only on the mean-field term. However, the form of state feedback strategy for LQ mean field social control is typically

$$u_i = K_i y_i + F_i y^{(N)} = K_i C x_i + F_i C x^{(N)}. \quad (32)$$

If the SOF feedback form is modified as in (32), the same optimisation problem is considered.

Then, the centralised form is given by

$$\boldsymbol{u}(t) = K_c \boldsymbol{y}(t) + F_c \boldsymbol{y}^{(N)}(t) = (K_c C_d + \varepsilon F_c C_c)\boldsymbol{x}(t), \quad (33)$$

where

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_N(t) \end{bmatrix} \in \mathbb{R}^{pN},$$

$$C_d = I_N \otimes C,\ C_c = J_N \otimes C,$$

$$K_c = \sum_{k=1}^{N} \delta_k^1 K_k \delta_k^{2\top} = \textbf{block diag}\begin{pmatrix} K_1 & \cdots & K_N \end{pmatrix},$$

$$F_c = \sum_{k=1}^{N} \delta_k^1 F_k \delta_k^{2\top} = \textbf{block diag}\begin{pmatrix} F_1 & \cdots & F_N \end{pmatrix}.$$

In this case, the closed-loop stochastic system and the cost function can be obtained as

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= [\{A_c + \varepsilon \boldsymbol{J}_N \otimes D \\ &\quad + B_c(K_c C_d + \varepsilon F_c C_c)\}\boldsymbol{x}(t) \\ &\quad + G_c \boldsymbol{f}_c(t, \boldsymbol{x}(t))]\ \mathrm{d}t \\ &\quad + \sum_{k=1}^{N} A_{pk}\boldsymbol{x}(t)\,\mathrm{d}w_k(t), \end{aligned} \quad (34a)$$

$$\begin{aligned} &J_{\mathrm{soc}}(\boldsymbol{u}(t), \boldsymbol{x}(0)) \\ &= \mathbb{E}\Big[\int_0^\infty \boldsymbol{x}^\top(t)\Big\{Q_c + (K_c C_d + \varepsilon F_c C_c)^\top \\ &\quad \times R_c(K_c C_d + \varepsilon F_c C_c)\Big\}\boldsymbol{x}(t)\,\mathrm{d}t\Big]. \end{aligned} \quad (34b)$$

Finally, the considered optimisation problem is stated as

$$J_{\mathrm{soc}}(\boldsymbol{x}(0), \boldsymbol{u}(t)) \le J^*_{\mathrm{soc}}(\boldsymbol{x}(0), \boldsymbol{u}^*(t)) = \mathbf{Tr}\,[\Lambda_N P_c], \quad (35)$$

where

$$\begin{aligned} \Sigma_{1c}(P_c, F_c, \lambda_c) &= P_c A_{\mathrm{CL}c} + A_{\mathrm{CL}c}^\top P_c \\ &\quad + \sum_{k=1}^{N} A_{kp}^\top P_c A_{kp} \\ &\quad + \lambda_c P_c \boldsymbol{\Omega}_c P_c + \frac{1}{\lambda_c} W_c^\top W_c \\ &\quad + Q_c + (K_c C_d + \varepsilon F_c C_c)^\top \\ &\qquad R_c(K_c C_d + \varepsilon F_c C_c), \\ A_{\mathrm{CL}c} &= A_c + \varepsilon \boldsymbol{J}_N \otimes D + B_c(K_c C_d + \varepsilon F_c C_c). \end{aligned} \quad (36)$$

Using the same technique as in Lemma 2.1, the necessary conditions for this optimisation problem are established by

$$0 = \Sigma_{1c}(P_c, F_c, \lambda_c), \quad (37a)$$

$$\begin{aligned} 0 &= S_c(A_{\mathrm{CL}c} + \lambda_c \boldsymbol{\Omega}_c P_c)^\top + (A_{\mathrm{CL}c} + \lambda_c \boldsymbol{\Omega}_c P_c) S_c \\ &\quad + \sum_{k=1}^{N} A_{kp} S_c A_{kp}^\top + \Lambda_N, \end{aligned} \quad (37b)$$

$$0 = B_i^\top P_c S_c \bar{C}_0^\top + RK^* \bar{C}_0 S_c \bar{C}_0^\top + \varepsilon RF^* \bar{C} S_c \bar{C}_0^\top, \quad (37c)$$

$$0 = B_i^\top P_c S_c \bar{C}^\top + RK^* \bar{C}_0 S_c \bar{C}^\top + \varepsilon RF^* \bar{C} S_c \bar{C}^\top, \quad (37d)$$

$$\lambda_c = \sqrt{\frac{\mathbf{Tr}[S_c W_c^\top W_c]}{\mathbf{Tr}[S_c P_c \boldsymbol{\Omega}_c P_c]}}, \quad (37e)$$

where

$$K^* = K_1 = \cdots = K_N, \quad (38a)$$

$$F^* = F_1 = \cdots = F_N \quad (38b)$$

from the uniform assumption and $\bar{C} = \begin{bmatrix} C & \cdots & C \end{bmatrix}$, $\bar{C}_0 = \begin{bmatrix} 0 & \cdots & C & \cdots & 0 \end{bmatrix}$.

Consequently, substituting partitioned P_c and S_c into (18a), the following reduced order SCMEs set can be obtained:

$$\begin{aligned} 0 &= [A + BKC + \varepsilon(D + BFC)]^\top P_d \\ &\quad + P_d[A + BKC + \varepsilon(D + BFC)] + A_p^\top P_d A_p \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon(1-\varepsilon)[P_o(D+BFC)+(D+BFC)^\top P_o] \\
&+\frac{1}{\lambda}W^\top W+\lambda P_d\Omega P_d+\lambda\varepsilon(1-\varepsilon)P_o\Omega P_o+Q_d \\
&+C^\top K^\top RKC \\
&+\varepsilon(C^\top F^\top RKC+C^\top K^\top RFC+C^\top F^\top RFC),
\end{aligned} \tag{39a}
$$

$$
\begin{aligned}
0=&[A+BKC+(1-\varepsilon)(D+BFC)+\lambda\Omega P_d]^\top P_o \\
&+P_o[A+BKC+(1-\varepsilon)(D+BFC)+\lambda\Omega P_d] \\
&+P_d(D+BFC)+(D+BFC)^\top P_d \\
&+\lambda(1-2\varepsilon)P_o\Omega P_o \\
&+Qo+C^\top F^\top RKC+C^\top K^\top RFC \\
&+C^\top F^\top RFC,
\end{aligned} \tag{39b}
$$

$$
\begin{aligned}
0=&[A+BKC+\varepsilon(D+BFC)+\lambda\Omega P_d]S_d \\
&+S_d[A+BKC+\varepsilon(D+BFC)+\lambda\Omega P_d]^\top \\
&+A_pS_dA_p^\top \\
&+\lambda(1-\varepsilon)(\Omega P_oS_o+S_oP_o\Omega) \\
&+(1-\varepsilon)[(D+BFC)S_o+S_o(D+BFC)^\top] \\
&+\Lambda_d,
\end{aligned} \tag{39c}
$$

$$
\begin{aligned}
0=&[A+BKC+(1-\varepsilon)(D+BFC) \\
&+(1-2\varepsilon)\lambda\Omega P_o+\lambda\Omega P_d]S_o \\
&+S_o[A+BKC+(1-\varepsilon)(D+BFC) \\
&+(1-2\varepsilon)\lambda\Omega P_o+\lambda\Omega P_d]^\top \\
&+\varepsilon[S_d(D+BFC)^\top+(D+BFC)S_d] \\
&+\lambda\varepsilon(\Omega P_oS_d+S_dP_o\Omega)+\Lambda_o,
\end{aligned} \tag{39d}
$$

$$
\begin{aligned}
0=&RKC(\mu S_d+S_o)C^\top+RFC(\mu S_d+S_o)C^\top \\
&+B^\top[\mu P_dS_d+P_dS_o+\varepsilon P_oS_d+(1-\varepsilon)P_oS_o]C^\top,
\end{aligned} \tag{39e}
$$

$$
\begin{aligned}
0=&\mu RKCS_dC^\top+\varepsilon RFC(\mu S_d+S_o)C^\top \\
&+B^\top(\mu P_dS_d+\varepsilon P_oS_o)C^\top,
\end{aligned} \tag{39f}
$$

$$
\begin{aligned}
0=&\lambda^2\mathbf{Tr}\,[[S_dP_d+(1-\varepsilon)S_oP_o]\Omega P_d \\
&+(1-\varepsilon)[\varepsilon S_dP_o+S_oP_d+(1-2\varepsilon)S_oP_o]\Omega P_o] \\
&-\mathbf{Tr}[S_dWW^\top].
\end{aligned} \tag{39g}
$$

Then, by letting $\varepsilon=0$, the following parameter independent SCMEs can be derived:

$$
\begin{aligned}
0=&(A+B\bar{K}C)^\top\bar{P}_d+\bar{P}_d(A+B\bar{K}C)+A_p^\top\bar{P}_dA_p \\
&+\frac{1}{\bar{\lambda}}W^\top W+\bar{\lambda}\bar{P}_d\Omega\bar{P}_d+Q_d+C^\top\bar{K}^\top R\bar{K}C,
\end{aligned} \tag{40a}
$$

$$
\begin{aligned}
0=&(A+B\bar{K}C+D+B\bar{F}C+\bar{\lambda}\Omega\bar{P}_d)^\top\bar{P}_o \\
&+\bar{P}_o(A+B\bar{K}C+D+B\bar{F}C+\bar{\lambda}\Omega\bar{P}_d) \\
&+\bar{P}_d(D+B\bar{F}C)+(D+B\bar{F}C)^\top\bar{P}_d \\
&+\bar{\lambda}\bar{P}_o\Omega\bar{P}_o+Q_o \\
&+C^\top\bar{F}^\top R\bar{K}C+C^\top\bar{K}^\top R\bar{F}C+C^\top\bar{F}^\top R\bar{F}C,
\end{aligned} \tag{40b}
$$

$$
\begin{aligned}
0=&(A+B\bar{K}C+\bar{\lambda}\Omega\bar{P}_d)\bar{S}_d \\
&+\bar{S}_d(A+B\bar{K}C+\bar{\lambda}\Omega\bar{P}_d)^\top \\
&+A_p\bar{S}_dA_p^\top+\bar{\lambda}(\Omega\bar{P}_o\bar{S}_o+\bar{S}_o\bar{P}_o\Omega) \\
&+\bar{S}_o(D+B\bar{F}C)^\top+(D+B\bar{F}C)\bar{S}_o+\Lambda_d,
\end{aligned} \tag{40c}
$$

$$
\begin{aligned}
0=&[A+B\bar{K}C+D+B\bar{F}C+\bar{\lambda}\Omega(\bar{P}_o+\bar{P}_d)]\bar{S}_o \\
&+\bar{S}_o[A+B\bar{K}C+D+B\bar{F}C+\bar{\lambda}\Omega(\bar{P}_o+\bar{P}_d)]^\top \\
&+\Lambda_o,
\end{aligned} \tag{40d}
$$

$$
0=R\bar{K}C\bar{S}_oC^\top+R\bar{F}C\bar{S}_oC^\top+B^\top(\bar{P}_d\bar{S}_o+\bar{P}_o\bar{S}_o)C^\top, \tag{40e}
$$

$$
0=R\bar{K}C\bar{S}_dC^\top+R\bar{F}C\bar{S}_oC^\top+B^\top(\bar{P}_d\bar{S}_d+\bar{P}_o\bar{S}_o)C^\top, \tag{40f}
$$

$$
\begin{aligned}
0=&\bar{\lambda}^2\mathbf{Tr}[(\bar{S}_d\bar{P}_d+\bar{S}_o\bar{P}_o)\Omega\bar{P}_d+(\bar{S}_o\bar{P}_d+\bar{S}_o\bar{P}_o)\Omega\bar{P}_o] \\
&-\mathbf{Tr}(\bar{S}_dWW^\top).
\end{aligned} \tag{40g}
$$

Finally, if $C(\bar{S}_o-\bar{S}_d)C^T$ is nonsingular, the feedback gains can be established as

$$
\bar{K}=-R^{-1}B^T\bar{P}_d(\bar{S}_o-\bar{S}_d)C^T[C(\bar{S}_o-\bar{S}_d)C^T]^{-1}, \tag{41a}
$$

$$
\begin{aligned}
\bar{F}=&R^{-1}B^T\bar{P}_d(\bar{S}_o-\bar{S}_d)C^T[C(\bar{S}_o-\bar{S}_d)C^T]^{-1} \\
&-R^{-1}B^T(\bar{P}_d\bar{S}_o+\bar{P}_o\bar{S}_o)C^T(C\bar{S}_oC^T)^{-1}.
\end{aligned} \tag{41b}
$$

It can be shown that the result of using feedback gains $\bar{K}$ and $\bar{F}$ has the same properties as Corollary 5.1, and it is omitted here because the proof follows the same approach.

6. Numerical examples

In this section, two numerical examples are presented to demonstrate the reliability and usefulness of the proposed approach in relation to the new reduced-order strategy set.

6.1. Example 1: SCMEs in (6) for one player case

Consider the following stochastic Riccati equation equation:

$$P(A+BFC)+(A+BFC)^\top P+\sum_{k=1}^{N} A_{kp}^\top P A_{kp} + P\Omega P + Q + C^\top F^\top RFC = 0 \quad (42)$$

The stochastic term $\sum_{k=1}^{N} A_{kp}^\top P A_{kp}$ exists in relation to the existing result in Gadewadikar et al. (2006). Furthermore, this is the special case with $\lambda = 1$ and $W \equiv 0$. The following algorithm can be performed:

$$\begin{aligned} 0 &= P^{(\ell+1)}A + A^\top P^{(\ell+1)} + \sum_{k=1}^{N} A_{kp}^\top P^{(\ell+1)} A_{kp} + P^{(\ell+1)}(\Omega - U)P^{(\ell+1)} + Q + L^{(\ell)T}R^{-1}L^{(\ell)}, && (43a)\\ F^{(\ell+1)} &= -R^{-1}\left(B^\top P^{(\ell+1)} - L^{(\ell)}\right)C^\top\left(CC^\top\right)^{-1}, && (43b)\\ L^{(\ell+1)} &= RF^{(\ell+1)}C + B^\top P^{(\ell+1)}, && (43c)\\ L^{(0)} &= 0, \quad \ell = 0,1,2,\ldots. && (43d)\end{aligned}$$

On the other hand, the following algorithm is also used by referring Kučera and De Souza (1995):

$$\begin{aligned} 0 &= P^{(\ell+1)}A + A^\top P^{(\ell+1)} + \sum_{k=1}^{N} A_{kp}^\top P^{(\ell+1)} A_{kp} + P^{(\ell+1)}(\Omega - U)P^{(\ell+1)} + Q + L^{(\ell)T}R^{-1}L^{(\ell)}, && (44a)\\ F^{(\ell+1)} &= -R^{-1}B^\top P^{(\ell+1)}C^\top\left(CC^\top\right)^{-1}, && (44b)\\ L^{(\ell+1)} &= RF^{(\ell+1)}C + B^\top P^{(\ell+1)}, && (44c)\\ L^{(0)} &= 0, \quad \ell = 0,1,2,\ldots. && (44d)\end{aligned}$$

It should be noted that the convergence of all of the numerical algorithms (10a), (43a), and (44a) are not guaranteed. The coefficient matrices are given below with the minor modification as the stochastic term (Gadewadikar et al., 2006):

$$A = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 & 0 & 0 \\ 0.82225 & -1.07741 & -0.17555 & 0 & 0 \\ 0 & 0 & -20.2 & 0 & 0 \\ 10 & 0 & 0 & -10 & 0 \\ -16.26 & -0.9788 & 0.4852 & 0 & 0 \end{bmatrix},$$

$$A_{1p} = 0.05A, \quad B = \begin{bmatrix} 0 \\ 0 \\ 20.2 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 57.2958 & 0 \\ 0 & 57.2958 & 0 & 0 & 0 \\ -16.26 & -0.9788 & 0.4852 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 264 & 16 & 1 & 0 & 0 \\ 16 & 60 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{bmatrix},$$

$$R = 0.1, \quad \Lambda = I_5, \quad N = 1.$$

It should be noted that these numeric values are based on the linearised F-16 aircraft (Gadewadikar et al., 2006) to confirm the effectiveness as well as the physical and application aspects. All algorithms, including Newton's method, have converged. In this case, the iteration is stopped when the convergence criterion, $\|L^{(\ell+1)} - L^{(\ell)}\| < 10^{-9}$, is satisfied. Specifically, the algorithm in (10a) converged at $\ell = 143$ iterations, and the algorithms in (43a) and (44a) both converged at $\ell = 9$ iterations. In addition, Newton's method and algorithm (10a) converge to the same value. The convergent solutions are

$$\begin{aligned} L^{(143)} &= [\ 1.0680 \quad 7.2215\times 10^{-1} \\ &\quad -1.0908\times 10^{-1} \\ &\quad 2.6115 \quad -2.2752\ ], \\ F^{(143)} &= [\ 6.8240\times 10^{-1} \quad 2.7358\times 10^{-1} \\ &\quad -4.4775 \quad -6.1830\times 10\ ], \\ \mathbf{Tr}[P^{(143)}] &= 2.1269\times 10^{2}. \end{aligned}$$

On the other hand, algorithms (43a) and (44a) also converge to the same value. The convergent solutions

Table 1. Compare with existing results.

Reference	[1]	[2]	[4]	Proposed
Problem	LQR	H_∞ Control	LQR	GCC
Algorithm	Riccati Base	Riccati Base	Online Learning	Newton's Method
Convergence	Yes	Yes	Yes	Yes (Local)
Convergence proof	No	Yes	No	Yes
Optimality	Not Always	Not Always	Suboptimal	Yes

LQR : linear quadratic control, GCC : Guaranteed Cost Control.

are

$$
\begin{aligned}
L^{(9)} &= [\ -3.3850 \times 10^{-3} \quad 0 \\
&\qquad -1.1344 \times 10^{-1} \quad 0 \quad 0\ ], \\
F^{(9)} &= [\ 0 \quad 1.4629 \times 10^{-1} \\
&\qquad -4.4526 \quad -3.1720 \times 10\ ], \\
\mathbf{Tr}[P^{(9)}] &= 2.4893 \times 10^2.
\end{aligned}
$$

Therefore, it is confirmed that the closed-loop stochastic system is stable for both feedback gains. Moreover, it can be seen that $\mathbf{Tr}[P]$, which represents cost, by the algorithms of (43a) and (44a) is smaller than the values obtained using Newton's method and the algorithm of (10a). From the above results, it is essential to select an effective algorithm. Furthermore, although matrix G is not included in SOF gain F in (43b) and (44b), it is confirmed that Lagrange multiplier S is necessary for optimisation because $C^\top(CC^\top)^{-1}$ and $SC^\top(CSC^\top)^{-1}$ are different in general cases. Finally, Table 1 shows the results of the comparison between the existing algorithms and newly proposed scheme. These results show that the proposed approach is reliable, as it guarantees both local convergence and optimality.

Remark 6.1: It is well known that many of the algorithms for solving the SOF problem using the BMI solver are NP-hard. On one hand, according to existing results in Movric et al. (2015), the SOF problem can be converted into a polynomial-time algorithm that solves the original problem. In the worst case, such an algorithm does not work well and fails (Fu & Luo, 1997). On the other hand, the algorithm proposed here can be solved using the Newton's method if a solution exists. It has the advantage of getting the solution in polynomial time in practical applications. However, it is known, when using the Newton's method, if the initial values are not properly chosen, it will not converge or it will not converge to an appropriate solution.

6.2. *Example 2: comparison of analytical solution*

Second, the following simple example is considered:

$$
\begin{aligned}
&A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad A_{1p} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \\
&B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ G = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \\
&C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad Q = C^\top C, \quad R = 1, \\
&\Lambda_d = \Lambda_o = I_2, \quad N = 1.
\end{aligned}
$$

When $\ell = 0$, the following stochastic Riccati equation should be solved:

$$
0 = PA + A^\top P + A_{1p}^\top PA_{1p} + P(\Omega - U)P + Q. \quad (45)
$$

The following analytical solution is obtained from a simple calculation:

$$
\begin{aligned}
&P = \begin{bmatrix} 2\alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad \alpha = \frac{1+\sqrt{7501}}{75}, \\
&F = -\alpha = -1.168110849182647, \\
&\mathbf{Tr}[P] = 3\alpha = 3.504332547543496.
\end{aligned}
$$

In this case, it is clear that $L = RFC + B^\top P = 0$. On the other hand, algorithms (10a), (44a) and (45) converge to the following same solutions respectively, after only one iteration: $L = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $F = -1.1681$, $\mathbf{Tr}[P] = 3.5043$. Consequently, it is shown that SOF gain F can be computed by the exiting algorithms in the special case where $L = RFC + B^\top P = 0$ holds. Finally, when using the Newton method, the necessity to select the initial conditions is a major drawback. Therefore, algorithm (10a) does not guarantee convergence, and convergence will be very slow, but it may be effective in selecting the initial gain. In other words, algorithm (10a) should be used first, and then the Newton method should be used.

Table 2. SOF strategy and $\phi(0)$.

N	F	$\phi(0)$
10	-2.7801×10^{-1}	5.1187×10^{-2}
10^2	-2.7929×10^{-1}	2.0682×10^{-1}
10^3	-2.7942×10^{-1}	2.3123×10^{-1}
10^4	-2.7943×10^{-1}	2.3366×10^{-1}
10^5	-2.7943×10^{-1}	2.3390×10^{-1}
10^6	-2.7943×10^{-1}	2.3392×10^{-1}
10^7	-2.7943×10^{-1}	2.3393×10^{-1}
10^8	-2.7943×10^{-1}	2.3393×10^{-1}
10^9	-2.7943×10^{-1}	2.3393×10^{-1}
10^{10}	-2.7943×10^{-1}	2.3393×10^{-1}
$\infty, (\varepsilon = 0)$	$\bar{F} = -2.7943 \times 10^{-1}$	2.3393×10^{-1}

6.3. Example 3: SCMEs in (18) with large N

The following constant coefficient matrices for the mean-field stochastic system (11a) are assumed:

$$n = 2, \quad x_i(t) = \begin{bmatrix} x_{1i}(t) \\ x_{2i}(t) \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix},$$
$$A_p = 0.1 \times A,$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$G = 0.85 \times I_2, \quad W = 0.5 \times I_2,$$
$$Q = 1.5 \times I_2, \quad R = 1,$$
$$\Phi = 1.5 \times I_2, \quad f(t, x_i(t)) = \begin{bmatrix} \sin(x_{1i}(t)) \\ \cos(x_{2i}(t)) \end{bmatrix}.$$

Using Newton's method (26), the SOF strategy and $\phi(0)$ are obtained as in Table 2.

To visualise the population size N and convergence values F, Figure 1 is given. As a result, it can be seen that when the population size N is above 1000, the feedback gain is almost constant; additionally, above this value it is sufficient to use the infinite gain $\bar{F} = -2.7943 \times 10^{-1}$. We see that the SOF gain can be computed for a large number of decision makers N. Furthermore, the degradation of cost in (31) is also shown. As a result, the numerical example shows that as the value of the number of decision makers N tends infinity, the resulting SOF gain converges to the zero-order gain in (23e). From the above, we can also see that the result in (31) is correct from the numerical example. This means that the zero-order gain in (23e) can be applied to a mean-field stochastic system with any large number of decision makers N. Finally, note that the low-dimensional SCME in (22a) can be solved instead of using (18a), and thus can be computed over $N = 10^{10}$. Thus, from a practical point of view, the low-dimensional decomposition computation method newly proposed in this paper is effective and efficient. Finally, in the previous studies (Moon & Başar, 2018; Wang et al., 2021a; Wang & Zhang, 2021; Wang, Zhang and Zhang, 2020), including the authors' result in Mukaidani, Irie et al. (2022), the degradation of the real cost functional has never been numerically evaluated. However, here the exact value of the cost functional can be computed without difficulty. Even for an sufficiently large N value, practically it is reliable to be able to calculate the cost functional. Finally, the simulation result is shown in Figure 2, where each state is approaching equilibrium point at $x_i(t) = 0, i = 1, \ldots, N$ when $N = 10^3$.

From the preceding results and discussions of the numerical experiments, it is clear that, even if N is sufficiently large, the full-order strategy set in (19) is useful because the proposed strategy is not cost-intensive and because all trajectories are stochastically stabilised.

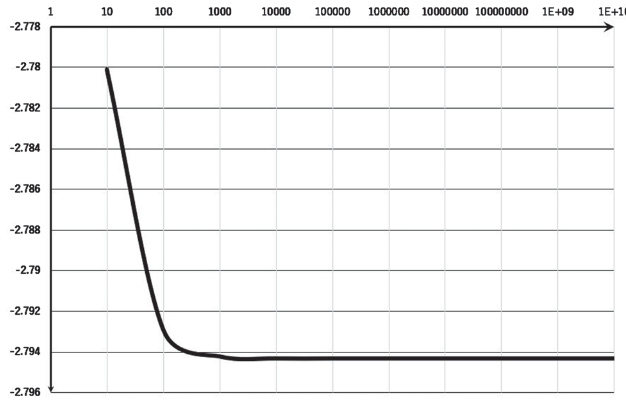

Figure 1. Population size versus feedback gain F.

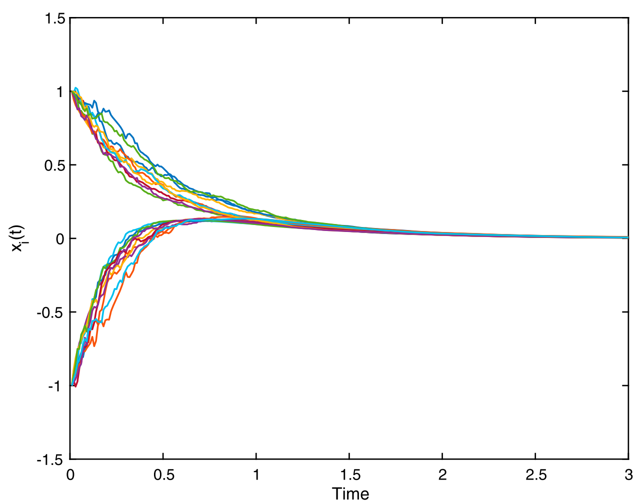

Figure 2. The trajectories of $x_i(t)$ for $N = 10^3$.

7. Conclusion

The mean-field games for nonlinear stochastic systems with many decision makers and unmodeled dynamics have been studied from the point of view of SOF strategies. After deriving some results on the SOF Pareto strategy, a large-scale SCME set was obtained as the necessary conditions for the original problem. To avoid computational limitations due to high-dimensional problems and resource constraints, a new numerical decomposition algorithm was proposed. It should be noted that a uniform structure based on mean-field systems plays an important role in the construction of this algorithm. As a result, even when the number of players approaches infinity, the proposed reduced-dimensional SOF strategy set can be computed as a low-dimensional problem. Finally, it has been shown that when the proposed high-order strategy is applied, the $O(\varepsilon^{2^{\ell}})$ approximation of the cost degradation is achieved for ℓ iterations using Newton's method. In particular, when the decentralised Pareto suboptimal strategy is used, the $O(1/N) = O(\varepsilon)$ approximation is obtained strictly in terms of the existing results. $O(1/\sqrt{N}) = O(\sqrt{\varepsilon})$. Furthermore, the numerical experiments have shown the effectiveness and usefulness of the proposed strategy set. Mean-field games for stochastic systems with many decision makers are an important and long-standing area of research. However, open problems remain. To the best of our knowledge, incentive Stackelberg games for mean-field stochastic nonlinear systems need further investigation. In particular, existence, solvability and uniqueness for the SOF case should be well organised when the population size approaches infinity. These problems will need to be addressed in future studies.

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Data availability statement

The following item can be used to confirm the SOF gain for Example 6.1. https://doi.org/10.24433/CO.8321065.v1

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Appendices

Appendix 1. Proof of Theorem 4.1

This can be proved by using the implicit function theorem (Gajic et al., 2008). Consider the system of equations for continuously differentiable function $\boldsymbol{F}(\boldsymbol{z})$ in (26). Let $(\boldsymbol{z}, \varepsilon) = (\boldsymbol{z}^{(0)}, 0)$ be a point on the function. Then, the Jacobian matrix at $\varepsilon = 0$

is derived as follows:

$$\Delta|_{\varepsilon=0} = \left.\frac{\partial \boldsymbol{F}(\boldsymbol{z})}{\partial \boldsymbol{z}^{\top}}\right|_{\varepsilon=0} = \Delta(\bar{z}, 0). \tag{A1}$$

Furthermore, using the assumption in (24), $\Delta|_{\varepsilon=0} = \Delta(\bar{z}, 0) \neq 0$ holds. Then for the function around $(\boldsymbol{z}^{(0)}, 0))$ we can write $\boldsymbol{z} = \boldsymbol{z}(\varepsilon) = \boldsymbol{z}^{(0)} + O(\varepsilon)$. That is, if Assumption (24) holds true, then the Jacobian matrix (A1) at $\varepsilon = 0$, related to the reduced-order SCMEs (22a) is nonsingular; then, by using the implicit function theorem, the asymptotic structure (25) can be established.

Appendix 2. Proof of Theorem 4.2

This can be proven directly by using the Newton–Kantorovich theorem (Yamamoto, 1986). It is shown that there exists a small σ value such that for $\varepsilon \in (0, \sigma)$, $\Delta(\boldsymbol{z}^{(0)}, \varepsilon)$ is nonsingular. Therefore, there exists ∇ such that $\nabla = \left\|[\Delta(\boldsymbol{z}^{(0)}, \varepsilon)]^{-1}\right\|$. In contrast, using Theorem 4.1, it is easy to verify that $\|\boldsymbol{F}(\boldsymbol{z}^{(0)})\| = O(\varepsilon)$. Hence, there exists γ such that $\gamma = \left\|[\Delta(\boldsymbol{z}^{(0)}, \varepsilon)]^{-1}\boldsymbol{F}(\boldsymbol{z}^{(0)})\right\| = O(\varepsilon)$. On the other hand, for any $\boldsymbol{z}^{a}$ and $\boldsymbol{z}^{b}$, there exists ρ such that $\left\|[\Delta(\boldsymbol{z}^{(0)}, \varepsilon)]^{-1}[\Delta(\boldsymbol{z}^{a}, \varepsilon) - \Delta(\boldsymbol{z}^{b}, \varepsilon)]\right\| = \rho\|\boldsymbol{z}^{a} - \boldsymbol{z}^{b}\|$. Thus, for a sufficiently small ε value, there exists θ such that $\theta = \rho\gamma = O(\varepsilon) < 2^{-1}$ because $\gamma = O(\varepsilon)$. Consequently, $\left\|\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{*}\right\| = 2^{1-\ell}(2\theta)^{2^{\ell}-1}\gamma = 2^{1-\ell} O(\varepsilon^{2^{\ell}}) \le O(\varepsilon^{2^{\ell}})$ is satisfied. Finally, (27) holds and Newton's method (26) has quadratic convergence. This is the desired result.

Appendix 3. Proof of Theorem 4.3

This result is proved by using the solution structure (27) of Theorem 4.2. Using the previous result, the following result holds:

$$\begin{aligned} &|J_{\mathrm{soc}}(\boldsymbol{x}(0), \boldsymbol{u}^{*}(t)) - J_{\mathrm{soc}}(\boldsymbol{x}(0), \boldsymbol{u}^{(\ell)}(t))| \\ &\quad = |\mathbf{Tr}[\Lambda_N P_c] - \mathbf{Tr}[\Lambda_N X_c]| \end{aligned} \tag{A2}$$

where $A_{\mathrm{CL}c}^{(\ell)} = A_c + \varepsilon \boldsymbol{J}_N \otimes D + \varepsilon B_c F_c^{(\ell)} C_c$,

$$\begin{aligned} 0 &= X_c A_{\mathrm{CL}c}^{(\ell)} + A_{\mathrm{CL}c}^{(\ell)T} X_c + \sum_{k=1}^{N} A_{kp}^{\top} X_c A_{kp} + \lambda^{(\ell)} X_c \Omega X_c \\ &\quad + \frac{1}{\lambda^{(\ell)}} W_c^{\top} W_c + Q_c + \varepsilon^2 C_c^{\top} F_c^{(\ell)T} R_c F_c^{(\ell)} C_c. \end{aligned} \tag{A3}$$

Then, substituting $F = F^{(\ell)} + O(\varepsilon^{2^{\ell}})$ and $\lambda_c = \lambda^{(\ell)} + O(\varepsilon^{2^{\ell}})$ into (A3) yields the following stochastic nonlinear matrix equation (SNME):

$$\begin{aligned} 0 &= X_c A_{\mathrm{CL}c} + A_{\mathrm{CL}c} X_c + \sum_{k=1}^{N} A_{kp}^{\top} X_c A_{kp} + \lambda_c X_c \Omega X_c \\ &\quad + \frac{1}{\lambda_c} W_c^{\top} W_c + Q_c + \varepsilon^2 C_c^{\top} F_c R_c F_c C_c + O\left(\varepsilon^{2^{\ell}}\right), \end{aligned} \tag{A4}$$

where $A_{\mathrm{CL}c} = A_c + \varepsilon \boldsymbol{J}_N \otimes D + \varepsilon B_c F_c C_c$.

Furthermore, the following SNME can be obtained by subtracting (A4) from (18a):

$$\begin{aligned} 0 &= Z_c A_{\mathrm{CL}c} + A_{\mathrm{CL}c} Z_c + \sum_{k=1}^{N} A_{kp}^{\top} Z_c A_{kp} \\ &\quad - \lambda_c (Z_c \Omega Z_c - Z_c \Omega P_c - P_c \Omega Z_c) + O\left(\varepsilon^{2^{\ell}}\right), \end{aligned} \tag{A5}$$

where $Z_c = P_c - X_c = \begin{bmatrix} Z_d & \varepsilon Z_o & \varepsilon Z_o & \cdots & \varepsilon Z_o \\ \varepsilon Z_o & Z_d & \varepsilon Z_o & \cdots & \varepsilon Z_o \\ \varepsilon Z_o & \varepsilon Z_o & Z_d & \cdots & \varepsilon Z_o \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon Z_o & \varepsilon Z_o & \varepsilon Z_o & \cdots & Z_d \end{bmatrix}$.

The following reduced-order stochastic nonlinear matrix equation can also be calculated by partitioning the SNME (A5):

$$\begin{aligned} 0 &= Z_d[A + \varepsilon(D + BFC)] + [A + \varepsilon(D + BFC)]^{\top} Z_d \\ &\quad + A_p^{\top} Z_d A_p + \epsilon(1-\epsilon)[Z_o(D + BFC) + (D + BFC)^{\top} Z_o] \\ &\quad - \lambda_c[Z_d \Omega Z_d + \varepsilon(1-\varepsilon) Z_o \Omega Z_o] \\ &\quad + \lambda_c[Z_d \Omega P_d + \varepsilon(1-\varepsilon) Z_o \Omega P_o \\ &\quad + P_d \Omega Z_d + \varepsilon(1-\varepsilon) P_o \Omega Z_o] + O\left(\varepsilon^{2^{\ell}}\right), \end{aligned} \tag{A6a}$$

$$\begin{aligned} 0 &= Z_d(D + BFC) + (D + BFC)^{\top} Z_d \\ &\quad + Z_o[A + (1-\varepsilon)(D + BFC)] \\ &\quad + [A + (1-\varepsilon)(D + BFC)]^{\top} Z_o \\ &\quad - \lambda_c[Z_o \Omega Z_d + Z_d \Omega Z_o + (1-2\varepsilon) Z_o \Omega Z_o] \\ &\quad + \lambda_c[Z_o \Omega P_d + Z_d \Omega P_o + (1-2\varepsilon) Z_o \Omega P_o \\ &\quad + P_o \Omega Z_d + P_d \Omega Z_o + (1-2\varepsilon) P_o \Omega Z_o] + O\left(\varepsilon^{2^{\ell}}\right). \end{aligned} \tag{A6b}$$

Letting $\varepsilon = 0$ in (A6a), the following zeroth-order solutions can be obtained:

$$\bar{Z}_d = \bar{Z}_o = 0, \tag{A7}$$

where

$$\begin{aligned} 0 &= \bar{Z}_d A + A^{\top} \bar{Z}_d + A_p^{\top} \bar{Z}_d A_p \\ &\quad - \lambda_c[\bar{Z}_d \Omega \bar{Z}_d - \bar{Z}_d \Omega \bar{P}_d - \bar{P}_d \Omega \bar{Z}_d], \\ 0 &= \bar{Z}_d(D + BFC) + (D + BFC)^{\top} \bar{Z}_d \\ &\quad + \bar{Z}_o[A + D + BFC] + [A + D + BFC]^{\top} \bar{Z}_o \\ &\quad - \lambda_c[\bar{Z}_o \Omega \bar{Z}_d + \bar{Z}_d \Omega \bar{Z}_o + \bar{Z}_o \Omega \bar{Z}_o] + \lambda_c[\bar{Z}_o \Omega \bar{P}_d + \bar{Z}_d \Omega \bar{P}_o \\ &\quad + \bar{Z}_o \Omega \bar{P}_o + \bar{P}_o \Omega \bar{Z}_d + \bar{P}_d \Omega \bar{Z}_o + \bar{P}_o \Omega \bar{Z}_o]. \end{aligned}$$

Hence, the following structures are satisfied:

$$Z_d = \bar{Z}_d + O(\varepsilon) = \varepsilon \bar{\bar{Z}}_d, \quad Z_o = \bar{Z}_o + O(\varepsilon) = \varepsilon \bar{\bar{Z}}_o. \tag{A8}$$

Subsequently, continuing the same procedure, the following sequences are derived easily:

$$Z_d = P_d - X_d = O\left(\varepsilon^{2^{\ell}}\right), \quad Z_o = P_o - X_o = O\left(\varepsilon^{2^{\ell}}\right). \tag{A9}$$

As a result,

$$\left|J_{\rm soc}(\boldsymbol{x}(0), \boldsymbol{u}^*(t)) - J_{\rm soc}(\boldsymbol{x}(0), \boldsymbol{u}^{(\ell)}(t))\right| = O\left(\varepsilon^{2^\ell - 1}\right),$$

$\ell = 0, 1, 2, \ldots$ is obtained. Finally, using the relation $\varepsilon = 1/N$, the equation in (29) is proved.

Appendix 4. Proof of Corollary 5.1

Using the similar technique to that in the proof of Theorem 4.3, the following relation holds:

$$\begin{aligned} \phi(0) &= \left|\frac{J_{\rm soc}(\boldsymbol{x}(0), \boldsymbol{u}^*(t))}{N} - \frac{J_{\rm soc}(\boldsymbol{x}(0), \bar{\boldsymbol{u}}(t))}{N}\right| \\ &= \left|\mathbf{Tr}\left[\Lambda_d(P_d - \bar{X}_d)\right] + (1-\epsilon)\mathbf{Tr}\left[\Lambda_o(P_o - \bar{X}_o)\right]\right|, \end{aligned} \tag{A10}$$

where

$$\bar{A}_{{\rm CL}c} = A_c + \varepsilon \boldsymbol{J}_N \otimes D + \varepsilon B_c \bar{F}_c C_c,$$

$$\bar{X}_c = \begin{bmatrix} X_d & \varepsilon X_o & \varepsilon X_o & \cdots & \varepsilon X_o \\ \varepsilon X_o & X_d & \varepsilon X_o & \cdots & \varepsilon X_o \\ \varepsilon X_o & \varepsilon X_o & X_d & \cdots & \varepsilon X_o \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_o & \varepsilon X_o & \varepsilon X_o & \cdots & X_d \end{bmatrix},$$

$$\bar{F}_c = I_N \otimes \bar{F},$$

$$\begin{aligned} 0 &= \bar{X}_c \bar{A}_{{\rm CL}c} + \bar{A}_{{\rm CL}c}^\top \bar{X}_c + \sum_{k=1}^{N} A_{kp}^\top \bar{X}_c A_{kp} + \bar{\lambda} \bar{X}_c \Omega \bar{X}_c \\ &\quad + \frac{1}{\bar{\lambda}} W_c^\top W_c + Q_c + \varepsilon^2 C_c^\top \bar{F}_c^\top R_c \bar{F}_c C_c. \end{aligned} \tag{A11}$$

Then, substituting $\bar{X}_c$ into (A11) and letting $\varepsilon = 0$ yields the following zeroth-order SCMEs:

$$0 = \bar{\bar{X}}_d A + A^\top \bar{\bar{X}}_d + A_p^\top \bar{\bar{X}}_d A_p + \frac{1}{\bar{\lambda}} W^\top W + \bar{\lambda} \bar{\bar{X}}_d \Omega \bar{\bar{X}}_d + Q_d, \tag{A12a}$$

$$\begin{aligned} 0 &= \bar{\bar{X}}_o (A + D + B\bar{F}C + \bar{\lambda}\Omega \bar{\bar{X}}_d) \\ &\quad + (A + D + B\bar{F}C + \bar{\lambda}\Omega \bar{\bar{X}}_d)^\top \bar{\bar{X}}_o \\ &\quad + \bar{\bar{X}}_d (D + B\bar{F}C) + (D + B\bar{F}C)^\top \bar{\bar{X}}_d \\ &\quad + \bar{\lambda} \bar{\bar{X}}_o \Omega \bar{\bar{X}}_o + Q_o + C^\top \bar{F}^\top R \bar{F} C. \end{aligned} \tag{A12b}$$

Comparing (23a) and (23b) with (A12a) and (A12b), it is easy to verify that the relations $\bar{P}_d = \bar{\bar{X}}_d$, $\bar{P}_o = \bar{\bar{X}}_o$ hold. Therefore, the result of $P_d = \bar{X}_d + O(\varepsilon)$, $P_o = \bar{X}_o + O(\varepsilon)$ can be obtained. Finally, the property of (31) can be obtained.