INT. J. PROD. RES., 1994, VOL. 32, NO. 1, 75–91

Control charts based on grouped observations

S. H. STEINER†, P. L. GEYER and G. O. WESOLOWSKY

It is often more economical to classify a continuous quality characteristic into several groups than it is to measure it exactly. We propose a control chart based on gauging theoretically continuous observations into multiple groups. This chart is designed to detect one-directional shifts in the mean of a normal distribution with specified operating characteristics. We show how to minimize the sample size required by optimizing the criteria used to group the quality characteristic. Control charts based on grouped observations may be superior to standard control charts based on variables when the quality characteristic is difficult or expensive to measure precisely but economical to gauge.

1. Introduction

It has long been recognized that it may be more economical to gauge observations into groups than to measure their quantities exactly. Stevens (1948) was the first to make a strong case for the use of a two-step gauge, which divides observations into three groups. He showed that, if the gauge limits are properly chosen, grouped data are an excellent alternative to exact measurement since the small loss in statistical efficiency may be more than offset by savings in the cost of measurement. In particular, it is often quicker, easier and therefore cheaper to gauge an article than it is to measure it exactly. Similarly, exact measurements occasionally require costly skilled personnel and sophisticated instruments (Ladany and Sinuary-Stern 1985). For example, in the manufacture of metal fasteners in a progressive die environment, good control of an opening gap dimension is required. However, using calipers distorts the measurements since the parts are made of rather pliable metal. As a result, the only economical alternative on the shop floor is to use step gauges.

In the area of acceptance sampling, plans to monitor the proportion of nonconforming units commonly require the classification of quality characteristics as acceptable or rejectable (Duncan 1986). Others have pointed out that, when the standard deviation of the variable of interest is known, savings in inspection costs can be realized by using an attributes plan with compressed specification limits. Compressed-limit sampling plans have been discussed by Ott and Mundel (1954), Dudding and Jennett (1944), Mace (1952), Ladany (1976) and Duncan (1986). Others have strived for greater efficiency by using three groups instead of two. Beja and Ladany (1974) proposed using three attributes to test for one-sided shifts in the mean of a normal distribution when the process dispersion is known. Ladany and Sinuary-Stern (1985) discuss the curtailment of artificial attribute sampling plans with two or three groups. The approach of Beja and Ladany and of Ladany and

Received January 1993.
Faculty of Business, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4M4, Canada.
†To whom correspondence should be addressed.

0020–7543/94 $10·00 © 1994 Taylor & Francis Ltd.

Sinuary-Stern is not easily extended to more than three groups, where gains in efficiency may be realized. Bray, Lyon and Burr (1973) consider distribution free three class attribute plans.

Stevens (1948) proposed two simple control charts for simultaneously monitoring the mean and standard deviation of a normal distribution using a two-step gauge. He also considered the optimal design of the gauge limits by maximizing the expected Fisher's information in a single observation. It is not straightforward to extend Stevens' methodology to more than three groups, and it is difficult to derive an operating characteristic (OC) curve for his charts.

Currently, in industry, multiple grouped data is handled in an *ad hoc* manner. Usually, for reasons of practicability, grouped observations are treated as if they were non-grouped, giving all the units that fall into a particular group a 'representative' value equal either to an end point or better to the central value of that group's interval. However, as stated by Kulldorff (1961), 'This procedure represents an approximation that often leads to considerable systematic errors.' Consequently, estimates of the process mean will be biased unless the distribution is uniform, and the error rates of a control chart based on this approach may be significantly higher than desired. In addition, assigning the interval end point or midpoint to observations cannot be done for end intervals, since for such intervals the end point and midpoint are equal to infinity. As a result, to utilize this *ad hoc* approach, we must use many groups to alleviate the bias problems and to ensure that no sample units fall into the end groups.

We propose a k-step gauged variable control chart to monitor shifts in the mean of a normal distribution when the process standard deviation is known. When observations are classified into groups, the appropriate model is multinomial with group probabilities being known functions of the unknown parameters. We shall consider testing whether or not the process mean has shifted. The uniformly most powerful test is based upon the likelihood ratio of the multinomial probabilities. This approach leads to optimal control charts that are simple to design and implement. Using the design methodology to be presented, the practitioner will be able to determine the required sample size n, control limit λ, and optimal group weights $z_1, z_2, \ldots,$ for any specific application. The resulting charting procedure is very similar to standard variables-based control charts and no more difficult to use. The implementation steps are as follows.

(1) Take a sample of size n each sample period.
(2) Gauge all the units into groups using a step gauge.
(3) Assign a weight z_i to each unit in group i.
(4) Plot the average weight of a sample, signalling 'out of control' if the average weight plots outside the control limit λ.

This new approach is designed specifically for grouped data and thus avoids bias problems; it can be effectively employed even with a few groups. Consequently, these proposed control charts have better OCs and lower measurement costs when compared with existing *ad hoc* control charts for grouped data.

In general, it is of interest to design a control chart such that it satisfies certain criteria with regards to its OC curve. More specifically, we may wish to design a control chart whose OC curve goes through the points $(\mu_0, 1-\alpha)$ and (μ_1, β), where μ_0 is the target value for the mean of the process and μ_1 is an undesirable mean value. With this interpretation we may consider α and β to be the error rates of chart, with α equal to the probability of a false alarm, and β equal to the probability that the chart will not immediately detect a shift in the mean to μ_1. This design problem corresponds to finding a sample size n and a critical value for the likelihood ratio λ such that

$$\alpha = \Pr\ (\text{chart signals} \mid \mu = \mu_0)$$

$$1 - \beta = \Pr\ (\text{chart signals} \mid \mu = \mu_1)$$

This paper is organized in the following manner. In § 2, we present the problem formulation. In § 3, we discuss the issues involved in the design of control charts for grouped data. Solution methodology for both the large- and the small-sample-size cases are presented. When α and β are small, or if the difference between μ_0 and μ_1 is small, then large sample sizes are required and we can appeal to the central limit theorem. If small sample sizes are required, the solution is more difficult. Section 3 also addresses the issue of discreteness. Since we are working with grouped (discrete) data and integer sample sizes the design problem is complicated. In § 4, we address the related but separate problem of step-gauge design. There are two decisions to be made in specifying the grouping criteria: we must decide how many groups are to be used, and how these groups are to be distinguished. In general, a k-step gauge classifies units into $k+1$ groups. As more groups are used, more information becomes available about the parameters of the underlying distribution. The limiting case occurs when the variable is measured to arbitrary precision. Given that a k-step gauge is to be used, not all gauge limits will provide the same amount of information about the parameters of the underlying distribution. It is not intuitively clear how to set the k steps of the gauge to minimize the sample size required. In § 4, we give tables of step-gauge limits that minimize the sample size required for tests with specific type I and II risks. We consider in detail the important special case where the error risks are equal and the gauge limits are placed symmetrically about $\frac{1}{2}(\mu_0 + \mu_1)$. It is well known that the optimal single-limit gauge should be placed at $\frac{1}{2}(\mu_0 + \mu_1)$ when the error risks are equal (Beja and Ladany 1974, Sykes 1981, Evans and Thyregod 1985). Beja and Ladany (1974) also suggested that the optimal gauge limits should be symmetrically placed about $\frac{1}{2}(\mu_0 + \mu_1)$ for a two-step gauge. Indeed, we show numerically that, if the error risks are equal, this strategy is optimal for k-step gauges. In § 5 we present an example from our work in the manufacture of metal fasteners to illustrate the use of the tables of optimal gauge limits in the design and implementation of step-gauge control charts. In summary, control charts based on grouped data are a viable alternative to variable control charts when it is expensive to measure the quality characteristic precisely.

2. A *k*-step gauge control chart

A control chart is a graphical representation of repeated hypothesis tests. We propose a k-step gauge control chart that uses the likelihood ratio of two specific hypotheses to create a control chart that can detect one-sided shifts in the mean from a normal distribution from grouped data. Suppose that the quality characteristic of interest is a random variable Y that has a normal distribution with probability density $\phi(Y; \mu, \sigma)$, where μ is the location parameter and σ is the known process standard deviation. Without loss of generality we shall assume that σ is unity. Suppose that the target value for the process is μ_0, and we wish our control chart to signal a false alarm with probability less than α, and to signal with probability at least $1-\beta$ whenever the process means shifts to μ_1. Assume $\mu_1 > \mu_0$ for convenience, noting that the solution presented can easily be adapted to opposite case. Our control chart will thus repeatedly test the hypothesis that $\mu = \mu_0$ against the alternative that $\mu = \mu_1$ with a level of significance of α and power $1-\beta$. The solution will have the property that for any mean value better than μ_0 (i.e. $<\mu_0$) the level of significance will be less than or equal to the level of significance at μ_0, and for mean values worse than μ_1 ($>\mu_1$) the power of the test is greater than or equal to power at μ_1. In this sense, our hypothesis test is equivalent to considering the composite hypothesis $\mu \leqslant \mu_0$ against $\mu \geqslant \mu_1$.

A k-step gauge classifies observations into one of $k+1$ distinct intervals. Let the k interval end points be denoted by t_j, $j = 1, 2, \ldots, k$, then the probability that an observation is classified as belonging to group j is given by

$$\begin{aligned} \pi_1(\mu) &= \int_{-\infty}^{t_1} \phi(y; \mu)\, \mathrm{d}y \\ \pi_j(\mu) &= \int_{t_{j-1}}^{t_j} \phi(y; \mu)\, \mathrm{d}y \qquad j = 2, \ldots, k \\ \pi_{k+1}(\mu) &= \int_{t_k}^{\infty} \phi(y; \mu)\, \mathrm{d}y \end{aligned} \tag{1}$$

Note that the definition of the gauge limits is totally general, and thus the distinct intervals need not be of equal size. In practice, most standard step gauges have intervals of equal size but, as will be shown in § 4, in some circumstances, step gauges with unequal intervals are optimal.

Let $\mathbf{X}$ be a $(k+1)$ column vector whose jth element X_j denotes the number of observations in a sample of size n that are classified into the jth group. Then the likelihood function for hypotheses regarding μ given a sample of size n is

$$L(\mu|\mathbf{X}) = c \prod_{j=1}^{k+1} \pi_j(\mu)^{X_j}$$

the constant c of proportionality being arbitrary. All the information which a sample of size n provides regarding the relative merits of our hypothesis is contained in the likelihood ratio of these hypotheses on the sample (Edwards 1972). In fact, by the Neyman–Pearson lemma (Kendall and Stuart 1979, p. 180) we know that for testing a simple hypothesis the optimal partitioning of the accept–reject region is based on the likelihood ratio of the two hypotheses.

The likelihood ratio for the two hypotheses of interest is given by

$$L(\mu|\mathbf{X}) = \frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} = \prod_{j=1}^{k+1} \left(\frac{\pi_j(\mu_1)}{\pi_j(\mu_0)}\right)^{X_j}$$

where

$$\sum_{j=1}^{k+1} X_j = n$$

To simplify subsequently notation considerably, we shall set the critical value for the likelihood ratio equal to $\exp(n\lambda)$. This way, as we shall see later, λ is the critical value, or control limit, for the statistic to be plotted on the chart. Therefore our control chart signals that the process mean has shifted whenever $LR(\mu|\mathbf{X})>\exp(n\lambda)$ or, equivalently, whenever $\sum_{j=1}^{k+1} X_j \ln[\pi_j(\mu_1)/\pi_j(\mu_0)]>n\lambda$. Define z_i as a random variable that is equal to $\ln[\pi_j(\mu_1)/\pi_j(\mu_0)]$ when the ith observation belongs to the jth group. Then our chart signals whenever the average likelihood ratio for a sample $\bar{z}$ is greater than λ.

If α and β are the desired error probabilities of our chart, our design problem is to find the sample size n and control limit λ so that

$$\alpha=\Pr\left(\sum_{i=1}^{n} z_i>n\lambda|\mu=\mu_0\right) \tag{2}$$

and

$$1-\beta=\Pr\left(\sum_{i=1}^{n} z_i>n\lambda|\mu=\mu_1\right) \tag{3}$$

The following lemma will be useful in simplifying the calculations for the important special case when $\alpha=\beta$.

Lemma

Suppose that the process has an underlying normal distribution. If $\alpha=\beta$, and the gauge limits are symmetrically placed about $\frac{1}{2}(\mu_0+\mu_1)$, then the distribution of z has moments that satisfy

$$E(z^r|\mu=\mu_0)=\begin{cases}-E(z^r|\mu=\mu_1) & \text{if } r \text{ is odd}\\ E(z^r|\mu=\mu_1) & \text{if } r \text{ is even}\end{cases}$$

Proof

For a k-step gauge where the gauge limits are symmetrically placed about $\frac{1}{2}(\mu_0+\mu_1)$ we have $\pi_i(\mu_0)=\pi_{k+2-i}(\mu_1)$ by the symmetry of the normal distribution. Since $z_j=\ln[\pi_j(\mu_1)/\pi_j(\mu_0)]$, we know that $z_i=-z_{k+2-i}$. Consequently,

$$\begin{aligned}E(z^r|\mu=\mu_1)&=\sum_{i=1}^{k+1}\pi_i(\mu_1)(z_i)^r\\&=\sum_{i=1}^{k+1}\pi_{k+2-i}(\mu_0)(-z_{k+2-i})^r\\&=(-1)^r\sum_{j=1}^{k+1}\pi_j(\mu_0)z_j^r\\&=(-1)^r E(z^r|\mu=\mu_0)\end{aligned}$$

3. Design of a *k*-step gauge control chart

Solving equations (2) and (3) for the required sample size and control limit is not straightforward. However, an approximate solution may be obtained by using the central limit theorem (CLT). This solution is presented in § 3.1 and is applicable when the required sample size is large. However, since grouped data are inherently discrete, the desired error rates will not be achieved exactly. This, coupled with small or moderate sample size, may cause the CLT solution to be not sufficiently accurate. In § 3.2, we give an algorithm that, for small and moderate sample sizes, can find the true error rates. In this way, it is possible to evaluate when the CLT solution is appropriate. Section 3.3 presents a procedure that can be used to design k-step gauge control charts for small sample sizes.

3.1. *Central limit theorem solution*

For large n, $\bar{z}=\sum_{i=1}^{n} z_i/n$ will have an approximate normal distribution. Since the random variables z_i, $i=1, \ldots, n$ are independent and identically distributed, we know that $E(\bar{z})=E(z)$, and $\text{var}(\bar{z})=\text{var}(z)/n$. To emphasize that the moments of $\bar{z}$ depend on the mean of the underlying distribution μ, define the mean and variance of $\bar{z}$ as $\delta(\mu)$ and $\tau^2(\mu)/n$ respectively. Then

$$\delta(\mu)=E(z)=\sum_{j=1}^{k+1} \pi_j(\mu)z_j$$

$$\tau^2(\mu)=\text{var}\ (z)=\sum_{j=1}^{k+1} \pi_j(\mu)z_j^2-\delta^2(\mu)$$

Using these definitions we can solve equations (2) and (3) for the required sample size and control limit:

$$n=\left(\frac{\Phi^{-1}(\alpha)\tau(\mu_0)-\Phi^{-1}(1-\beta)\tau(\mu_1)}{\delta(\mu_0)-\delta(\mu_1)}\right)^2 \tag{4}$$

and

$$\lambda=\frac{\Phi^{-1}(\alpha)\tau(\mu_0)\delta(\mu_1)-\Phi^{-1}(1-\beta)\tau(\mu_1)\delta(\mu_0)}{\Phi^{-1}(\alpha)\tau(\mu_0)-\Phi^{-1}(1-\beta)\tau(\mu_1)} \tag{5}$$

where $\Phi^{-1}()$ denotes the inverse of the cumulative distribution function of the standard normal distribution.

If we decide that $\alpha=\beta$ and the gauge limits are symmetric about $\frac{1}{2}(\mu_0+\mu_1)$, then equations (4) and (5) can be considerably simplified.

Theorem

If $\alpha=\beta$ and the gauge limits are symmetrically placed about $\frac{1}{2}(\mu_0+\mu_1)$, then

$$n=\left(\frac{\Phi^{-1}(\alpha)\tau(\mu_0)}{\delta(\mu_0)}\right)^2$$

and

$$\lambda=0$$

Proof

By the lemma in the previous section, we know that $\delta(\mu_1)=-\delta(\mu_0)$, and the variance of z under μ_0 is equal to the variance of z under μ_1 since

$$\begin{aligned}\tau^2(\mu_1)&=\sum_{j=1}^{k+1} \pi_j(\mu_1)z_j^2-\delta^2(\mu_1)\\&=\sum_{j=1}^{k+1} \pi_j(\mu_0)z_j^2-[-\delta(\mu_0)]^2\\&=\tau^2(\mu_0)\end{aligned}$$

When $\alpha=\beta$, $\Phi^{-1}(\alpha)=-\Phi^{-1}(1-\beta)$, then the denominator of equations (4) and (5) is given by $\delta(\mu_0)-\delta(\mu_1)=2\delta(\mu_0)$ which is clearly non-zero. The numerators of equations (4) and (5) are

$$\Phi^{-1}(\alpha)\tau(\mu_0)\delta(\mu_1)-\Phi^{-1}(1-\beta)\tau(\mu_1)\delta(\mu_0)=0$$

and

$$\Phi^{-1}(\alpha)\tau(\mu_0)-\Phi^{-1}(1-\beta)\tau(\mu_1)=2\Phi^{-1}(\alpha)\tau(\mu_0)$$

The result follows immediately from substitution into equations (4) and (5).

3.2. *Determination of exact α and β*

Since the sample size must be an integer, and the random variable z_i is discrete, the actual error levels will not be exactly as desired. In general, for small sample sizes, the distribution of $\bar{z}$ will be positively skewed when the mean is equal to μ_0 and negatively skewed when the mean is equal to μ_1 since most of the gauge limits are placed between μ_0 and μ_1. Note that this skewness problem will be most pronounced when trying to detect large shifts with few groups. In cases where the distribution of $\bar{z}$ is skewed, the actual error rates will be larger than the normal approximation would suggest. For example, if we use the methodology of § 3.1 to design a two-step gauge chart to detect a 2σ shift in the mean with desired error rates $\alpha=\beta=0{\cdot}001$, the actual error risks calculated with the branch-and-bound algorithm presented below are $\alpha=0{\cdot}002$ and $\beta=0{\cdot}006$. Figure 1 illustrates the significant extent to which, in this case, the distribution of $\bar{z}$ deviates from normality.

To compute the exact α and β for a given sample of size n, critical value λ and gauge steps vector $\mathbf{t}$, we must determine all partitions of the n observations into $k+1$ groups, without regard to order, such that $LR(\mu|\mathbf{X})>\exp(N\lambda)$. Let $\{\mathbf{K}_s\}$ represent the set of all such partitions that cause the chart to signal. The probability that the chart signals when the true value of the mean is μ is given by

$$\Pr[LR(\mu|\mathbf{X})>\exp(N\lambda)]=\sum_{\{\mathbf{K}_s\}}\frac{n!}{X_1!X_2!\ldots X_{k+1}!}\prod_{j=1}^{k+1}\pi_j(\mu)^{X_j} \tag{6}$$

Then $\alpha=1-\Pr[LR(\mu_0|\mathbf{X})>\exp(N\lambda)]$ and $\beta=\Pr[LR(\mu_1|\mathbf{X})>\exp(N\lambda)]$. The number of such partitions grows exponentially as the number of groups increases and polynomially as the sample size increases. Fortunately, for large n and k the CLT is applicable and we may use the methodology of the preceding section. For

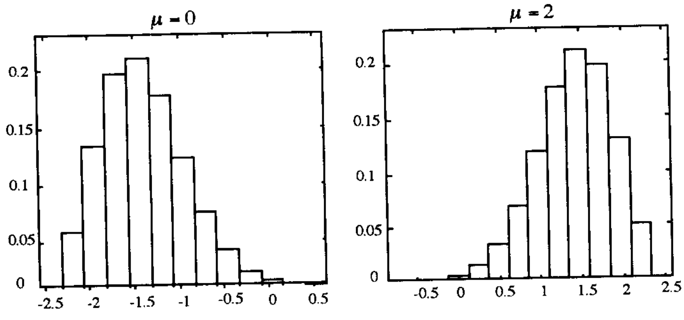

Figure 1. Exact distribution of $\bar{z}$ ($t=[0{\cdot}5,\ 1{\cdot}5]$; $z=[-2{\cdot}2344,\ 0,\ 2{\cdot}2344]$; $\lambda=0$; $n=9$).

moderately large n and k, we give an algorithm, similar in spirit to the branch-and-bound algorithm, which finds the set $\{\mathbf{K}_s\}$, and the probability that the sample belongs to $\{\mathbf{K}_s\}$ when the mean is at the target value μ_0 and the unacceptable value u_1. The algorithm avoids total enumeration by fathoming samples which could not possibly lead to a signal.

Before presenting the algorithm, we would like to offer some of the intuition behind the fathoming rule. Note that the z_i values are ordered in the sense that they increase from negative values in the lower tail to positive values in the upper tail. Suppose that a partial sample of n' observations covering only the first k' groups in the left tail has n_i observations in the ith group. The maximum possible value of $\bar{z}$ will occur if the remaining $n-n'$ observations fall into the $(k+1)$th group. Hence, if at such a stage in the enumeration, we find that

$$\sum_{i=1}^{k'} n_i z_i + (n-n')z_{k+1} < n\lambda$$

then we know that any sample containing this partial sample does not belong to $\{\mathbf{K}_s\}$. Hence, we begin considering partial samples by first allocating observations to the lower tail. As soon as a partial sample can be fathomed (rejected), we stop allocating observations to that group and begin allocating observations to the next group. This continues until all samples have been either fathomed or included in the set $\{\mathbf{K}_s\}$.

3.2.1. *Algorithm for the determination of* $\{\mathbf{K}_s\}$ *and* $\Pr(\mathbf{X} \in \{\mathbf{K}_s\} \mid \mu)$
This is as follows.

Branch step. For each group, starting with the first and proceeding to the $(k+1)$th group, consider all partial allocations to previous groups not yet fathomed. From each unfathomed node create a new branch for every possible allocations of the remaining observations to the next group.

Bound step. For every branch, calculate the bound equal to $\sum_{i=1}^{k'} n_i z_i + (n-n')z_{k+1}$. This is an upper bound on the value of $\bar{z}$ for the current partial sample.

Fathoming step. Fathom all branches where the bound is less than $n\lambda$.

Summary step All branches not fathomed form the set $\{\mathbf{K}_s\}$. Use equation (6) to obtain $\Pr(\mathbf{X} \in \{\mathbf{K}_s\} \mid \mu_0)$ and $\Pr(\mathbf{X} \in \{\mathbf{K}_s\} \mid \mu_1)$.

3.2.2. *Use of the algorithm*

This algorithm substantially reduces the amount of computation required; in most cases, over 50% of the total possible allocations were eliminated. An important special case occurs when type I and type II errors (α and β) are considered equally important and gauge limits are equally spaced symmetric about $\frac{1}{2}(\mu_0+\mu_1)$. In that case, by the lemma, the distribution of $\bar{z}$ when $\mu=\mu_0$ is the mirror reflection about zero of the distribution of $\bar{z}$ when $\mu=\mu_1$. As a consequence, the group weights are symmetric, and the CLT solution suggests that we may achieve approximately equal error risks when $\lambda=0$ (see the example presented in Fig. 1). Note, however, that the error rates achieved will not be exactly equal since the distribution of β is discrete and will admit zero as a possible value, since the weights are perfectly symmetric. While we have arbitrarily decided not to signal if $\bar{z}=0$, the sample offers no information regarding the relative merits of the possibilities that $\mu=\mu_0$ or $\mu=\mu_1$.

Exactly equal error probabilities may be achieved by sampling another observation in the event that $\bar{z}=0$. If this is desired, then the set $\{\mathbf{K}_s\}$ and equation (6) need to be redefined accordingly.

The question of how large a sample is required for the CLT solution to be sufficiently accurate is important. Clearly the answer depends on many factors, including the number of gauge limits used, the location of gauge limits, the magnitude of mean shift that we wish to detect readily, and the accuracy required for a particular application. Given that the gauge limits are chosen primarily between the mean levels of the null and alternative hypothesis, the most important factor becomes the number of gauge limits. Figures 2–4 show actual error rates for plans designed with the CLT solution to detect mean shifts with two gauge limits [0·25, 1·25], three gauge limits [0, 0·75, 1·5] and five gauge limits [−0·25, 0·25, 0·75, 1·75] respectively and error rates $\alpha=\beta=0{\cdot}005$. True error rates, obtained from the branch-and-bound algorithm, are plotted for various sample sizes.

Figures 2–4 all show that, because of the skewness in the distribution of $\bar{z}$, the true error rates are always somewhat larger than expected. As this underestimation of the error rates by the CLT solution is expected, we may determine that the CLT solution is appropriate if the deviations of the true error rates from the desired rates is small. With this evaluation criterion, the effect of the number of gauge limits, as expressed in the figures, is significant. In the two-gauge-limit case, the fluctuations in the true error rates are still apparent at a sample size of 50. However, for more gauge limits the CLT solution performs much better. When using three gauge limits, the error rates become quite stable and close to the desired levels by a sample size of 20. In the five-gauge-limit case, the same is true at a sample size of 15. These results are of course dependent on the location of the gauge limits, and the accuracy required in any particular situation, but they do provide some insight into the usefulness of the CLT solution.

3.3. *Design of a small-sample step-gauge chart*

If a small sample size is required, the solution presented in § 3.1 may not be sufficiently accurate, and the true error rates may be significantly higher than the desired levels. However, by using the CLT solution as a starting solution, and utilizing the algorithm from § 3.2 which calculates the exact true error rates, we can employ the following iterative procedure to design an appropriate chart.

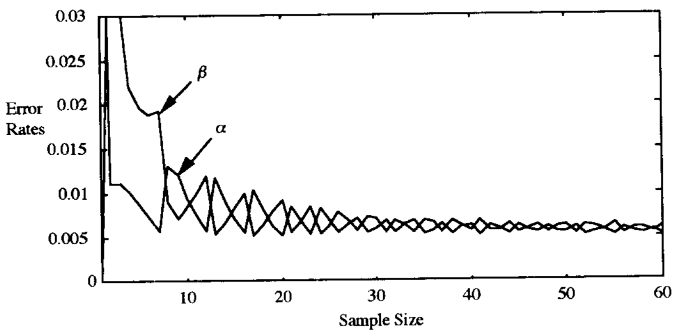

Figure 2. True error rates with two gauge limits ($t=[0{\cdot}25, 1{\cdot}25]$).

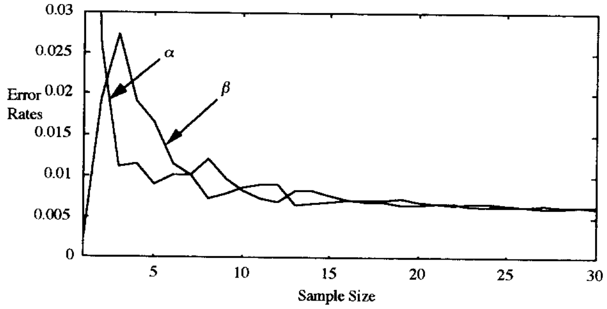

Figure 3. True error rates with three gauge limits ($t=[0, 0{\cdot}75, 1{\cdot}5]$).

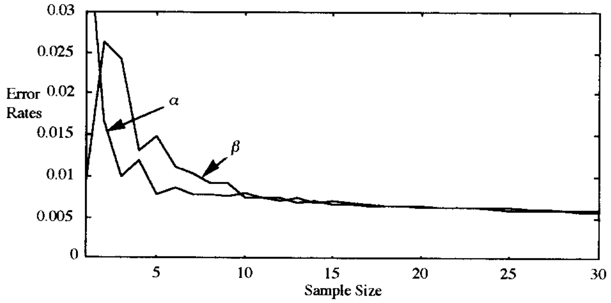

Figure 4. True error rates with five gauge limits ($t=[-0{\cdot}25, 0{\cdot}25, 0{\cdot}75, 1{\cdot}25, 1{\cdot}75]$).

(1) Use equations (4) and (5) to determine an initial solution for n, n^* say, and control limit λ.
(2) Use the branch-and-bound algorithm to compute the exact α and β for sample size $\lceil n^* \rceil$ and control limit λ.
(3) Incrementally increase n until satisfactory error rates are achieved.

To illustrate this, assume that we desired a chart to detect a shift of 2σ units in a standard normal distribution with $\alpha=0{\cdot}001$ and $\beta=0{\cdot}005$. Using the normal approximation and optimal gauge limits, $\mathbf{t}'=[0{\cdot}1636, 0{\cdot}8762, 1{\cdot}6076]$, derived in §4, the initial solution is $\lambda=0{\cdot}0717$ and $n=14{\cdot}3$. If we increment n from 15 and use the exact algorithm, we obtain the actual error rates given in Table 1. Because of the discreteness problem, the solution using the CLT results in higher α and β than planned. We require $n=17$ to obtain error rates $\alpha<0{\cdot}001$ and $\beta<0{\cdot}005$. Note that this incremental strategy for designing small-sample-size control charts will not necessarily find the solution with the smallest sample size. This is because we are not simultaneously adjusting the control limit as we increment the sample size. It may be possible that at some lower sample size an adjustment of the control limit may change the actual error rates in such a way that they both satisfy the requirements.

4. Optimal step-gauge design

Until now we have assumed that the gauge limits are fixed. Although this is often true in most industrial environments, in some circumstances it may be possible and

n	α	β
15	0·0017	0·0064
16	0·0015	0·0045
17	0·00099	0·0038

Table 1. Exact α and β values as n increases.

desirable to design the step gauge specifically for a control chart. In this section we shall determine, by minimizing the required sample size from the CLT solution, the optimal step-gauge limits. This procedure will also allow us to compare the efficiency of the optimal limits with fixed limits.

Beja and Ladany (1974), Sykes (1981) and Evans and Thyregod (1985) have shown that, when the error risks α and β are equal, the optimal single-step gauge should be placed at $\frac{1}{2}(\mu_0+\mu_1)$. Beja and Ladany (1974) also suggested that the optimal gauge steps for a two-step gauge should be symmetrically placed about $\frac{1}{2}(\mu_0+\mu_1)$. Using this rule of thumb, a one-dimensional search for the optimal steps is possible. This solution will only be optimal if the error risks are equal.

Suppose that we are given the magnitude of the mean shift that is to be readily detected, that is $\mu_1-\mu_0$, and the error rates for a chart. If the error rates are small and/or the shift to be readily detected is small, then the required sample size will be large enough that $\bar{z}$ is approximately normally distributed. In this case we can determine optimal gauge limits by minimizing equation (4), subject to the constraint that the gauge limits remain ordered. Formally, let $\mathbf{t}$ be the k-dimensional vector of gauge limits. Then we have the multidimensional minimization problem

$$\min\,[n(\mathbf{t})+m(\mathbf{t})]$$

where

$$m(\mathbf{t})=\begin{cases} M, & \text{if } t_j>t_{j+1} \text{ for all } j=1,\ldots,k \\ 0, & \text{otherwise} \end{cases}$$

and M is a large number. This optimization problem can easily and efficiently be solved by the Nelder–Mead multidimensional simplex algorithm (Press *et al.* 1988).

Tables 2 and 3 give the resulting optimal steps, and the corresponding weights for step gauges with one to seven steps, and for charts that should readily detect shifts in the mean of $\frac{1}{2}\sigma$, 1σ or $\frac{3}{2}\sigma$ units. It should be noted that the required sample size obtained from equation (4) is fairly insensitive to slight deviations from the optimal step gauge design. This is illustrated in Figure 5 for a two-step gauge, where the gauge limits remain symmetric about 0·5. The figure shows the sample size required to detect a 1σ mean shift with error rates of 0·005 as a function of gauge limits. The horizontal axis of Fig. 5 represents the amount that each gauge limit deviates from 0·5, that is the gauge limits are placed at $0{\cdot}5-\gamma$ and $0{\cdot}5+\gamma$, where γ is the amount of deviation from the one-step gauge case. The optimal value for γ, from Table 2, is $=0{\cdot}5424$. Near this value the required sample size increases only slowly.

For small sample sizes the distribution of $\bar{z}$ is skewed when the mean is at either μ_0 or μ_1; as a result, the optimal gauge limits and weights presented in Tables 2 and 3 are no longer optimal. Theoretically, it is possible to find the optimal gauge limits

μ_1	k	n/λ		Values of t_i and z_i for the following i 1	2	3	4	5	6	7	8
0·5	1	$n=235·5$	t_i	0·25							
		$\lambda=0$	z_i	−0·4001	0·4001						
	2	$n=186·0$	t_i	−0·3417	0·8417						
		$\lambda=0$	z_i	−0·6052	0	0·6052					
	3	$n=179·6$	t_i	−0·6925	0·2500	0·925					
		$\lambda=0$	z_i	−0·74	−0·2188	0·2188	0·74				
	4	$n=164·6$	t_i	−0·9384	0·1139	0·6139	1·4384				
		$\lambda=0$	z_i	−0·8395	−0·3667	0	0·3667	0·8395			
	5	$n=161·1$	t_i	−1·1254	−0·3743	0·2500	0·8743	1·6254			
		$\lambda=0$	z_i	−0·9172	−0·4771	−0·1511	0·1511	0·4771	0·9172		
	6	$n=158·9$	t_i	−1·2749	−0·5751	−0·0142	0·5142	1·0751	1·7749		
		$\lambda=0$	z_i	−0·9804	−0·5642	−0·2653	0	0·2653	0·5642	0·9804	
	7	$n=157·5$	t_i	−1·3987	−0·7372	−0·2202	0·2500	0·7202	1·2373	1·8986	
		$\lambda=0$	z_i	−1·0334	−0·6356	−0·3563	−0·1154	0·1154	0·3563	0·6356	1·0334
1	1	$n=55·6$	t_i	0·5000							
		$\lambda=0$	z_i	−0·8070	0·8070						
	2	$n=44·4$	t_i	−0·0424	1·0424						
		$\lambda=0$	z_i	−1·1789	0	1·1789					
	3	$n=41·2$	t_i	−0·3428	0·5000	1·3428					
		$\lambda=0$	z_i	−1·4062	−0·3972	0·3972	1·4062				
	4	$n=40·0$	t_i	−0·5373	0·1813	0·8187	1·5373				
		$\lambda=0$	z_i	−1·5600	−0·6495	0	0·6495	1·5600			
	5	$n=39·2$	t_i	−0·6723	−0·0357	0·5000	1·0357	1·6723			
		$\lambda=0$	z_i	−1·6692	−0·8257	−0·2615	0·2615	0·8257	1·6692		
	6	$n=38·8$	t_i	−0·7697	−0·1941	0·2767	0·7233	1·1941	1·7697		
		$\lambda=0$	z_i	−1·7492	−0·9553	−0·4503	0	0·4503	0·9553	1·7492	
	7	$n=38·6$	t_i	−0·8417	−0·3149	0·1093	0·5000	1·8907	1·3149	1·8417	
		$\lambda=0$	z_i	−1·8090	−1·0537	−0·5938	−0·1929	0·1929	0·5938	1·0537	1·8090
1·5	1	$n=22·4$	t_i	0·7500							
		$\lambda=0$	z_i	−1·2275	1·2275						
	2	$n=18·1$	t_i	0·2661	1·2339						
		$\lambda=0$	z_i	−1·7172	0	1·7172					
	3	$n=17·0$	t_i	0·0273	0·7500	1·4727					
		$\lambda=0$	z_i	−1·9817	−0·5190	0·5190	1·9817				
	4	$n=16·6$	t_i	−0·1068	0·4829	1·0171	1·6068				
		$\lambda=0$	z_i	−2·1358	−0·8189	0	0·8189	2·1358			
	5	$n=16·4$	t_i	−0·1867	0·3132	0·7500	1·1868	1·6867			
		$\lambda=0$	z_i	−2·2293	−1·0090	−0·3225	0·3225	1·0090	2·2293		
	6	$n=16·3$	t_i	−0·2365	0·1971	0·5706	0·9294	1·3029	1·7365		
		$\lambda=0$	z_i	−2·2882	−1·1366	−0·5429	0	0·5429	1·1366	2·2882	
	7	$n=16·2$	t_i	0·2688	0·1135	0·4413	0·7500	1·0587	1·3865	1·7688	
		$\lambda=0$	z_i	−2·3268	−1.2265	−0·7025	−0·2297	0·7025	1·2265	2·3268	

n is calculated with the assumption that the type 1 and 2 errors are $=0·001$.

Table 2. Optimal steps and weights for the standard normal distribution, $\alpha=\beta$.

μ_1	k	n/λ		Values of t_i and z_i for the following i							
				1	2	3	4	5	6	7	8
0·5	1	n=198·1	t_i	0·2889							
		λ=0·0070	z_i	−0·3878	0·4125						
	2	n=156·3	t_i	−0·2954	0·8870						
		λ=0·0090	z_i	−0·5880	0·0204	0·6220					
	3	n=144·0	t_i	−0·6405	0·3009	1·2428					
		λ=0·0099	z_i	−0·7197	−0·1949	0·2423	0·7603				
	4	n=138·4	t_i	−0·8812	0·0587	0·6685	1·4929				
		λ=0·0103	z_i	−0·8162	−0·3402	0·0263	0·3925	0·8620			
	5	n=135·4	t_i	−1·0634	−0·3151	0·3079	0·9321	1·6839			
		λ=0·0106	z_i	−0·8913	−0·4483	−0·1227	0·1791	0·5048	0·9418		
	6	n=133·6	t_i	−1·2084	−0·5121	−0·0468	0·5746	1·1359	1·8372		
		λ=0·0107	z_i	−0·9521	−0·5333	−0·2351	0·0297	0·2948	0·5936	1·0070	
	7	n=132·4	t_i	−1·3276	−0·6706	−0·1561	0·3129	0·7829	1·3009	1·9645	
		λ=0·0109	z_i	−1·0029	−0·6026	−0·3244	−0·0843	0·1462	0·3872	0·6667	1·0618
1	1	n=46·6	t_i	0·5725							
		λ=0·0256	z_i	−0·7618	0·8533						
	2	n=37·3	t_i	0·0459	1·1288						
		λ=0·0333	z_i	−1·1147	0·0792	1·2429					
	3	n=34·6	t_i	−0·2387	0·5968	1·4438					
		λ=0·0367	z_i	−1·3259	−0·3028	0·4901	1·4854				
	4	n=33·5	t_i	−0·4178	0·2891	0·9254	1·6528				
		λ=0·0385	z_i	−1·4649	−0·5414	0·1037	0·7552	1·6533			
	5	n=32·9	t_i	−0·5384	0·0827	0·6141	1·1525	1·8020			
		λ=0·0395	z_i	−1·5608	−0.7049	−0·1481	0·3741	0·9436	1·7760		
	6	n=32·6	t_i	−0·6230	−0·0658	0·3983	0·8443	1·3207	1·9127		
		λ=0·0402	z_i	−1·6290	−0·8229	−0·3278	0·1193	0·5716	1·0847	1·8684	
	7	n=32·4	t_i	−0·6838	−0·1776	0·2380	0·6261	1·0188	1·4507	1·9969	
		λ=0·0406	z_i	−1·6786	−0·9111	−0·4631	−0·0671	0·3183	0·7234	1·1939	1·9396
1·5	1	n=18·8	t_i	0·8471							
		λ=0·0498	z_i	−1·1378	1·3202						
	2	n=15·2	t_i	0·3034	1·3495						
		λ=0·0654	z_i	−1·5924	0·1616	1·8456					
	3	n=14·3	t_i	0·1636	0·8762	1·6076					
		λ=0·0717	z_i	−1·8291	−0·3309	0·7057	2·1367				
	4	n=13·9	t_i	0·0443	0·6198	1·1540	1·7579				
		λ=0·1082	z_i	−1·9625	−0·6099	0·2005	1·0273	2·3134			
	5	n=13·7	t_i	−0·0250	0·4587	0·8917	1·3331	1·8502			
		λ=0·0762	z_i	−2·0414	−0·7843	−0·1104	0·5349	1·2347	2·4245		
	6	n=13·6	t_i	−0·0676	0·3493	0·7170	1·0763	1·4567	1·9090		
		λ=0·0771	z_i	−2·0903	−0·9006	−0·3216	0·2176	0·7654	1·3757	2·4958	
	7	n=13·6	t_i	−0·0949	0·2708	0·5917	0·8987	1·2101	1·5462	1·9479	
		λ=0·0776	z_i	−2·1220	−0·9821	−0·4740	−0·0072	0·4529	0·9334	1·4757	2·5433

Table 3. Optimal steps and weights for the standard normal distribution, α=0·001, β=0·005.

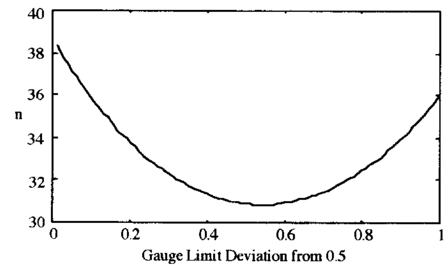

Figure 5. Required sample size as a function of gauge limit design.

Interval	Weight z_i
1: $(-\infty, 73, 69)$	$-1{\cdot}3259$
2: $(73{\cdot}69, 74{\cdot}78)$	$-0{\cdot}3028$
3: $(74{\cdot}78, 75{\cdot}88)$	$0{\cdot}4901$
4: $(75{\cdot}88, \infty)$	$1{\cdot}4854$

Table 4. Example gauge limit and weight design.

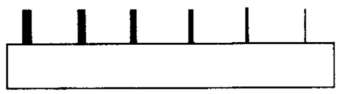

Figure 6. An idealized six-step gauge.

for the small-sample-size case. However, in general, for small sample sizes this is computationally expensive, and the gauge limits given in Tables 2 and 3 are nearly optimal.

Tables 2 and 3 were computed for a standard normal in-control process, although they may be used to calculate the optimal steps for any normal process by noting that, if $\mathbf{t}$ is the vector of optimal steps to detect the mean shift of a $N(0,1)$ process, then $\mathbf{t}' = \mathbf{t}\sigma + \mu\mathbf{l}$ will be the corresponding vector of optimal steps for an $N(\mu, \sigma)$ process. For example, suppose that we wish to detect a 1σ shift in the mean of a $N(74, 1{\cdot}3)$ process with error rates $\alpha = 0{\cdot}001$ and $\beta = 0{\cdot}005$ using a three-step gauge. From Table 3, we should use a sample of size 35, and a critical value of 0·0367. The optimal gauge steps in Table 3 are $-0{\cdot}2387$, $-0{\cdot}5968$, $1{\cdot}4438$. After multiplying by σ and adding μ, the three-step gauge will classify observations into the four groups with the corresponding weights given in Table 4. The process is deemed 'out of control' if the average weight in the sample of size 35 is $>0{\cdot}0367$.

5. Metal-fasteners example

This example is motivated by our work in the manufacture of metal fasteners in a progressive die environment. It was desired to monitor increases in the width of an important gap dimension of a fastener. Since the metal used is pliable, calipers distort the gap measurement, and the only economical alternative on the shop floor is to use a step gauge. Figure 6 shows an example of a step gauge with six pins of different diameters. These pins are used to gauge parts into different groups based on the smallest pin that a part's opening gap does not fall through.

The target value for the process mean is 0·074 inch, and previous studies using a precision optical measurement device suggest that the standard deviation is constant and equal to 1·3. We wish to create a control chart that has an OC curve that passes through the points (74, 0·995) and (75·3, 0·005) or better (i.e. $\alpha=\beta=0{\cdot}005$, $\mu_1=1$) that uses a six-step gauge. With reference to Table 2, the optimal gauge limit design suggests classifying units into the seven intervals, with their corresponding weights, given in Table 5.

As an aside, note that the optimal weights given in this case, and in general, can be rounded off to two or three significant digits to ease implementation without a significant effect on the OCs of the resulting control chart. Using this step-gauge design and solving equations (4) and (5) for the sample size and critical value gives $n=27$ and $\lambda=0$. The resulting control charting procedure for each sample can be summarized as follows.

(1) Take a sample of 27 units from the process.
(2) Assign each of the 27 units a weight based on Table 5 above.
(3) Calculate the average weight for the sample.
(4) Plot the average weight on the control chart.
(6) Search for an assignable cause if the point plots above 0.

The above procedure was simulated and results are illustrated in Fig. 7. The first ten samples were in control, that is the mean was 74; the next three samples were taken after a mean shift of 1σ unit, that is the mean shifted to 75·3. Note that, owing to the small error rates that were chosen, the control chart detected the mean shift

Interval	Weight z_i
1: $(-\infty$, 73·00)	−1·7492
2: (73·00, 73·75)	−0·9553
3: (73.75, 74·35)	−0·4503
4: (74·35, 74·94)	0·0
5: (74·94, 75·55)	0·4503
6: (75·55, 76·30)	0·9553
7: (76·30, ∞)	1·7492

Table 5. Step-gauge design for metal-fasteners example.

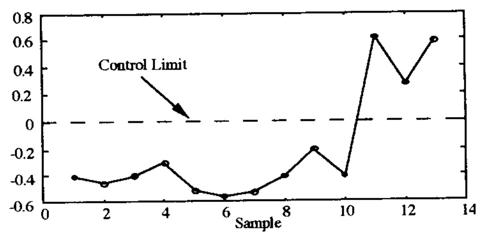

Figure 7. Control chart with 1σ shift in mean at sample 11.

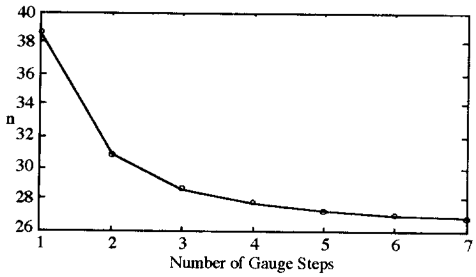

Figure 8. Plot of sample size required against the number of gauge limits using $\alpha = \beta = 0{\cdot}005$ and the optimal gauge limits derived from Table 2.

immediately. To ease implementation on the shop floor, it is possible to round off the weights. The loss in efficiency to go to two significant digits in this case is almost negligible and can be precisely evaluated using the branch-and-bound algorithm.

For the purposes of comparison, we have calculated the sample size required using the optimal gauge limits from Table 2 for one to seven gauge limits and have plotted the results in Fig. 8. From Fig. 8, it is clear that our six-step gauge compares very favourably with the optimal binomial approach (a one-step gauge) requiring samples of size 27 rather than 39. In addition, we calculated that a variables control chart approach would also require samples of size 27 (round up from 26·55); this agrees with the asymptotic nature of the curve as the number of step gauges increases in Fig. 7. Our purposed six-step-gauge control chart is thus virtually idential in terms of power with a control chart based on variables and is thus an excellent alternative in this situation.

6. Conclusion

We present a multiple-step-gauge control chart that is applicable for detecting shifts in a mean of a normal distribution when observations are classified into one of several groups. We show how the control chart can be designed to satisfy specified operating characteristics. We develop design methodology for both large and small sample sizes. We also address the question of optimal gauge design, deriving the optimal gauge limits for the normal approximation solution. The results show that the k-step-gauge control chart is a viable alternative to other control charts, approaching the variables-based control charts in efficiency. These charts are applicable in situations where variables measurements are expensive or impossible, and yet classifying units in groups is economical.

Acknowledgements

This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada. The authors would also like to thank two anonymous referees. The comments and suggestions that they made greatly improved the presentation of the material.

References

Beja, A., and Ladany, S. P., 1974, Efficient sampling by artificial attributes. *Technometrics*, **16** (4), 601–611.

Bray, D. F., Lyon, D. A., and Burr, I. W., 1973, Three class attributes plans in acceptance sampling. *Technometrics*, **15** (3), 575–585.

Dudding, B. P., and Jennett, W. J., 1944, *Control Chart Technique when Manufacturing to a Specification* (London: British Standards Institute).

Duncan, A. J., 1986, *Quality Control and Industrial Statistics*, 5th Edn (Homewood, IL: Richard D. Irwin).

Edwards, A. W. F., 1972, *Likelihood* (Cambridge University Press).

Evans, I. G., and Thyregod, P., 1985, Approximately optimal narrow limit gauges. *Journal of Quality Technology*, **17,** 63–66.

Kendall, M., and Stuart, A., 1979, *The Advanced Theory of Statistics*, Vol. 2, *Inference and Relationships*, 4th edition (London: Charles Griffin).

Kulldorff, G., 1961, *Contributions to the Theory of Estimation from Grouped and Partially Grouped Samples* (New York: Wiley).

Ladany, S. P., 1976, Determination of optimal compressed limit gaging sampling plans. *Journal of Quality Technology*, **8** (4), 225–231.

Ladany, S. P., and Sinuary-Stern, Z., 1985, Curtailment of artifical attribute sampling plans. *International Journal of Production Research*, **23** (5), 847–856.

Mace, A. E., 1952, The use of limit gages in process control. *Industrial Quality Control*, **8** (4) January, 24–31.

Ott, E. R., and Mundel, A. B., 1954, Narrow-limit gaging. *Industrial Quality Control*, **10** (5) March, 21–28.

Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., 1988, *Numerical Recipes*, (Cambridge University Press).

Stevens, W. L., 1948, Control by gauging. *Journal of the Royal Statistical Society, Series B*, **10** (1), 54–108.

Sykes, J., 1981, A nomogram to simplify the choice of a sampling plan using a single gauge. *Journal of Quality Technology*, **13,** 36–41.