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Statistical Process Control Using Two Measurement Systems

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Often in industry, critical quality characteristics can be measured by more than one measurement system. Typically, in such a situation, there is a fast but relatively inaccurate measurement system that may be used to provide some initial information and a more accurate and expensive, and possibly slower, alternative measurement device. In such circumstances, it is desirable to determine the minimum cost-control chart for monitoring the production process using some combination of the measurement systems. This article develops such a procedure. An example of its use in the automotive industry is provided.

KEY WORDS: Control chart; Measurement costs.

Metrology is an important aspect of manufacturing because measurements are necessary for monitoring and controlling production processes. In many situations, however, there is more than one way to measure an important quality dimension. Frequently the choice between the different measurement systems is not clear due to trade-offs with respect to measurement cost, time, and accuracy. One particular situation that is explored in this article occurs when there is a "quick and dirty" measurement device that is inexpensive and relatively fast but is not the most accurate way to measure and a slower, more accurate and expensive measurement device or method. Good examples of this situation occur in many manufacturing plants. For example, in foundries, the chemistry of molten iron may be checked using a quick method, called a "quick lab," or it may be sent to a laboratory. In the foundry application, the quick measurement is used to monitor and control the process because adjustments to composition are required immediately and the lab measurement takes several hours. The slower lab measurements are used only for after-the-fact confirmation. Another example is the use of in-line fixture gauges to monitor the production of engine covers. The fixture gauges provide approximate measurements for some critical dimensions, and a coordinate measurement machine (CMM) can be used to determine more precise values. This engine-covers example is discussed in more detail later.

When two measurement devices are available, the current process-monitoring approach is to use results from each measurement device separately and often for different purposes. From cost and efficiency considerations, however, it is not optimal in most cases to use only one of the measurement devices to monitor the process output. In this article a method for using both measurement systems together to monitor the process mean and process variability is pro-

posed. The basic idea is straightforward. The first measurement device is inexpensive and quick, so I try initially to make a decision regarding the state of control of the process based on results from the first measurement device. If the results are not decisive, I measure the same sample of units again using the more accurate measurement device. I assume that the testing is not destructive or intrusive. Notice that this procedure does not require additional sampling because the same sample is measured again if the initial results were not conclusive. Not requiring an additional independent sample is an advantage because obtaining another independent sample may be difficult and/or time consuming.

This idea of using the second measurement device only in cases in which the first measurement does not yield clear-cut results is motivated by earlier work by Croasdale (1974) and Daudin (1992). Croasdale and Daudin developed double-sampling control charts as an alternative to traditional $\bar{X}$ control charts. Double-sampling charts add warning limits to the traditional control charts in addition to control limits. The warning limits are used to decide when a second independent sample is needed to reach a conclusion regarding the process's stability. Double-sampling charts, however, are not applicable in the two-measurement-devices problem because they assume that the same measurement device measures all samples and that measurement error is negligible.

The article is organized in the following manner. In Section 1, control charts for detecting changes in the process mean or variability using two measurement devices in com-

bination are defined. An example of their use is given in Section 2. In Section 3, two-measurement control charts are designed to minimize measurement costs subject to a statistical constraint in terms of the false-alarm rate and power of the resulting charts. Finally, in Sections 4 and 5, some implementation issues are discussed and a summary of the results is given.

1. CONTROL CHARTS FOR TWO MEASUREMENT SYSTEMS

The results from the two measurement systems are modeled as follows. Let

$$Y_{ij} = X_i + e_{ij}, \qquad i = 1, \ldots, n, j = 1, 2, \qquad (1)$$

where X_i is the true dimension of the ith unit, Y_{i1} and Y_{i2} are the results when measuring the ith unit with the first and second measurement devices, respectively, and e_{ij} is the measurement error. We assume that the e_{ij}'s are independent and normally distributed with mean 0 and variance σ_j^2 and that X_i and e_{ij} are independent of each other. The assumption that the mean of e_{ij} equals 0 implies that I have compensated for any long-term bias of the measurement device. The variability of the two measurement devices (σ_1, σ_2) is assumed to be well known. This is a reasonable assumption because regular-gauge repeatability and reproducibility studies for all measurement devices are often required in industry and in any case may be easily performed. Because each sample may be measured twice, I assume that the measurement is nondestructive. I also assume that the actual dimensions of the quality characteristic of interest are normally distributed with mean and standard deviation equal to μ and σ, respectively. Thus, $X \sim N(\mu, \sigma^2)$, and $\bar{X} \sim N(\mu, \sigma^2/n)$. Moreover, without loss of generality, I assume that the in-control process has zero mean and standard deviation equal to 1. In other words, for the in-control process, the X variable represents a standardized variable. For nonnormal quality characteristics, a transformation to near normality would allow the use of the results presented here.

I begin by defining some terms. Measuring the n units in the sample with the first measurement device, I may calculate $\bar{Y}_1 = \sum_{i=1}^{n} Y_{i1}/n$. If the same sample is measured with the second measurement device, I obtain $\bar{Y}_2 = \sum_{i=1}^{n} Y_{i2}/n$. Based on the distributional assumptions, it can be shown that $\bar{Y}_1$ and $\bar{Y}_2$ are bivariate normal with $E(\bar{Y}_1) = E(\bar{Y}_2) = \mu$, $\text{var}(\bar{Y}_1) = (\sigma^2 + \sigma_1^2)/n$, $\text{var}(\bar{Y}_2) = (\sigma^2 + \sigma_2^2)/n$, and $\text{cov}(\bar{Y}_1, \bar{Y}_2) = E(\text{cov}(\bar{Y}_1, \bar{Y}_2|\bar{X})) + \text{cov}(E(\bar{Y}_1|\bar{X}), E(\bar{Y}_2|\bar{X})) = 0 + \sigma^2/n = \sigma^2/n$. Note that $\bar{Y}_1$ and $\bar{Y}_2$ are not independent because they represent the sample averages obtained by the first and second measurement device, respectively, on the *same* sample of size n. Assuming $\sigma_2 < \sigma_1, \bar{Y}_2$ provides more precise information about the true process mean than $\bar{Y}_1$. A weighted average of $\bar{Y}_1$ and $\bar{Y}_2$, however, provides even more information. Define $\bar{w}$ as the average of the i weighted sums given by

$$w_i = kY_{i1} + (1 - k)Y_{i2}. \qquad (2)$$

Based on the moments of $\bar{Y}_1$ and $\bar{Y}_2$, I get

$$E(\bar{w}) = \mu$$

$$\text{var}(\bar{w}) = \frac{1}{n} \left(k^2(\sigma_1^2 + \sigma^2) \right.$$
$$\left. + (k-1)^2(\sigma_2^2 + \sigma^2) + 2k(1-k)\sigma^2 \right)$$

$$\text{cov}(\bar{Y}_1, \bar{w}) = (\sigma^2 + \sigma_1^2 k)/n.$$

We obtain the most information about the true process mean when the weighting constant k is chosen so as to minimize $\text{var}(\bar{w})$. Denoting this best value for k as k_{opt} and solving gives

$$k_{opt} = \sigma_2^2/(\sigma_1^2 + \sigma_2^2). \qquad (3)$$

Using k_{opt}, the variance of $\bar{w}$ and the correlation coefficient relating $\bar{Y}_1$ and $\bar{w}$, denoted ρ_w, are given by the following equations, respectively:

$$\text{var}(\bar{w}|k_{opt}) = \left(\sigma^2 + \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right) / n \qquad (4)$$

and

$$\rho_w = \rho(\bar{Y}_1, \bar{w}|k_{opt}) = \frac{(\sigma^2[\sigma_1^2 + \sigma_2^2] + \sigma_1^2 \sigma_2^2)^{1/2}}{([\sigma^2 + \sigma_1^2][\sigma_1^2 + \sigma_2^2])^{1/2}}. \qquad (5)$$

The value of k_{opt} will be close to 0 if the second measurement system is much more precise than the first device. In that case, $\bar{w}$ almost equals $\bar{Y}_2$. In general, the bigger the discrepancy between σ_1 and σ_2, the less there is to gain from using $\bar{w}$ over $\bar{Y}_2$.

The proposed two-measurement $\bar{X}$ chart operates as follows. In every sampling interval, take a rational sample of size n from the process. Measure all units with the first measurement device to obtain $Y_{11}, Y_{21}, \ldots, Y_{n1}$. Calculate $\bar{Y}_1$, and if $\bar{Y}_1$ falls outside the interval $[-c_1, c_1]$, where c_1 is the control limit for the first measurement device, conclude that the process is out of control. If, on the other hand, $\bar{Y}_1$ falls within the interval $[-r_1, r_1]$, where r_1 is the extra measurement limit $(r_1 \leq c_1)$, conclude that the process is in control. Otherwise, the results from the first measurement device are inconclusive, and I must measure the n sample units again using the second measurement device. Combining the information from the two measurements on each unit in the sample together, I base my decisions on $\bar{w}$. If $\bar{w}$ falls outside the interval $[-c_2, c_2]$, where c_2 is the control limit for the combined sample, I conclude that the process is out of control; otherwise I conclude the process in in control. This decision process is summarized as a flowchart in Figure 1.

In many situations it is reasonable to simplify this procedure by setting c_1 equal to infinity. As a result of this restriction, based only on the results from the first measurement device, I can conclude that the process is in control or that I need more information, but not that the process is out of control. In applications this restriction is reasonable so long as the time delay for the second measurements is not overly large.

A two-measurement control chart designed to detect changes in process variability, similar to a traditional S
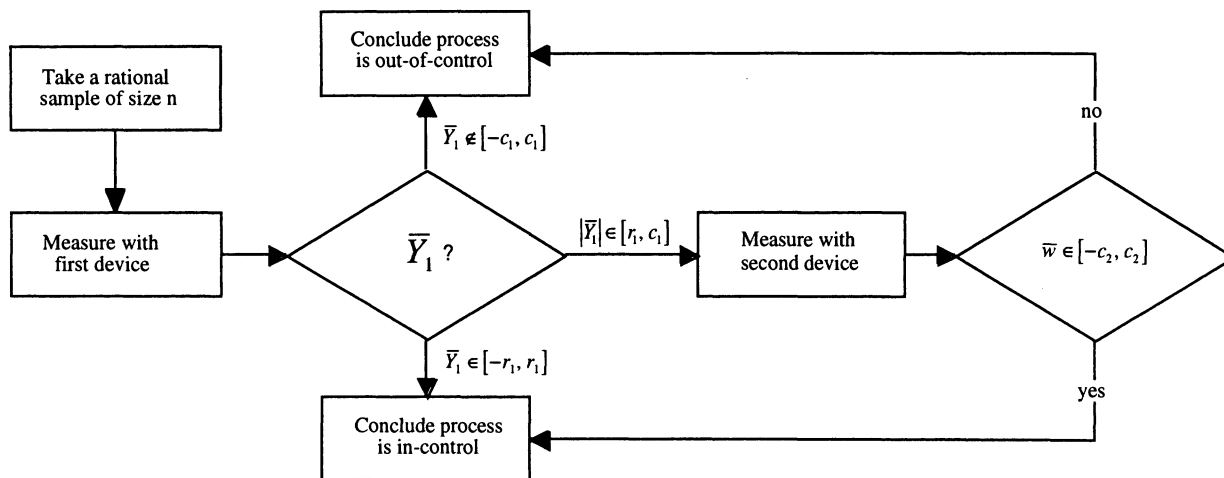
Figure 1. Decision Process for Control Charts for the Process Mean Using Two Measurement Systems.

chart, is also possible. If the measurement variability is substantial, however, decreases in the process variability are very difficult to detect. Thus, I consider a chart designed to detect only increases in variability. Moreover, to simplify the calculations somewhat, I do not allow signals based on only the first measurement device. This simplification is analogous to the version of the chart for the process mean in which set $c_1 = \infty$. The chart is based on two sample standard deviations, defined as

$$s_1 = \sqrt{\sum_{i=1}^{n} (y_{1i} - \bar{y}_1)^2/(n-1)}$$

and

$$s_w = \sqrt{\sum_{i=1}^{n} (w_i - \bar{w})^2/(n-1)},$$

where w_i is given by (2). The two-measurement-system control chart for detecting increases in standard deviation operates as follows. If $s_1 < d_1$, conclude that the process is in control with respect to variability. Otherwise, measure the sample again with the second measurement system. If $s_w < d_w$, conclude that the process is in control; otherwise conclude that the process variability has increased.

In any application involving two measurement devices, the first question that needs to be answered is whether just one of the measurement devices should be used or if using them in combination will result in substantially lower costs. It is difficult to provide simple general rules because there are many potentially important factors. If the cheaper measurement device is quite accurate, however, say $\sigma_1 < .4$ (relative to a process standard deviation of unity), then there is little to be gained by considering the second measurement device, and it is probably best to use only the first measurement device. When the measurement variability is larger, a fairly simple decision rule for whether a control chart based on two measurement systems is preferable can be obtained by considering only the variable measurement cost associated with each measurement device. To match

the performance of a traditional Shewhart $\bar{X}$ control chart with subgroups of size 5 with measurement device i, I need samples of size $5(1 + \sigma_i^2)$. If the variable measurement cost associated with the second measurement device is ν_2 times the amount for the first measurement device, then the ratio of the variable measurement costs for the charts based on measurement systems 1 and 2 is $R = \nu_2(1 + \sigma_2^2)/(1 + \sigma_1^2)$. Based on experience, the greatest gains from using the two-measurement-device control chart arise when R is close to 1. Generally for a substantial reduction in costs, say greater than around 10%, the value of R should lie between .6 and 8. Otherwise, using only the second measurement device is preferred if $R < .6$, and using only the first measurement device would be better if $R > 8$. More specific cost comparisons are considered at the end of Section 3.

2. EXAMPLE

The manufacture of engine front covers involves many critical dimensions. One such critical dimension is the distance between two bolt holes in the engine cover used to attach the cover to the engine block. This distance may be measured accurately using a CMM, which is expensive and time consuming. An easier, but less accurate, measurement method uses a fixture gauge that clamps the engine cover in a fixed position while measuring hole diameters and relative distances.

In this example, the fixture gauge is the first measurement device and the CMM is the second measurement device. Previous measurement-system studies determined that, for standardized measurements, $\sigma_1 = .5$ and $\sigma_2 = .05$ approximately; that is, the CMM has less measurement variability than the fixture gauge. I also know that, on a relative-cost basis, using the CMM is six times as expensive as the fixture gauge in terms of personnel time. I shall assume that the fixed costs associated with the two measurement methods is 0. Thus, in terms of the notation from the sample cost model presented in Section 3, I have $f_1 = f_2 = 0, \nu_1 = 1$, and $\nu_2 = 6$. The main goal in this example was to control the process mean. As such, in this example I use a two-measurement-system control chart only to detect changes in the process mean. Process variability is monitored us-

ing a traditional S chart with the results only from the first measurement system.

Solving Expression (9) given in Section 3 of this article, with the additional simplification that $c_1 = \infty$, gives $r_1\sqrt{n} = 2.80$ and $c_2\sqrt{n} = 2.92$, with $n = 5.26$ for a relative cost of 5.65. These values are given approximately in Figure 3, Section 3. In this optimal solution the values for r_1 and c_2 are almost equal. From an implementation perspective, setting r_1 and c_2 equal is desirable because it simplifies the resulting control chart, as will be shown. With the additional constraint that $r_1 = c_2$, the optimal solution to (9) is $r_1\sqrt{n} = c_2\sqrt{n} = 2.89, n = 5.36$, with a corresponding cost of 5.67. For implementation, the sample size is rounded off to 5. Thus, the control limits r_1 and c_2 are set at ± 1.3. The measurement costs associated with this plan are around 10% less than the measurement costs associated with the current plan that uses only the first measurement device and around 80% less than the cost associated with using only the CMM machine.

Figure 2 gives an example of the resulting two-measurement $\bar{X}$ control chart. On the chart the sample averages based on the first measurement device are shown with an "o," and the sample average of the combined first and second measurements (if the second measurement is deemed necessary) are shown with an "x." The extra measurement limit ($\pm r_1$) for the results from the first measurement device and the control limit ($\pm c_2$) for the combined sample are given by the solid horizontal lines on the chart. If the sample average based on the first measurement lies between the solid horizontal lines on the chart, I conclude that the process is in control. Otherwise, if the initial point lies outside the extra measurement limits, a second measurement of the sample is required. Using the second measurement, I calculate the combined sample weighted average $\bar{w} = .01\bar{Y}_1 + .99\bar{Y}_2$ (based on this weighting, I could use just $\bar{Y}_2$ rather than $\bar{w}$ without much loss of power in this example). If $\bar{w}$ falls outside the solid horizontal lines, I conclude that the process shows evidence of an assignable cause; otherwise the process appears to be in control. The dashed/dotted line denotes the center line of the control chart. In this example, for illustration, the value 1.0 was added to all the measurements after the 19th observation to simulate a one-sigma shift in the process mean. Figure 2 shows that, among these 25 measurements, a sec-ond sample was required six times, at sample numbers 7, 20, 21, 22, 24, and 25. Only samples 21, 22, 24, and 25 yield an out-of-control signal, however. In the other cases, the second measurement of the sample suggests that the process is still in control. Of course, the number of times the second measurement was needed after observation 19 is also an indication that the process has shifted. In this application, using two-measurement control charts results in a reduction in the measurement costs without affecting the ability of the monitoring procedure to detect process changes.

3. DESIGN OF CONTROL CHARTS USING TWO MEASUREMENT SYSTEMS

Determining the optimal design for two-measurement control charts involves determining the best values for the control limits and sample size. As pointed out by Woodall (1986, 1987), however, purely economic models of control charts may yield designs that are unacceptable in terms of operating characteristics. For example, the "optimal" design from a purely cost perspective may have such a large false-alarm rate that the chart is routinely ignored. For this reason, in this article, the optimal designs for two-measurement control charts are constrained to satisfy certain minimum operating characteristics. I first consider the design of two-measurement $\bar{X}$ charts and then look at two-measurement S charts. The MATLAB® computer code that determines the optimal design in both cases is available from me.

3.1 Design of Two-Measurement $\bar{X}$ Charts

Using the assumption of normality, it is possible to determine the probabilities of making the various decisions illustrated in Figure 1. Let $\phi(z) = e^{-z^2/2}/\sqrt{2\pi}$ and $Q(z) = \int_{-\infty}^{z} \phi(x)\, dx$ be the probability density function and cumulative density function of the standard normal, respectively. Moreover, denote the probability density function of the standardized bivariate normal as $\phi(z_1, z_2, \rho) = (2\pi\sigma_1\sigma_2\sqrt{1-\rho^2})^{-1}\exp(-(z_1^2 - 2\rho z_1 z_2 + z_2^2)/(2 - 2\rho^2))$. Then, (6), (7), and (8) give expressions for the probabilities that the following events occur: The procedure concludes that the process is out of control (i.e., the procedure signals) based on results from the first measurement, measuring the sample with the second measurement is necessary, and the
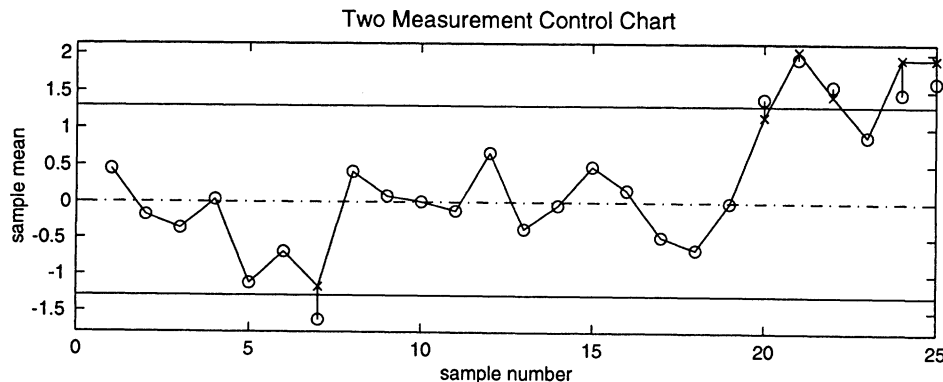
Two Measurement Control Chart

Figure 2. Two-Measurement Control Chart for the Process Mean.

combined results from the first and second measurement devices lead to a signal:

$$p_1(\mu) = \text{Pr(signal on first measurement)}$$

$$= \text{Pr}(\bar{Y}_1 > c_1 \text{ OR } \bar{Y}_1 < -c_1)$$

$$= 1 + Q([-c_1 - \mu]/\sigma_1^*) - Q([c_1 - \mu]/\sigma_1^*), \quad (6)$$

$$q_1(\mu) = \text{Pr(second measurement needed)}$$

$$= \text{Pr}(r_1 < \bar{Y}_1 < c_1 \text{ OR } - r_1 > \bar{Y}_1 > -c_1)$$

$$= Q([c_1 - \mu]/\sigma_1^*) - Q([r_1 - \mu]/\sigma_1^*)$$

$$+ Q([-r_1 - \mu]/\sigma_1^*) - Q([-c_1 - \mu]/\sigma_1^*), \quad (7)$$

and

$$p_2(\mu) = \text{Pr(signal on combined measurements)}$$

$$= \text{Pr}((\bar{w} > c_2 \text{ OR } \bar{w} < -c_2)$$

$$\text{and } (r_1 < \bar{Y}_1 < c_1 \text{ OR } - r_1 > \bar{Y}_1 > -c_1))$$

$$= \iint\limits_{\substack{z_1 \in [(r_1-\mu)/\sigma_1^*, (c_1-\mu)/\sigma_1^*] \\ z_2 \in [-\infty, (-c_2-\mu)/\sigma_w^*]}} \phi(z_1, z_2, \rho_w)\, dz_1\, dz_2$$

$$+ \iint\limits_{\substack{z_1 \in [(r_1-\mu)/\sigma_1^*, (c_1-\mu)/\sigma_1^*] \\ z_2 \in [(c_2-\mu)/\sigma_w^*, \infty]}} \phi(z_1, z_2, \rho_w)\, dz_1\, dz_2$$

$$\times \iint\limits_{\substack{z_1 \in [(-c_1-\mu)/\sigma_1^*, (-r_1-\mu)/\sigma_1^*] \\ z_2 \in [-\infty, (-c_2-\mu)/\sigma_w^*]}} \phi(z_1, z_2, \rho_w)\, dz_1\, dz_2$$

$$+ \iint\limits_{\substack{z_1 \in [(-c_1-\mu)/\sigma_1^*, (-r_1-\mu)/\sigma_1^*] \\ z_2 \in [(c_2-\mu)/\sigma_w^*, \infty]}} \phi(z_1, z_2, \rho_w)\, dz_1\, dz_2, \quad (8)$$

where $\sigma_1^* = \sqrt{(\sigma^2 + \sigma_1^2)/n}$ and $\sigma_w^* = \sqrt{(\sigma^2\sigma_1^2 + \sigma^2\sigma_2^2 + \sigma_1^2\sigma_2^2)/(\sigma_1^2 + \sigma_2^2)n}$. Note that p_1, p_2, and q_1 depend on the true process mean and standard deviation. Setting c_1 equal to infinity implies that $p_1(\mu) = 0$ for all μ.

In this article a cost model based on measurement costs is developed. This measurement-cost model is easy to use because it requires only estimates of the fixed and variable measurement costs for the two measurement devices. A more complex cost model that considers that all the production costs could be developed based on the general framework of Lorenzen and Vance (1986). The production-cost model is often difficult to apply, however, because costs due to false alarms, searching for assignable causes, and so forth are difficult to estimate in many applications.

The goal is to minimize the measurement costs while maintaining the desired minimum error rates of the procedure. Let f_i and v_i denote the fixed and variable measure-

ment costs for the ith measurement system, respectively ($i = 1, 2$). In my analysis, without loss of generality, I may set $v_1 = 1$ because the results depend only on the relative values of the measurement costs. In addition, to restrict the possibilities somewhat, the fixed cost associated with the first measurement device is set to 0; that is, $f_1 = 0$. This restriction is justified because typically the first measurement device is very easy and quick to use and would not require much setup time or expense. Then, the measurement cost per sample is $n + (f_2 + v_2 n)q_1(\mu)$. The best choice for the sampling interval must be determined through some other criterion, such as the production schedule. There are several ways to define an objective function using the measurement costs. Because the process will (it is hoped) spend most of its time in control, I minimize the in-control measurement costs. Using this formulation, the optimal design of the control chart using two measurement devices is determined by finding the design parameters that

$$\text{minimize } n + (f_2 + v_2 n)q_1(0)$$

$$\text{subject to } \alpha = p_1(0) + p_2(0) \le .0027$$

$$\text{and } \beta = 1 - p_1(2) + p_2(2) \le .0705, \quad (9)$$

where α is the false-alarm rate—that is, the probability that the chart signals when the process mean is in control—and $1 - \beta$ is the power of the probability that the chart signals when the process mean shifts to $\mu_1 = \pm 2$. These particular choices for maximum false-alarm rate and minimum power to detect two sigma shifts in the mean are based on at least matching the operating characteristics of a Shewhart $\bar{X}$ chart with samples of size 5.

Optimal values for the design parameters c_1, c_2, r_1, and n that satisfy (9) can be determined using a constrained minimization approach such as applying the Kuhn–Tucker conditions. This solution approach was implemented using the routine "constr" in the optimization toolbox of MATLAB®.

Figures 3 and 4 show the optimal design parameters for two measurement charts that satisfy (9) for different measurement cost parameters when setting c_1 equal to infinity. Figure 3 gives results when the second measurement device also has no fixed costs, and Figure 4 considers the situation in which the fixed cost associated with the second measurement device is relatively large. Figures 3 and 4 may be used to determine the design parameter values that are approximately optimal for two-measurement $\bar{X}$ charts in terms of in-control measurement costs. For measurement costs in between those given, interpolation can be used to determine reasonable control-limit values. In practice, the sample size, n, must be rounded off to the nearest integer value. Rounding off the sample size affects the power of the control chart but has no effect on the false-alarm rate of the procedure. Of course, rounding down the sample size decreases the procedure's power, but rounding up increases the power.

Figures 3 and 4 each consist of four subplots that show contour plots of the optimal design parameters— $r_1\sqrt{n}, c_2\sqrt{n}$, and n as a function of σ_1 and σ_2, the variability inherent in the two measurement devices. Each subplot represents four different values of v_2, the variable measure-

$$v_2 = 1$$

$$v_2 = 2$$

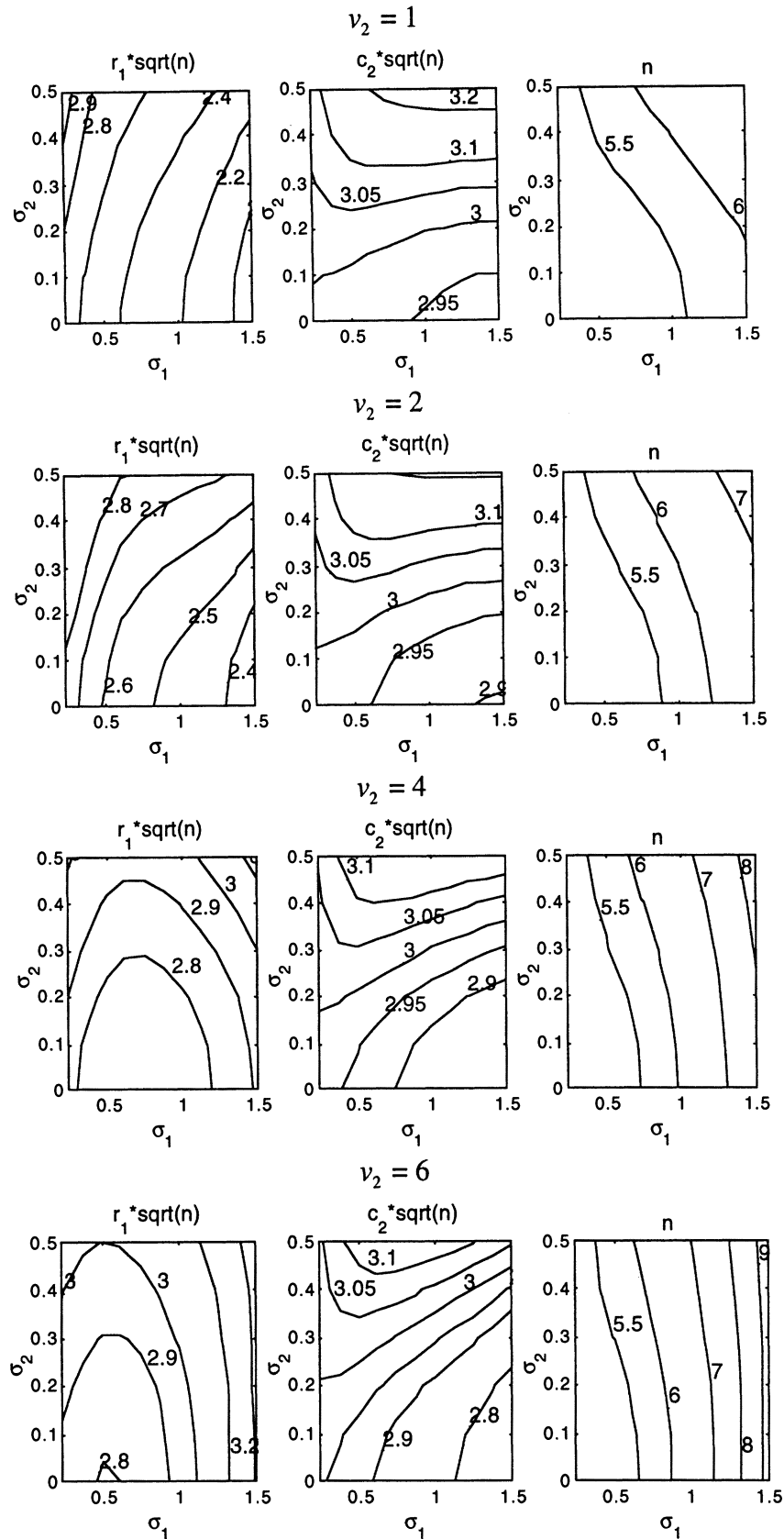$$v_2 = 4$$

$$v_2 = 6$$

Figure 3.   Contour Plots of the Design Parameters for the No-Fixed-Cost Case: $f_1 = 0$, $v_1 = 1$, $f_2 = 0$.

ment cost associated with the second measurement device. Optimal values for $r_1 \sqrt{n}$, $c_2 \sqrt{n}$, and n in the general case in which c_1 is allowed to vary are very similar to those given in Figures 3 and 4. In general, the optimal value of c_1 is large and consequently does not affect the procedure much unless there is a large shift in the process mean.
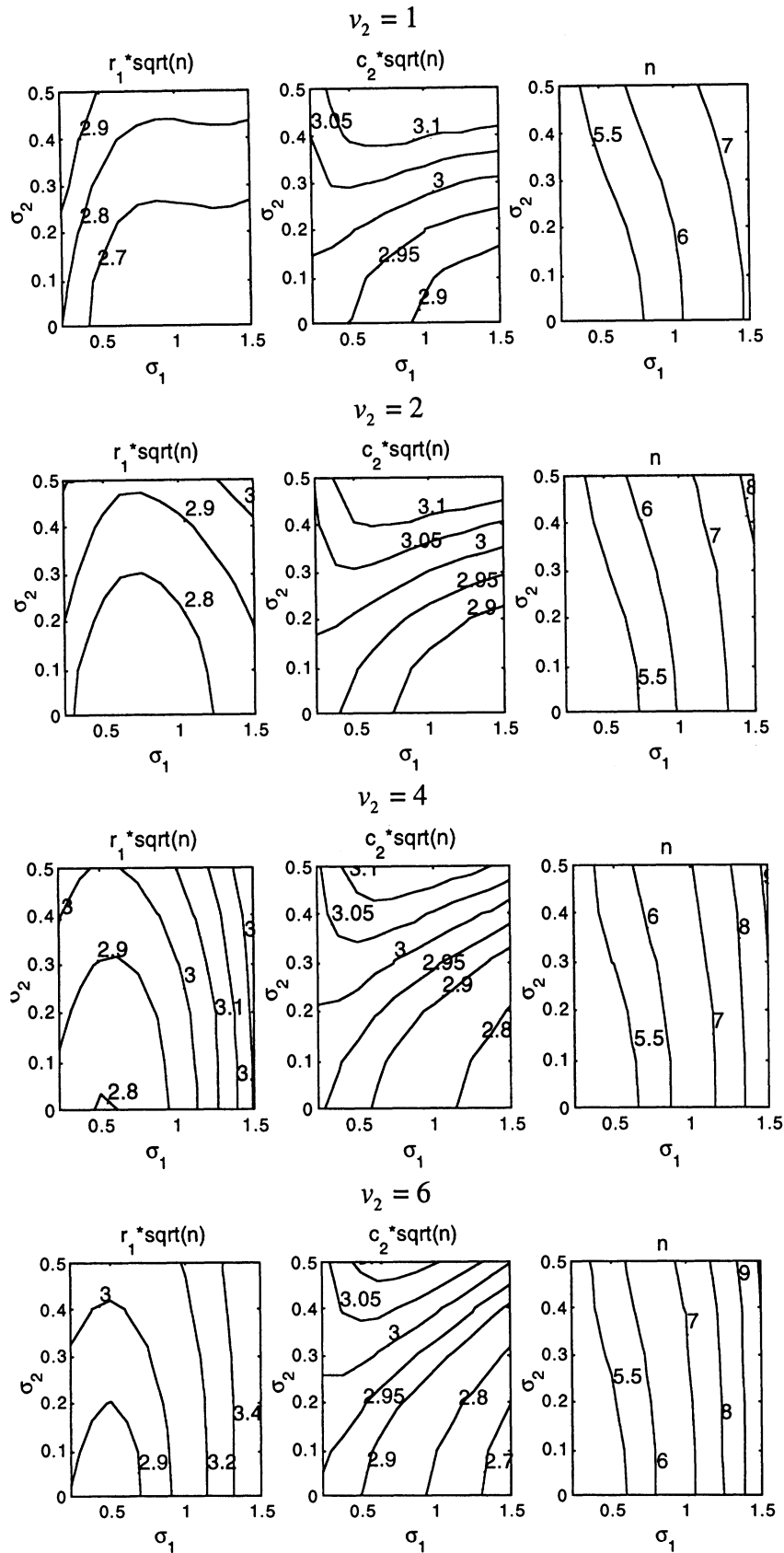
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$$v_2 = 1$$

$$v_2 = 2$$

$$v_2 = 4$$

$$v_2 = 6$$

Figure 4. Contour Plots of the Design Parameters for the Large-Fixed-Cost Case: $f_1 = 0$, $v_1 = 1$, $f_2 = 10$.

Figures 3 and 4 suggest that the parameters $r_1\sqrt{n}$ and $c_2\sqrt{n}$ are the most sensitive to changes in the variability of the measurement devices. In general, when the measure- ment costs of the two measurement devices are comparable, as the first measurement device becomes more variable (σ_1 increases), n increases, while $r_1\sqrt{n}$ decreases. This result
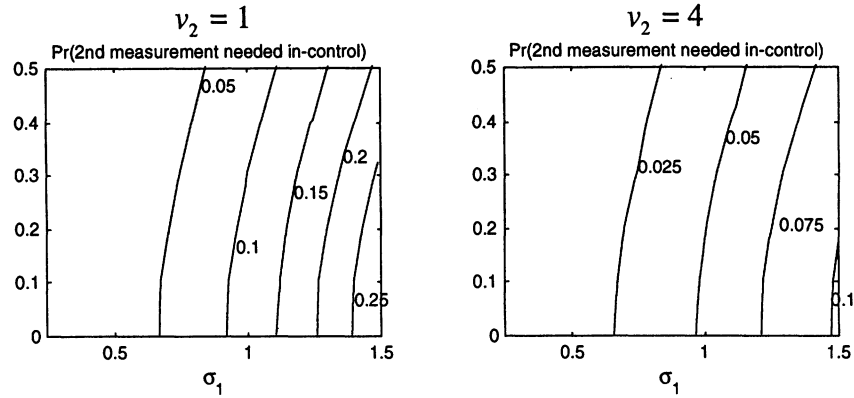
$$\nu_2 = 1 \qquad\qquad \nu_2 = 4$$

Figure 5. Contour Plots of the Probability That the Second Measurement Is Required With the Process in Control: $f_1 = 0$, $v_1 = 1$, $f_2 = 0$.

makes sense because it indicates that I rely more on the second measurement device when the first device is less precise. Conversely, as the second measurement device becomes more variable (σ_2 increases), $c_2\sqrt{n}$ and n increase while $r_1\sqrt{n}$ increases marginally because I rely more on the first measurement device.

Now consider the case in which the second measurement device is expensive (f_2 or ν_2 large). As the second measurement device becomes less reliable (σ_2 increases), again I observe that $c_2\sqrt{n}$ increases while n and $r_1\sqrt{n}$ increase marginally, which makes sense. The pattern appears to be counterintuitive, however, when the first measurement device becomes less reliable (σ_1 increases) because n and $c_2\sqrt{n}$ decrease marginally, but $r_1\sqrt{n}$ increases! Does this mean that I rely more heavily on the inaccurate first measurement device? Looking more closely, this apparent contradiction disappears. As σ_1 increases, the optimal $r_1\sqrt{n}$ also increases, but this does not imply that the decisions are more likely to be based on only the first measurement device. When the accuracy of a measurement device is poor, I expect to observe large deviations from the actual value. Thus, the observed increase in $r_1\sqrt{n}$ is only taking this into account. Consider Figure 5, which shows contours of the probability that the second measurement is needed in the two cases $f_2 = 0$ and $\nu_2 = 1$ or 4. The plots in Figure 5 show clearly that, as the first measurement device becomes less accurate, I rely on it less, even though, as shown in Figure 3, $r_1\sqrt{n}$ increases.

I may also compare the performance of using two measurement charts with traditional $\bar{X}$ using only one of the measurement systems. Figure 6 shows the percent reduction in measurement costs attainable through the use of both the measurement systems as compared with the best of the two individual measurement systems. In the case in which ν_2 equals 2, the dotted line shows the boundary between the points at which each individual measurement system is preferred. To the right of the dotted line (where the measurement variability of the first measurement system is large), the second measurement system is preferred. When ν_2 equals 4 and 6, the first measurement device on its own is preferred over the second measurement device over the whole range of the plot.

3.2 Design of Two Measurement S Charts

Now consider deriving the optimal two-measurement control chart to detect increases in the process variability. Mathematically, the optimal two-measurement S chart that minimizes in-control measurement costs is determined by finding the control limits d_1 and d_w that

minimize $n + \nu_2 n p_{s_1}(1)$

subject to $p_s(1) \le .001$ and $p_s(2) \ge .33$, $\qquad$ (10)

where $p_{s_1}(1) = \Pr(s_1 \ge d_1|\sigma = 1) = 1 - \chi^2_{n-1}(d_1^2(n - 1)/(1 + \sigma_1^2))$ and $p_s(\sigma)$ equals the probability that the two-measurement S chart signals; that is, $p_s(\sigma) = \Pr(s_w > d_w|s_1 > d_1, \sigma)$. $\chi^2_{n-1}(x)$ is the cumulative density func-

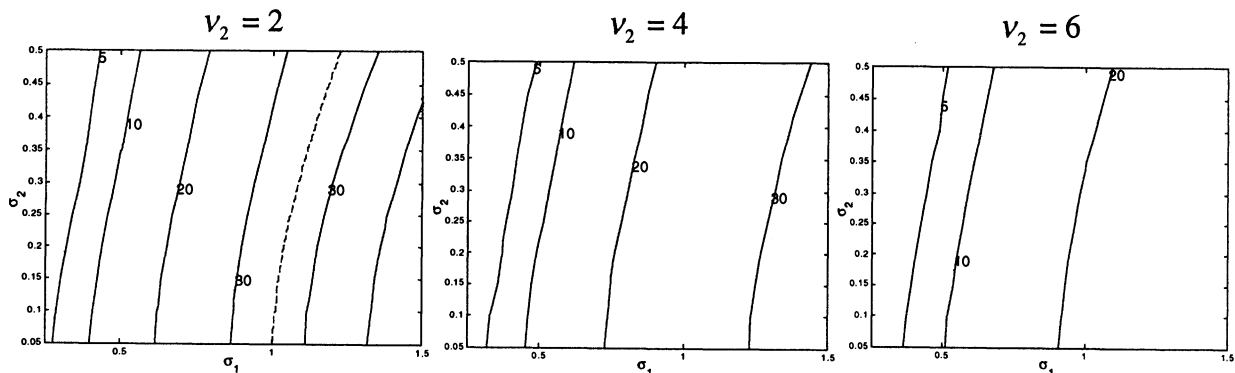$$\nu_2 = 2 \qquad\qquad \nu_2 = 4 \qquad\qquad \nu_2 = 6$$

Figure 6. Contour Plots Showing the Percent Reduction in In-Control Measurement Costs Possible Using the Two-Measurement $\bar{X}$ Control Chart.
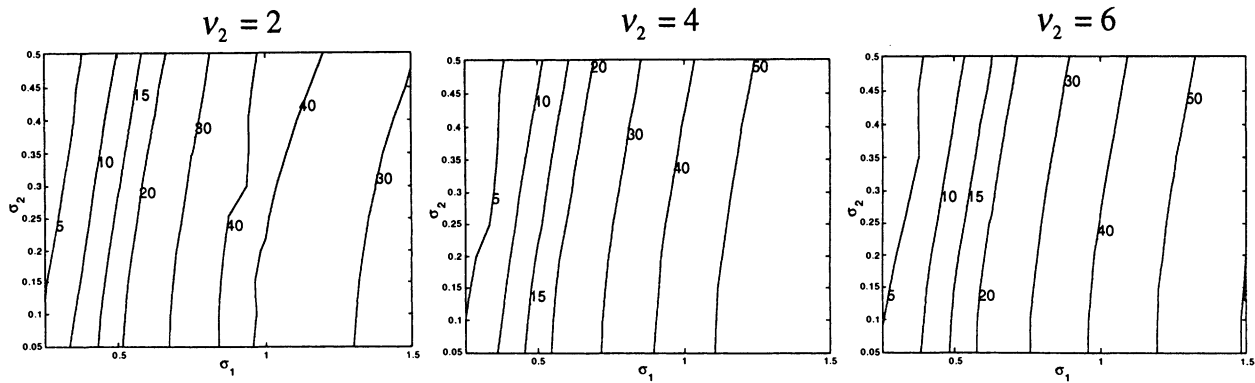
Figure 7. Percentage Decrease in In-Control Measurement Costs Possible With Two-Measurement S Chart, Assuming $f_2 = 0$.

tion of a central chi-squared distribution with $n - 1$ df. Using results presented in the Appendix, I may accurately approximate $p_s(\sigma)$ for any given actual process standard deviation. The choice of .33 is based on the power that can be attained using a traditional S chart with no measurement error, samples of size 5, and a false-alarm rate of .001.

Figure 7 shows the expected percent decrease in measurement costs that results when using the optimal two-measurement S chart rather than the lowest cost traditional S chart based on only one of the measurement systems. When $\nu_2 = 1$—that is, both measurement systems are equally expensive—using just the more accurate measurement device is always preferred and it is not beneficial to use the two-measurement-system approach. Figure 7 suggests that large potential savings in measurement costs are possible using the two-measurement approach to detect increases in process variability.

In practice, a process is typically monitored using both $\bar{X}$ and S charts. Thus, from an implementation perspective using the same sample size for both charts is highly desirable. For two-measurement charts, because typically detecting changes in the process mean is a higher priority, I use the sample size suggested by the optimal two-measurement $\bar{X}$ chart. Solving (10) shows that the optimal sample size for the two-measurement S chart is usually smaller than the sample size suggested for the two-measurement $\bar{X}$ chart. As a result, by using the larger sample size the resulting two-measurement S chart will have better than the minimum defined operating characteristics.

Deriving the best values for n, d_1, and d_w from (10), I could prepare plots similar to those in Figures 3 and 4. To simplify the design, however, I consider an approximation. Based on the range of typical values for measurement costs and the measurement variability and assuming $f_2 = 0$, I obtain, using regression analysis, the following approximations for the optimal control limits:

$$\hat{d}_1 = 1.94 - .18\sigma_1 + .28\sigma_1^2 + .03\nu_2$$

$$\hat{d}_w = 2.7 - .11\sigma_1 + .22\sigma_2 - .01\nu_2 - .27\hat{d}_1. \qquad (11)$$

These approximately optimal limits give good results over the range of typical measurement variability.

4. IMPLEMENTATION ISSUES

An alternative approach to process monitoring in this context is to use a second sample that is different from the first sample—that is, to take a completely new sample rather than to measure the first sample again. This approach is of course a necessity if the testing is destructive, but it leads to increased sampling costs, as well as difficulties in obtaining a new independent sample in a timely manner, due to autocorrelation in the process. If these sampling concerns can be overcome, however, the advantage of using an additional sample is that more information about the true nature of the process is available in two independent samples than in measuring the same sample twice. If feasible, taking a new *independent* sample would be preferred; however, in many cases it is not possible in a timely manner.

In a similar vein, I may consider situations in which repeated measurements with a single measurement system are feasible. If repeated *independent* measurements are possible, then, by averaging the results, I can reduce the measurement variability by a factor of $\sqrt{n}$. If the measurements are very inexpensive, then repeated *independent* measurement with one device will eventually yield (using enough measurements) a measurement variability so small that it may be ignored. Alternately, I could apply the methodology developed in this article in which I consider the second measurement to be simply the results of repeated measurements on the units with the first measurement device. If repeated inexpensive *independent* measurements using the first measurement device are possible, then using those measurements would be the preferred approach. This approach will work only, however, if I can obtain repeated *independent* measurements of the units, which is often not the case.

5. SUMMARY

This article develops a measurement-cost model that can be used to determine an optimal process-monitoring control chart that uses two measurement devices. It is assumed that the first measurement device is fast and cheap, but relatively inaccurate, whereas the second measurement device is more accurate but also more costly. The proposed monitoring procedure may be considered an adaptive monitoring method that provides a reasonable way to compromise between measurement cost and accuracy.

ACKNOWLEDGMENTS

APPENDIX: DERIVATION OF THE PROBABILITY THAT THE S CHART SIGNALS

Using the notation of the article,

$$A = \begin{bmatrix} \sum_{i=1}^{n}(y_{1i} - \bar{y}_1)^2, & \sum_{i=1}^{n}(y_{1i} - \bar{y}_1)(w_i - \bar{w}) \\ \sum_{i=1}^{n}(y_{1i} - \bar{y}_1)(w_i - \bar{w}), & \sum_{i=1}^{n}(w_i - \bar{w})^2 \end{bmatrix}$$

has a central Wishart distribution with $n - 1$ df and covariance matrix given by

$$\sum = \begin{bmatrix} \sigma^2 + \sigma_1^2, & \sigma^2 + k\sigma_1^2 \\ \sigma^2 + k\sigma_1^2, k^2(\sigma^2 + \sigma_1^2) & \\ +(k - 1)^2(\sigma^2 + \sigma_2^2) + 2k(1 - k)\sigma^2 & \end{bmatrix}$$

(Arnold 1988). Denoting the elements of the matrix A as a_{ij}, it can be shown that

$$\Pr(0 \le a_{11} \le c_1\Sigma_{11}, 0 \le a_{22} \le c_2\Sigma_{22})$$

$$= \frac{(1 - \rho^2)^{(n-1)/2}}{\Gamma((n - 1)/2)} \sum_{j \ge 0} \frac{\rho^{2j}}{\Gamma(j + (n - 1)/2)\Gamma(j + 1)}$$

$$\times I\left(j + (n - 1)/2, \frac{c_1}{2(1 - \rho^2)}\right)$$

$$\times I\left(j + (n - 1)/2, \frac{c_2}{2(1 - \rho^2)}\right),$$

where Σ_{ij} is an element of the covariance matrix, $\rho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$ is the correlation coefficient, $\Gamma(x)$ is the gamma function, and $I(d, g) = \int_0^g t^{d-1}e^{-t}\,dt$ is the incomplete gamma function. This infinite sum converges quickly unless ρ is very close to 1 (or -1).

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