CHAPTER 1

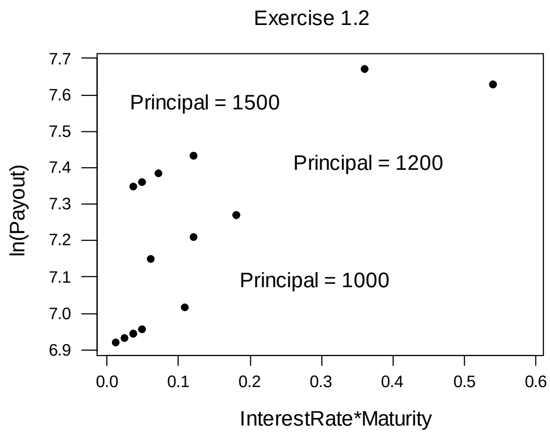
1.1 Tensile strength of an alloy can be expected to increase with increasing hardness and density of the stock. Bivariate scatter plots of tensile strength against hardness and of tensile strength against density of the stock are useful. Such scatter plots indicate whether the relationship is linear, or more complicated.

Bivariate scatter plots are unable to reveal 3-dimensional relationships. For that one needs to consider three-dimensional graphs. Alternatively, one can proceed as follows. If measurements on the tensile strengths of several different alloys of a given density but of changing values of hardness are given, one can plot tensile strength against hardness at this one fixed level of density. Furthermore, if tensile strength and hardness data for alloys of a second different density are available, one can construct a similar scatter plot for that other level of density. If the two scatter plots (scatter plots of tensile strength against hardness, at the two different levels of density) show different slopes, then the effect of hardness on tensile strength depends on the level of density. The factors hardness and density of the stock are said to interact in their effect on tensile strength.

Data from experiments are usually more informative as one can control the conditions under which the experimental runs are carried out. Experimentation is probably not possible in case (f). The relative humidity conditions in the plant can not be varied according to a fixed experimental plan. Instead, one takes measurements in the plant on the relative humidity, and at the same time on the output (performance) of the process. A danger with such data is that the relative humidity in the plant may be affected by unknown factors that also affect the output. The root cause is not the humidity of the plant, but these other "lurking" variables.

1.2 The graph given below indicates a linear relationship between ln(Payout) and the product of interest rate and maturity, with an intercept that depends on the invested principal. Note that the linear model in the transformed variables fits perfectly.

This is expected from the model Payout = Pexp(RT). Taking the logarithm on both sides of the equation, leads to ln(Payout) = ln(P) + RT. The intercept changes with the logarithm of the invested principle; the regression coefficient of RT is one.



1.3 Selected examples are:

Exercise 2.9: MBA grade point average and GMAT score: observational study
Exercise 2.10: Fuel efficiency and car characteristics: observational study of 45 cars
Exercise 2.24: Thickness of egg shell and PCB: observational study on pelicans
Exercise 2.27: Absorption of chemical liquid; experimental data
Exercise 4.12: Amount of plant water usage: observational study
Exercise 4.14: Survival of bull semen: experimental data
Exercise 4.15: Toxic action of a certain chemical on silkworm larvae: experimental data
Exercise 4.21: Abrasion as function of hardness and tensile strength of rubber: experimental data
Exercise 6.14: Tear properties of paper: experimental data
Exercise 6.17: Rigidity, elasticity and density of timber: observational study
Exercise 8.1: Incumbent vote share in US presidential elections: observational study
Exercise 8.2: Height and weight of boys: observational study
Exercise 8.3: Soft drink sales: observational study

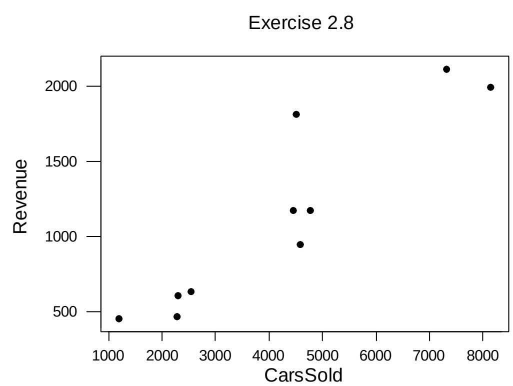
1.4 The response variable may be the breaking strength of a viscose fiber, and the explanatory variables may be the percentage of certain chemicals in the spin bath and the speed at which the liquid viscose is pressed through the nozzles into the spin bath. A designed experiment varies the explanatory variables (the design factors) according to a well thought-out plan and randomizes the arrangement of the experimental runs. The breaking strength of the resulting material is measured for each experimental run. In this case the data arise from a designed experiment.

However, the data could also be obtained through an observational study. The plant manager may take readings on the process – measurements on the breaking strength of the fiber and on the chemicals present in the spin bath, as well as the speed of the process. The manager may do this every 4 hours, collect observational data, and construct a regression model relating the response to the explanatory variables. However, several problems may arise with such observational data. First, the variation in the explanatory factors may not be large enough to actually affect the response. Second, and more importantly, the response may be affected by other variables that one has failed to control and account for. For example, the relative humidity may play a role. With observational data such as these one is never sure whether a "lurking" variable may be present. With designed experiments, and proper randomization of the experimental runs, such problems are much smaller.

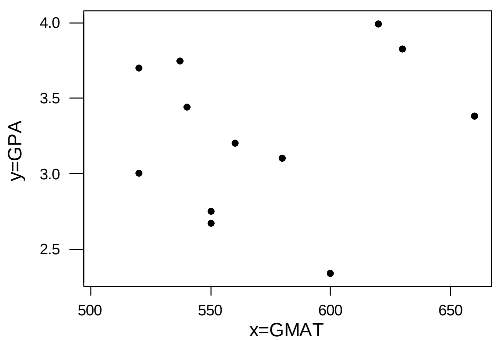
Monthly macroeconomic data on interest rates, GNP, and unemployment are examples of observational data. The data are given to the analyst who has no opportunity to affect the way the data are obtained.

Survey data are other examples of observational data; for example, survey data that involve observations on brand choices and features of products. Alternatively, brand preferences can be assessed through designed experiments. Participants in such experiments are presented a sequence of brands with various characteristics, arranged in a random sequence, and their brand selections are measured. In this case the data arise from an experiment.

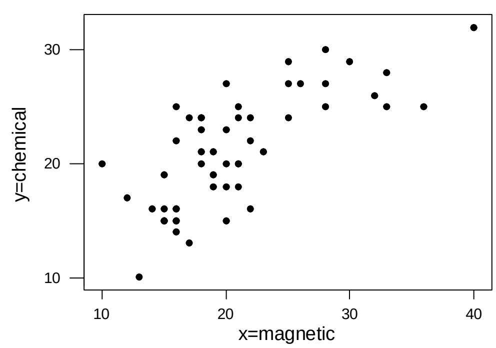
1.5 Scatter plots for the data in Exercises 2.8, 2.9, 2.21, and 2.25 are given below. We notice a linear relationship in Exercise 2.8. There is no strong (linear) relationship in Exercise 2.9. The relationship in Exercise 2.21 may involve a quadratic component; more information on the response when x is in the range from 30 to 40 would be helpful. We notice an approximate linear relationship in Exercise 2.25. However, note that the two responses between 3 and 4 at the high level of x are somewhat different from the rest.



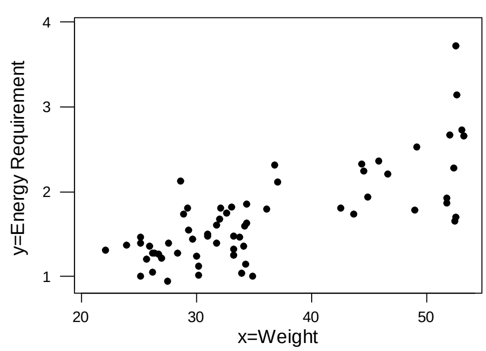
Exercise 2.9



Exercise 2.21

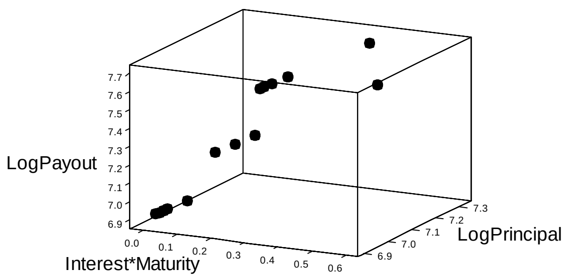


Exercise 2.25

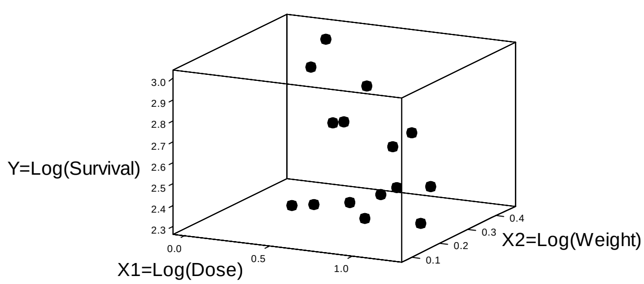


1.6 Usually it is not very easy to spot relationships from 3-dimensional graphs; see the two examples shown below. The bivariate scatter plots for the silkworm data set are easier to interpret.

3-Dimensional Plot: Investment Data



3-Dimensional Plot: Silkworm Data



1.7 Consider models with a single explanatory variable *x*. The quadratic model,

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon \ ,$$

is nonlinear in the explanatory variable *x*, but linear in the three parameters β_0, β_1 and β_2. The polynomial model (with $p > 1$),

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + ... + \beta_p x^p + \varepsilon ,$$

is nonlinear in the explanatory variable *x*, but linear in the parameters.

The quadratic model with two explanatory variables,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11}(x_1)^2 + \beta_{22}(x_2)^2 + \beta_{12} x_1 x_2 + \varepsilon \quad ,$$

is nonlinear in x_1 and x_2, however it is linear in the parameters. The equation describes a quadratic function in two variables. For certain values of the parameters the expected response looks like a bowl with a unique minimum, an upside bowl with a unique maximum, or a saddle point.

1.8 Consider a response *y* and a single explanatory variable *x*. The following models are nonlinear in the parameters. You may want to consider one of these models and trace out the mean response for changing levels of *x*. For example, take the first model with $\alpha = 0.39$ and $\beta = 0.10$ and consider *x* values between 8 and 40. This particular model is studied in Chapter 9; *x* is the age of a chemical product in weeks, and the response *y* is its remaining chlorine.

$$y = \alpha + (0.49 - \alpha)\exp[-\beta(x - 8)] + \varepsilon$$

$$\mathrm{y} = \beta_1 + \frac{\beta_2}{1 + \exp[-\beta_3(\mathrm{x} - \beta_4]} + \varepsilon$$

$$y = \frac{\alpha}{1 + \beta\exp(-\gamma x)} + \varepsilon \quad \alpha > 0, \beta > 0, \gamma > 0$$

$$y = \frac{\beta_1 x}{\beta_2 + x} + \varepsilon$$

1.9 Sales may increase linearly with time, but the variability may depend on the level (the mean) of sales. If sales are very small, one can not expect tremendous variability. Sales can not be negative, so the variability is automatically bounded from below. On the other hand there is more room for bigger variability if the level of the sales is high. It is useful to think in terms of percentages. One may expect a variability (expressed as a standard deviation) of ± 10 percent. If sales are at level 10, this implies an uncertainty of ± 1 units. On the other hand, if the level is at 1000, the uncertainty is ± 100 units. If the variability (standard deviation) is proportional to the level, one should analyze the logarithm of sales, and not the sales. You will learn in Chapter 6 that this transformation stabilizes the variance. In this situation the variability in the logarithms of sales does not depend on the level of the sales.

Another situation, where the variability of the response can be expected to depend on the explanatory variable is when measuring distance. Assume that we want to determine the distances between pairs of points (where some are close together, while others are far apart). We can expect that the error in measuring close distances is smaller than the error in measuring points that are far apart. The variability in the measurements can be expected to increase with distance.

1.10 Economic "well-being" has an impact on people's decision to have children. During the post World War II period, a period characterized by rapid economic growth, many young Europeans affected by the war delayed their decision to have children. Economic activity of the post World War II period also had an impact on the breeding space for storks and led to a decrease in the number of storks. Considering annual numbers of births and annual numbers of storks, one can observe a strong positive correlation. However, no one - except young children - would interpret this correlation as a causal effect.

Poverty of a school district affects the number of students in subsidized lunch programs, with poorer districts having more children in these nationally subsidized programs. Poverty also affects the scholastic test scores in these districts. The strong positive correlation between the number of children in subsidized lunch programs and test achievement scores in these districts does not imply that there is a causal connection between subsidized lunch and test scores. It is poverty that is the driving causal factor.

High summer temperatures are related to high beer sales. High summer temperatures are also related to increased sales of suntan lotion. Daily sales of suntan lotion and beer sales are positively correlated. This, however, does not imply a causal connection. It is not that people who drink require more sun tan lotion.

1.11 Contact your state to obtain this information.

1.12 (a) Ignoring variability, we find that for the ith subject: $\text{RelativeRaise}_i = \beta\text{Performance}_i$. All points in the graph of RelativeRaise against Performance are on a straight line through the origin.

The absolute raise (that is, the raise in terms of dollars earned) can be written as

$$\text{AbsoluteRaise} = (R)(\text{PreviousSalary}) = (\beta\text{PreviousSalary})\text{Performance}$$

A graph of AbsoluteRaise against Performance does not exhibit a perfect linear association as the slope depends on the previous salary that changes from person to person. A regression of AbsoluteRaise on Performance may not provide the correct estimate of the parameter β. Take two workers; the previous salary of the first worker is half the salary of the second one, but the first worker is twice as productive. Their absolute raises are the same. The slope in the plot of AbsoluteRaise against Performance is zero, and not the desired parameter β.

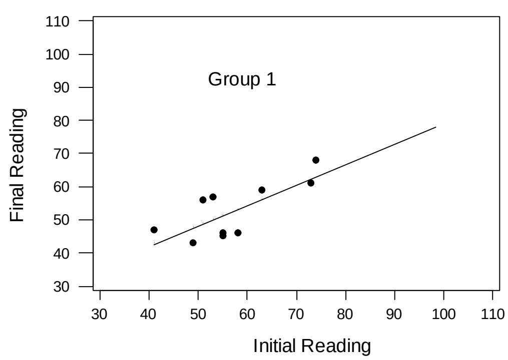
(b) Let R = Relative Raise, where R is a small number such as 0.03 (3 percent). The ratio

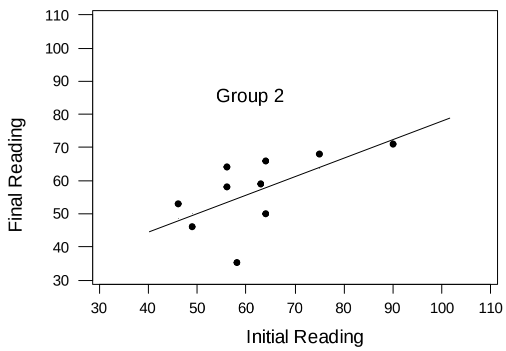
CurrentSalary/PreviousSalary = [(1 + R)PreviousSalary]/PreviousSalary = 1 + R . A first-order Taylor series expansion of $\ln(1+R) \approx R$ is valid for small R . Hence $\ln(\text{CurrentSalary/PreviousSalary}) = \ln(1+R) \approx R = \beta\text{Performance}$ is linearly related to Performance. A regression of ln(CurrentSalary/PreviousSalary) on Performance provides an estimate of β.

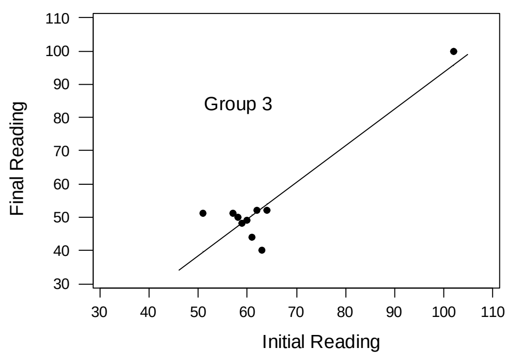
1.13 The five separate scatter plots of final reading *y* against initial reading *z*, one for each contraceptive group, are given below. The graphs have identical scales on both axes, and the "best fitting" straight lines have been added to the plots.

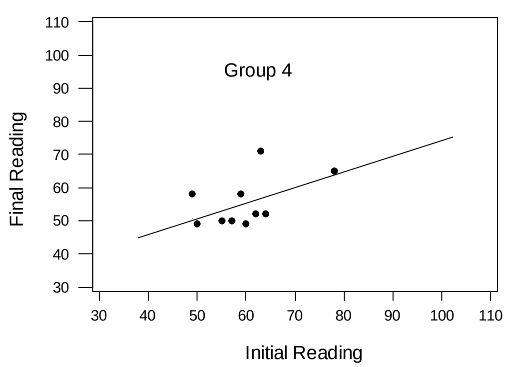
Model 1.8 assumes that the slopes in these five graphs are the same. The five graphs show that this may be a reasonable assumption. For the third group the slope is difficult to estimate. Apart from one subject with a very large initial reading ($z = 102$) there is little variation among the initial readings (all other *z*'s are between 50 and 65). It is difficult to pin down the value of the slope as the estimate is heavily influenced by the one subject with the high initial reading ($z = 102$) and response $y = 100$. Chapter 6 discusses influential observations in detail.

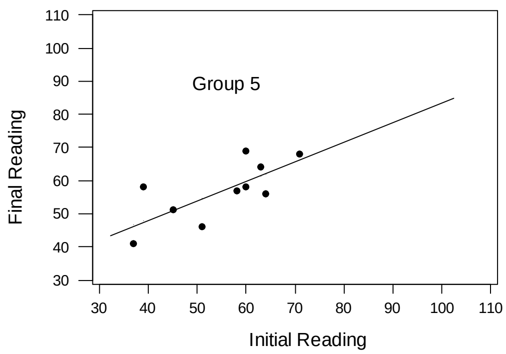
Assuming that $\alpha = 1$, one can look at the changes, $y - z$ = final reading - initial reading. This implies that we compare five groups (samples), with the objective to test whether the means of the changes are the same. That is, H_0: $\beta_1 = \beta_2 = \ldots = \beta_5$.




Group 2
Final Reading
Initial Reading
110
100
90
80
70
60
50
40
30
30
40
50
60
70
80
90
100
110


Group 3
Final Reading
Initial Reading
110
100
90
80
70
60
50
40
30
30
40
50
60
70
80
90
100
110


Group 4
Final Reading
Initial Reading
110
100
90
80
70
60
50
40
30
30
40
50
60
70
80
90
100
110


Group 5
Final Reading
Initial Reading
110
100
90
80
70
60
50
40
30
30
40
50
60
70
80
90
100
110

CHAPTER 2

Many excellent computer programs are available for plotting the data and for carrying out the regression calculations. Here we use S-Plus, R, Minitab, SAS, and SPSS. Most programs work the same and it is not difficult to switch from one program to the other. Most packages are spreadsheet programs. You enter the data into the various columns of a spreadsheet and use simple commands to carry out the operations. The results (fitted values, residuals, …) can be stored in unused columns of the worksheet. Many options are available within all programs. You need to consult the on-line help for detailed discussion and examples.

The Minitab software is very easy to use. Minitab works like a spreadsheet program. We enter the data into columns of a spreadsheet and use the tabs: Stat > Regression > Regression. We specify the response variable and the explanatory (regressor) variables and execute the regression command. The output provides the estimates, standard errors, t-ratios and probability values. It displays the ANOVA table and the coefficient of determination. The output (residuals and fitted values) can be stored in unused columns of the worksheet.

A note on computing with R

R is a free software which is available through the internet; it can be downloaded from http://cran.us.r-project.org/. It is very similar to the commercial package S-Plus. R is a language and an environment for statistical computing and graphics. It can be used with Windows 95 or later versions, a variety of Unix and Linux platforms, and Apple Macintosh (OS versions later than 8.6).

The most convenient way to use R is at a graphics work station running a windowing system. We have used R on UNIX machines to solve several of the exercises, and the following discussion assumes this set-up. If you are running R under Windows, you will need to make some minor adjustments.

R issues the prompt **" >"** whenever it expects input commands. Let us assume that the UNIX shell prompt is %. You can start the R program with the command **%R**. Then R will return with a banner line, and R commands may be issued at this point. The command

>help.start()

starts the HTML interface for on-line help, using the web browser that is available at your computer. You can use the mouse to explore features of the help facility. The command for quitting an R session is

>q()

At this point you will be asked whether you want to save the data from your R session.

R has an extensive help facility. You can get information on any specific function – for example the natural logarithm – by typing

>help(log) or >?log

R is case-sensitive, so x and X refer to different variables. R operates on named data structures. Data can be entered at the terminal or can be read from an external file. Entering the elements of a vector x – consisting of the four numbers 2, 4, 5, and 7 – one uses the R command

>x <- c(2,4,5,7) or >x = c(2,4,5,7)

This is an assignment statement using the function c(). Notice that the assignment operator "<-" (which is the same as the "=" operator) consists of the two characters < ("less than") and - ("minus") and points to the object receiving the value of the expression. For simplicity we use "=".

For the exercises in this book we read the data from an external file (a text file in UNIX). In exercise 2.6, for example, we have modified the file **hooker** so that the first four lines are as follows:

Temp AP
210.8 29.211
210.2 28.559
208.4 27.972

The first line of the file specifies a name for each variable in the data frame. The subsequent lines include the values for each variable. To read an entire data frame, we use the command

>hook = read.table("hooker",header=T)

The filename **hooker** is in quotes; header =T indicates that the first line includes the names of the variables. The commands

>Temp = hook[,1]; >AP=hook[,2]

define the first column of the matrix "hook" as Temp and the second column as AP. The statement

>LnAP = 100*log(AP)

results in a transformation of the variable AP; log(AP) is the natural log of AP.

The function for fitting simple or multiple linear regression models is lm(). For instance, a simple linear regression of Temp on LnAP can be fit by issuing the command

>hookfit = lm(Temp~LnAP)

The output object from the lm() command, "hookfit", is a fitted model object. Information about the fitted model can be extracted from this file. For example,

>summary(hookfit)

prints a comprehensive summary of the results of the regression analysis including the estimated coefficients, their standard errors, t–values and p-values (see the solution to exercise 2.6).

The command

>anova(hookfit)

supplies the analysis of variance (ANOVA) table. The command

>plot(LnAP,Temp)

plots Temp (the y-coordinate) against LnAP (the x-coordinate). A graphics window opens automatically. The fitted line can be superimposed on the scatter plot by issuing the command

>abline(hookfit)

The command

>qqnorm(hookfit$residuals)

leads to a normal probability plot of the residuals where "residuals" is in the fitted model object "hookfit".

Our discussion has focused on the free software package R. Note that the commands and the output of S-Plus are pretty much the same.

In subsequent chapters (Chapters 4 - 8) we consider multiple linear regression models. These models can be fit quite easily with R (and S-Plus). Suppose we have data in the vectors y, x1, x2 and x3. We can fit a multiple linear regression of y on x1, x2, and x3 by using the command

>mregfit=lm(y~x1+x2+x3)

Information about the model is in the fitted model object "mregfit". Note that an intercept term is included by default. One can restrict the intercept to be zero through

>mulregfit=lm(y~x1+x2+x3-1)

The above commands can be fine-tuned according to specific requirements. Many other commands are available to perform various statistical analyses and plots (such as residual analysis, leverages, Cook's D, various residual plots). This note is meant as a brief introduction to R. You should use the on-line help mentioned above to obtain more details.

2.1

(a) 95^{th} percentile = 10 + 3(1.645) = 14.93; 99^{th} percentile = 10 + 3 (2.326) = 16.98

(b) $t(0.95;10) = 1.812$; $t(0.95;25) = 1.708$; $t(0.99;10) = 2.764$; $t(0.99;25) = 2.485$

(c) $\chi^2(0.95;1) = 3.84$; $\chi^2(0.95;4) = 9.49$; $\chi^2(0.95;10) = 18.31$

$\chi^2(0.99;1) = 6.63$; $\chi^2(0.99;4) = 13.28$; $\chi^2(0.99;10) = 23.21$

(d) $F(0.95;2,10) = 4.10$; $F(0.95;4,10) = 3.48$; $F(0.99;2,10) = 7.56$;

$F(0.99;4,10) = 5.99$

2.2 Computer programs can be used to calculate the percentiles. Or, they can be looked up in the tables given in the appendix. The rounding errors are due to the number of digits displayed in various tables (and programs).

(a) $z(0.95) = 1.645$; $\chi^2(0.90;1) = 2.706$: $(1.645)^2 = 2.706$

$z(0.975) = 1.96$; $\chi^2(0.95;1) = 3.841$: $(1.96)^2 = 3.841$

$z(0.99) = 2.326$; $\chi^2(0.98;1) = 5.412$: $(2.326)^2 = 5.412$

$z(0.995) = 2.576$; $\chi^2(0.99;1) = 6.635$: $(2.576)^2 = 6.635$

(b) $t(0.95;4) = 2.132$; $F(0.90;1,4) = 4.545$: $(2.132)^2 = 4.545$

$t(0.975;4) = 2.776$; $F(0.95;1,4) = 7.709$: $(2.776)^2 = 7.709$

$t(0.99;10) = 2.764$; $F(0.98;1,10) = 7.638$: $(2.764)^2 = 7.638$

$t(0.995,10) = 3.169$; $F(0.99;1,10) = 10.044$: $(3.169)^2 = 10.044$

2.3 Correlation = 0.816; R^2 = 0.867; Estimated equation: $\hat{\mu} = 3 + 0.5x$

Same (linear regression) results for all four data sets. However, scatter plots in Figure 4.10 of the text show that linear regression is only appropriate for first data set. The correlation coefficients and the least squares estimates can be obtained by computer programs such as S-Plus, R, Minitab, SPSS, Minitab and others.

2.4

(a) Scatter plot shows an approximate linear relationship

(b) $\hat{\beta}_1 = 40/12.8 = 3.125$; $\hat{\beta}_0 = 13 - (3.125)(4.2) = -0.125$

(c) Fitted equation: $\hat{\mu} = -0.125 + 3.125x$

(d) $\hat{\mu}(x = 5) = -0.125 + 3.125(5) = 15.5$

(e)

X = Sales People	Y = Cars Sold	Fitted Value	Residual
6	20	18.625	1.375
6	18	18.625	-0.625
4	10	12.375	-2.375
2	6	6.125	-0.125
3	11	9.250	1.750

(f) $s^2 = 11/3 = 3.67$

(g) 95% confidence interval for β_1 : $3.125 \pm (3.182)(0.5352)$ or (1.42, 4.83). Since zero is not in this interval, we reject $\beta_1 = 0$.

(h) Significant relationship between the number of cars sold and the number of sales people. Number of cars sold increases as the number of sales people increases.

(i) If you know (can predict) sales, you can solve the equation in (c) to obtain the number of sales people that are required. However, only five weeks of data was available to estimate the model. Also, we do not know whether this period is representative for the whole year. Advisable to collect more data before using this model for decision making.

2.5 Minitab Output:

```
The regression equation is
Cars Sold = - 0.12 + 3.12 Sales People

Predictor        Coef     SE Coef          T        P
Constant       -0.125       2.406      -0.05    0.962
Sales People    3.1250      0.5352      5.84    0.010

S = 1.915       R-Sq = 91.9%     R-Sq(adj) = 89.2%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      125.00      125.00     34.09    0.010
Residual Error     3       11.00        3.67
Total              4      136.00
```

2.6
(a) Scatter plot (not shown here) indicates that a linear model is not appropriate. A quadratic component or a transformation are needed.
(b) Scatter plot confirms linear relationship between y = TEMP and x = 100ln(AP).
(c) R (S-Plus) output from the function ‘lm’:

```
                  Value  Std. Error    t value  Pr(>|t|)
(Intercept)     49.2684      1.1990    41.0925    0.0000
100ln(AP)        0.4782      0.0040   119.0838    0.0000

Residual standard error: s = 0.4016 with 29 degrees of freedom
Multiple R-Squared: 0.998
F-statistic: 14,180 with 1 and 29 degrees of freedom; the p-value is 0
```

(c) Estimated equation: $\hat{\mu} = 49.268 + 0.478\ln(\text{AP})$; $R^2 = 0.998$; $s = \sqrt{\text{MSE}} = 0.402$. The model is appropriate since there is small random scatter around the fitted line;

(d) (i) $\hat{\beta}_1 = 0.4782$ and $\text{s.e.}(\hat{\beta}_1) = 0.0040$. Since t(0.975;29) = 2.045, a 95% confidence interval for β_1: 0.4782 – 2.045(0.0040), 0.4782 + 2.045(0.0040), or (0.470 , 0.486)

(ii) $\hat{\mu} = 49.268 + 0.478(100\ln(25)) = 203.195$;

$$\text{s.e.}(\hat{\mu}) = \sqrt{s^2\left[\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}\right]} = \sqrt{(0.402)^2/31 + (0.0040)^2(321.888 - 298.041)^2} = 0.1196$$

95% confidence interval:
[203.195 – 2.045 (0.1196), 203.195 + 2.045 (0.1196)], or (202.950, 203.440)

(e) Estimates and standard errors of β_0 and β_1 change by factor of 5/9.

2.7

(a) $\hat{\beta} = \bar{y} = \sum y_i / n$; $s^2 = \sum (y_i - \bar{y})^2 / (n-1)$

(b) (i) Prediction interval is wider
(ii) 99% percent prediction interval is wider
(iii) Calculation error

2.8 Minitab output:

```
The regression equation is
Revenue = 32 + 0.263 Cars

Predictor        Coef     SE Coef          T        P
Constant         31.9       185.2       0.17    0.867
Cars          0.26251     0.03930       6.68    0.000

S = 264.0       R-Sq = 84.8%     R-Sq(adj) = 82.9%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1     3109923     3109923     44.62    0.000
Residual Error     8      557529       69691
Total              9     3667452
```

(a) Estimated equation: $\hat{\mu} = 31.9 + 0.2625x$; t-ratio($\hat{\beta}_1$) = 0.2625/0.0393 = 6.68; p-value = 0.0002; number of cars sold is a significant predictor variable.

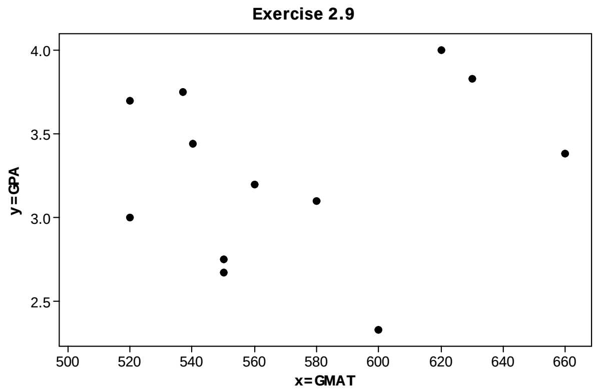
(b) 95% confidence interval for β_1: $0.2625 \pm (2.306)(0.0393)$ or (0.172, 0.353)

(c) $R^2 = 0.848$

(d) Standard deviation of y after factoring in x is $s = \sqrt{MSE} = 264.0$; standard deviation of y (without factoring x) is 638.3531.

(e) $\hat{\mu}(x = 1187) = 343.5$

2.9 The scatter plot of y = GPA against x = GMAT score shows considerable variability.



The Minitab regression output is given below:

```
The regression equation is
GPA = 2.16 + 0.00193 x=GMAT

Predictor      Coef   SE Coef     T      P
Constant      2.158     2.014  1.07  0.309
GMAT     0.001931  0.003510  0.55  0.594

S = 0.532633   R-Sq = 2.9%   R-Sq(adj) = 0.0%

Analysis of Variance

Source          DF      SS      MS     F      P
Regression       1  0.0858  0.0858  0.30  0.594
Residual Error  10  2.8370  0.2837
Total           11  2.9228
```

(a) Estimated equation: $\hat{\mu} = 2.158 + 0.0019x$; $R^2 = 0.029$; the model explains only 2.9% of the variability in y; not much of a relationship over the limited range of GMAT scores; other factors may be more important

(b) $\hat{\mu}(x = 540) = 2.158 + 0.001931(40) = 3.23$

(c) t-ratio($\hat{\beta}_1$) = 0.001931/0.00351 = 0.55; p-value = 0.594; conclude $\beta_1 = 0$

2.10

(a) Prediction at weight 2000 is 0.5598 + (0.001024)(2000) = 2.6078. Since n is large and the estimation error can be ignored, s.e(prediction error) = s = $\sqrt{0.066}$ = 0.2569. Thus, an approximate 95% prediction interval is 2.6078 $\pm$ (1.96)(0.2569), or (2.104, 3.111). Note that 1.96 is from the standard normal table.

(b) The prediction at weight 1500 is 0.5598 + (0.001024)(1500) = 2.0958. Thus, an approximate 95% prediction interval is 2.09 $\pm$ (1.96)(0.2569) = (1.592, 2.599)

2.11

$$\frac{1}{R^2} = \frac{SST}{SSR} = \frac{SSR + SSE}{SSR} = 1 + \frac{SSE}{SSR} = 1 + \frac{n-p-1}{p}\frac{1}{F}$$

Hence, $R^2 = \left[1 + \frac{n-p-1}{pF}\right]^{-1}$.

2.12

(a) $\hat{\beta}_1 = \sum x_i y_i \big/ \sum x_i^2$; $s^2 = \sum (y_i - \hat{\beta}_1 \bar{y})^2 \big/ (n-1)$

(b) $\sum e_i x_i = 0$, but not necessarily $\sum e_i = 0$

(c) $V(\hat{\beta}_1) = \frac{1}{\left[\sum x_i^2\right]^2} \sigma^2 \sum x_i^2 = \sigma^2 \frac{1}{\left[\sum x_i^2\right]}$

2.13

(a) Estimated equation: $\hat{\mu} = 0.520x$; $s^2 = 46.2/16 = 2.89$;

$\hat{\beta}_1 = 0.520$; s.e.$(\hat{\beta}_1) = 0.0132$; 95% confidence interval: (0.492, 0.548)

(b) Estimated equation: $\hat{\mu} = 0.725 + 0.498x$; $\hat{\beta}_0 = 0.725$; s.e.$(\hat{\beta}_0) = 1.549$;

$\hat{\beta}_0$ / s.e.$(\hat{\beta}_0) = 0.725/1.549 = 0.47$; p-value = 0.65; conclude $\beta_0 = 0$

2.14 Minitab output:

```
The regression equation is
y = - 0.228 + 0.995 x

Predictor        Coef     SE Coef         T        P
Constant      -0.2281      0.1378     -1.65    0.137
x            0.994757    0.005219    190.59    0.000

S = 0.2067      R-Sq = 100.0%    R-Sq(adj) = 100.0%

Analysis of Variance

Source           DF          SS          MS         F        P
Regression        1      1552.2      1552.2  36322.72    0.000
Residual Error    8         0.3         0.0
Total             9      1552.6
```

(a) Fitted equation: $\hat{\mu} = -0.228 + 0.995x$

(b) 95% confidence interval for β_0: $-0.2281 \pm (2.306)(0.1378)$ or $(-0.546, 0.090)$

(c) 95% confidence interval for β_1: $0.9948 \pm (2.306)(0.005219)$ or $(0.983, 1.007)$

(d) (i) Test $\beta_0 = 0$: 95% confidence interval for β_0 covers 0;

(ii) Test $\beta_1 = 0$: 95% confidence interval for β_1 covers 1

(e) Minitab output

```
The regression equation is
y = 0.987 x

Predictor        Coef     SE Coef          T        P
Noconstant
x            0.987153    0.002704     365.09    0.000

S = 0.2258

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      6796.2      6796.2 133292.08    0.000
Residual Error     9         0.5         0.1
Total             10      6796.7
```

95% confidence interval for β_1: $0.9872 \pm (2.262)(0.002704)$ or $(0.981, 0.993)$; does not cover 1

(e) Restriction $\beta_0 = 0$. The estimate of β_1 depends on the estimate of β_0. Thus the estimates of β_1 with β_0 restricted at 0 and with unrestricted β_0 are not necessarily the same.

2.15 R output:

```
Residual Standard Error = 4.5629
R-Square = 0.6767
F-statistic (df=1, 5) = 10.4657
p-value = 0.0231

            Estimate   Std.Err   t-value   Pr(>|t|)
Intercept    68.4459   12.9270    5.2948     0.0032
x            -0.4104    0.1268   -3.2351     0.0231

ANOVA
Source          DF        SS        MS      F      P
Regression       1    217.90    217.90  10.47  0.023
Residual Error   5    104.10     20.82
Total            6    322.00
```

(a) Estimated equation: $\hat{\mu} = 68.45 - 0.41x$; $R^2 = 0.677$; $s = 4.563$.

F-statistic = 10.47; p-value = 0.023; reject $\beta_1 = 0$

(b) $\text{s.e.}(\hat{\beta}_0) = 12.93$; $\hat{\beta}_0 / \text{s.e.}(\hat{\beta}_0) = 68.45/12.93 = 5.29$; p-value = 0.003

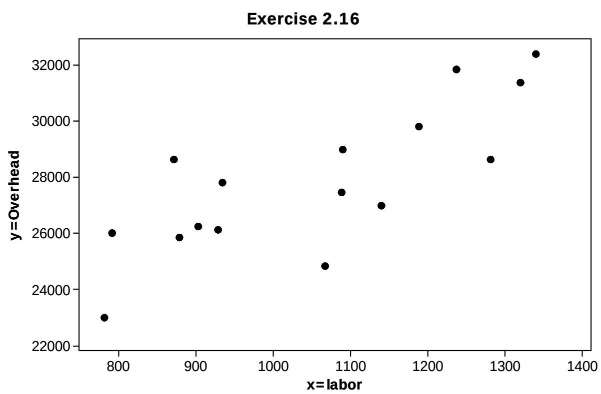
$\text{s.e.}(\hat{\beta}_1) = 0.127$; $\hat{\beta}_1 / \text{s.e.}(\hat{\beta}_1) = -0.41/0.127 = -3.23$; p-value = 0.023;

reject $\beta_0 = 0$ and $\beta_1 = 0$ at the 5 percent significance level.
99% confidence interval for β_1: (-0.92, 0.11).

(c) $\hat{\mu}(x = 100) = 27.41$; $s.e.(\hat{\mu}(x = 100)) = 1.73$;
95% confidence interval: (22.97,31.86).

(d) $\hat{\mu}(x = 84) = 33.98$; $s.e.(\hat{\mu}(x = 84)) = 2.76$;
95% confidence interval: (26.88, 41.07).
Note that $\bar{x}$ =101 and $s.e.(\hat{\mu}_0)$ is smallest when $x_0 = \bar{x}$. As x_0 moves away from $\bar{x}$, $s.e.(\hat{\mu}_0)$ becomes larger and the corresponding confidence interval becomes wider.

2.16 The scatterplot of overhead against labor hours shows a linear relationship




```
The regression equation is
Overhead = 16310 + 11.0 Labor

Predictor    Coef  SE Coef     T      P
Constant    16310     2421  6.74  0.000
Labor      10.982    2.268  4.84  0.000

S = 1645.61   R-Sq = 62.6%   R-Sq(adj) = 60.0%

Analysis of Variance

Source          DF         SS        MS      F      P
Regression       1   63517077  63517077  23.46  0.000
Residual Error  14   37912232   2708017
Total           15  101429309
```

The fitted values are the estimates of the expected total departmental overhead; they can be used as the predictions of the total departmental overhead for these given labor hours. Prediction intervals can be calculated. For example, for a new month with

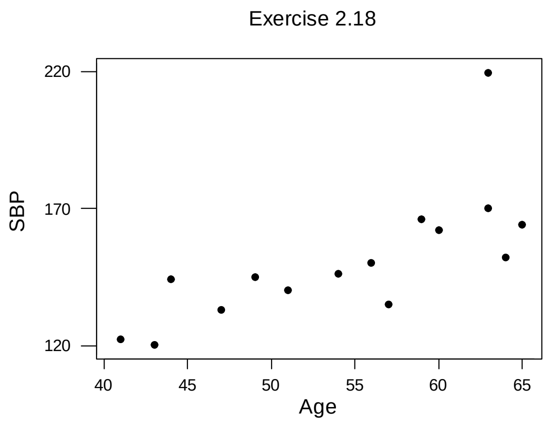
$x_i = 1{,}000$ labor hours, the prediction is $\hat{y}_i = 428$ and the 95% prediction interval is (23645, 30939).

2.17
(a) The scatter plot shows that length (y) increases with increasing width (x).

```
Residual Standard Error = 4.295
R-Square = 0.9555
F-statistic (df=1, 8) = 171.7821
p-value = 0

             Estimate  Std.Error     t-value    Pr(>|t|)
Intercept    -46.4359    13.4161     -3.4612      0.0086
Width (x)      1.7924     0.1368     13.1066      0.0000
```

(b) Estimated equation: $\hat{\mu} = -46.44 + 1.792x$;
95% confidence interval for β_0 : (-77.37, -15.50);
95% confidence interval for β_1 : (1.48, 2.11).
(c) Good fit; $R^2 = 0.956$
(d) $\hat{\mu}(x = 100) = 132.8$; 95% prediction interval: (122.39,143.22)
(e) Strong linear relationship

2.18
(a) The plot of SBP against age indicates that there is a linear relationship between SBP and age.



(b) Estimated equation: $\hat{\mu} = 33.31 + 2.168x$;

(c) Analysis of variance

Source	DF	SS	MS	F	P
Regression	1	4361.5	4361.5	14.58	0.002
Residual Error	13	3889.4	299.2		
Total	14	8250.9			

(d) F = 14.58; p-value = 0.002; reject $\beta_1 = 0$

(e) $\text{s.e.}(\hat{\beta}_1) = 0.568$; $\hat{\beta}_1 / \text{s.e.}(\hat{\beta}_1) = 2.168/0.568 = 3.82$; same p-value = 0.002; reject $\beta_1 = 0$

(f) Individual with $x = 63$ and $y = 220$ unusual. Estimates and standard errors change; R^2 increases. See R output shown below.

```
Residual Standard Error = 8.9007
R-Square = 0.7019
F-statistic (df=1, 12) = 28.2562
p-value=2e-04

           Estimate  Std.Error   t-value   Pr(>|t|)
Intercept  58.9876    16.6075    3.5519    4e-03
Weight      1.6244     0.3056    5.3157    2e-04

ANOVA
Source           DF        SS      MS      F      P
Regression        1    2238.5  2238.5  28.26  0.000
Residual Error   12     950.7    79.2
Total            13    3189.2
```

2.19 R Output:

```
Residual Standard Error = 0.1512
R-Square = 0.9496
F-statistic (df=1, 4) = 75.4083
p-value = 0.001

            Estimate   Std.Error   t-value   Pr(>|t|)
Intercept    3.7073      0.0955   38.8347      0.000
Mol.weight  -0.0123      0.0014   -8.6838      0.001
```

(a) Estimated equation: $\hat{\mu} = 3.707 - 0.0123x$; $R^2 = 0.950$

(b) F-statistic = 75.41; p-value = 0.001; reject $\beta_1 = 0$ at the 0.01 significance level. Significant linear relationship.

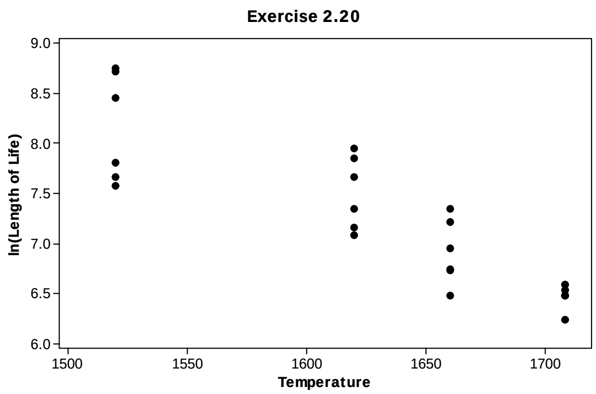
(c) Response is average of 3 observations. Use of individual values would improve the sensitivity of the analysis.

(d) No; molecular weight 200 far outside the region of experimentation; one does not know whether the linear relationship will continue to hold.

2.20

(a) Scatterplot of y = length of life against x = temperature shows: (i) length of life decreases with increasing temperature; (ii) variability in y is related to the level of y.

(b) Logarithmic transformation, ln(y), goes a long way toward stabilizing the variability.



(c) Minitab output

```
The regression equation is
ln(Life) = 22.1 - 0.00911 temp

Predictor       Coef   SE Coef       T      P
Constant      22.084     1.773   12.46  0.000
temp       -0.009110  0.001088   -8.37  0.000

S = 0.368943   R-Sq = 76.1%   R-Sq(adj) = 75.0%
```

```
Analysis of Variance
Source          DF       SS      MS      F      P
Regression       1   9.5347  9.5347  70.05  0.000
Residual Error  22   2.9946  0.1361
Total           23  12.5293
```

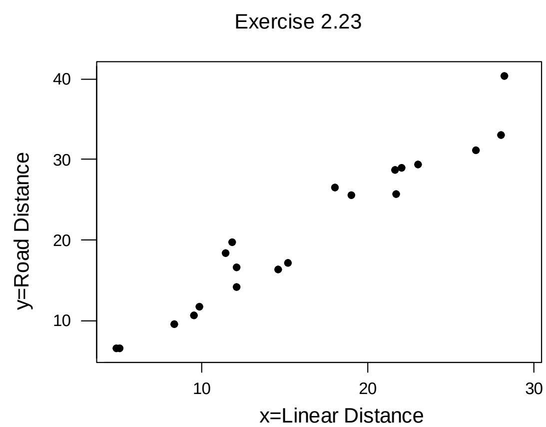
2.21 Plot of the chemical test against the magnetic test (not shown) indicates a linear relationship. Results of fitting a linear regression model are given below (R output):

```
Residual Standard Error = 3.4636
R-Square = 0.5372
F-statistic (df=1, 51) = 59.2056
p-value = 0
          Estimate  Std.Err   t-value  Pr(>|t|)
Intercept   8.9565   1.6523   5.4205     0
Mag Test    0.5866   0.0762   7.6945     0
```

Estimated equation: $\hat{\mu} = 8.957 + 0.587x$; R^2 = 0.537; F = 59.21; reject $\beta_1 = 0$
Significant linear relationship between the tests. However, variability large and predictive power low.

2.22 Plot of y (memory retention) against x (time) shows a nonlinear (exponentially decaying) pattern. Graphs of ln(y) against x and ln(y) against ln(x) show similar patterns. Plot of y against ln(x) shows a linear pattern.
Estimated equation: $\hat{\mu} = 0.846 - 0.079\ln(x)$; R^2 = 0.990; good model

2.23 The graph of road distance against linear distance shows an approximate linear relationship

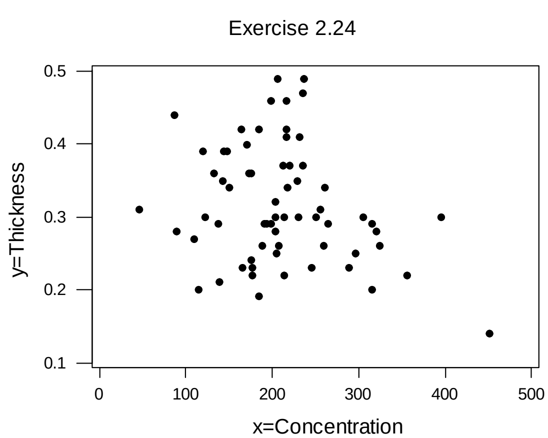


.

Estimated equation: $\hat{\mu} = 0.375 - 0.000279x$; $R^2 = 0.939$; s = 2.436;
$t(\hat{\beta}_1) = 0.379/1.26943 = 16.67$; p-value 0.000; conclude that $\beta_1 > 0$. Interesting fact that the confidence interval for β_1 does not cover one; $1.269 \pm (2.10)(0.076)$ or (1.109, 1.429)

```
The regression equation is
y=Road = 0.38 + 1.27 x=Linear

Predictor        Coef     SE Coef          T        P
Constant        0.379       1.344       0.28    0.781
x=Linear      1.26943     0.07617      16.67    0.000

S = 2.436       R-Sq = 93.9%     R-Sq(adj) = 93.6%

Analysis of Variance

Source            DF          SS          MS        F        P
Regression         1      1648.3      1648.3   277.73    0.000
Residual Error    18       106.8         5.9
Total             19      1755.1
```

2.24 The graph of concentration against thickness shows considerable scatter. Also the first egg with concentration = 452 and thickness = 0.14 is unusual and somewhat different from the rest (more on outlying cases in Chapter 6).



Estimated equation: $\hat{\mu} = 0.375 - 0.000279x$; $R^2 = 0.064$ small;
$t(\hat{\beta}_1) = -0.000279/0.000135 = -2.07$ with p-value 0.042 is barely significant at the 0.05 significance level.
Without the first case, the estimated equation is: $\hat{\mu} = 0.357 - 0.000184x$; $R^2 = 0.025$ is

small; $t(\hat{\beta}_1) = -0.000184/0.000146 = -1.26$ with p-value = 0.214. We conclude that $\beta_1 = 0$.

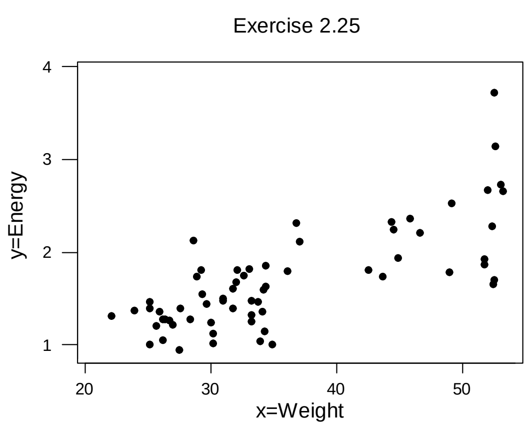
<u>With all observations:</u>

```
The regression equation is
Thickness = 0.375 -0.000279 Concentration

Predictor        Coef     SE Coef          T        P
Constant      0.37494     0.02990      12.54    0.000
Concentr   -0.0002790   0.0001345      -2.07    0.042

S = 0.07848     R-Sq = 6.4%      R-Sq(adj) = 4.9%

Analysis of Variance
Source            DF          SS          MS         F        P
Regression         1    0.026493    0.026493      4.30    0.042
Residual Error    63    0.388021    0.006159
Total             64    0.414514
```

<u>With the first observation omitted:</u>

```
The regression equation is
Thickness = 0.357 -0.000184 Concentration

Predictor        Coef     SE Coef          T        P
Constant      0.35700     0.03174      11.25    0.000
Concentr   -0.0001838   0.0001464      -1.26    0.214

S = 0.07761     R-Sq = 2.5%      R-Sq(adj) = 0.9%
```

2.25 The scatter plot of energy requirement against weight shows a linear relationship.



Estimated equation: $\hat{\mu} = 0.133 - 0.0434x$; $R^2 = 0.563$; s = 0.3662;

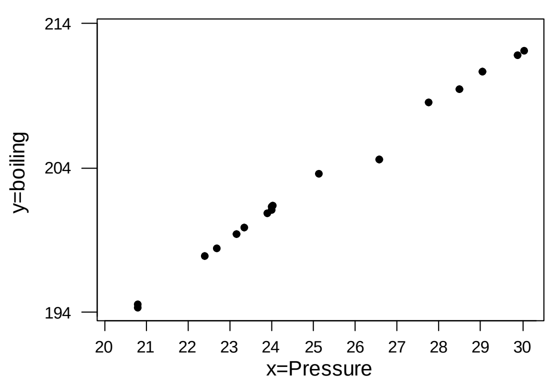
$t(\hat{\beta}_1) = 0.04342/0.004857 = 8.94$ with p-value 0.000 is significant; we conclude that $\beta_1 > 0$ and that weight has a significant influence. Energy requirement increases by 0.0434 Mcal/Day for each kg of body weight.

The 11th observation (weight = 52.6; y = 3.73) should be scrutinized it is the observation that seems somewhat different from the pattern exhibited by the majority of the cases (more on outlying cases in Chapter 6).

```
The regression equation is
Energy = 0.133 + 0.0434 Weight

Predictor         Coef     SE Coef          T        P
Constant        0.1329      0.1804       0.74    0.464
Weight        0.043416    0.004857       8.94    0.000

S = 0.3662      R-Sq = 56.3%     R-Sq(adj) = 55.6%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      10.718      10.718     79.91    0.000
Residual Error    62       8.316       0.134
Total             63      19.034
```

2.26 The scatter plot of boiling point against barometric pressure shows a strong linear relationship.

Figure 2.26



Estimated equation: $\hat{\mu} = 155.296 + 1.902x$; $R^2 = 0.994$; s = 0.444;

$t(\hat{\beta}_1) = 1.90178/0.03676 = 51.74$ with p-value 0.000; we conclude $\beta_1 > 0$; barometric pressure has a significant influence on boiling point. The boiling point

increases by 1.92 degrees F when barometric pressure increases by one inch of mercury.
The observation y = 204.6, x = 26.57 should be scrutinized as it seems different from the pattern that is exhibited by the rest (more on outlying cases in Chapter 6).

```
The regression equation is
boiling = 155 + 1.90 Pressure

Predictor        Coef     SE Coef          T        P
Constant      155.296       0.927     167.47    0.000
Pressure      1.90178     0.03676      51.74    0.000

S = 0.4440      R-Sq = 99.4%     R-Sq(adj) = 99.4%

Analysis of Variance
Source            DF          SS          MS         F        P
Regression         1      527.82      527.82   2677.11    0.000
Residual Error    15        2.96        0.20
Total             16      530.78
```

The data set in Exercise 2.6 includes cases where barometric pressure < 20. The graph with both data sets (not given) shows that the estimated models are quite similar.

2.27

(a) Response y = takeup(kg). Scatter plot indicates a linear relationship. R output:

```
Residual Standard Error = 3.3945
R-Square = 0.9858
F-statistic (df=1, 22) = 1530.289    p-value = 0

              Estimate  Std.Error    t-value    Pr(>|t|)
Intercept      -9.8960     1.6887    -5.8602       0
x               0.0753     0.0019    39.1189       0
```

y = Takeup(kg): $\hat{\mu} = -9.896 + 0.0753x$; $R^2 = 0.986$; F = 1,530.3; reject $\beta_1 = 0$

(b) Response y = takeup(kg). Scatter plot indicates a linear relationship. R output:

```
Residual Standard Error = 0.3952
R-Square = 0.703
F-statistic (df=1, 22) = 52.068
p-value = 0
           Estimate  Std.Error    t-value   Pr(>|t|)
Intercept    4.7372     0.1966    24.0973      0
 x           0.0016     0.0002     7.2158      0
```

y = Takeup(%): $\hat{\mu} = 4.737 + 0.00162x$; $R^2 = 0.703$; F = 52.07; reject $\beta_1 = 0$

Both models fit well. However, the first one seems to be better (larger R^2).

CHAPTER 3

3.1

(a) $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}$; $A' = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

(b) $A'A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 8 & 16 \\ 8 & 5 & 8 \\ 16 & 8 & 21 \end{bmatrix}$

(c) tr(A) = 2 + 2 + 4 = 8 ; $\text{tr}(A'A) = 17 + 5 + 21 = 43$

(d) det(A) = (2)(2)(4) + (3)(1)(1) + (0)(2)(2) – (2)(2)(1) – (1)(2)(2) – (3)(0)(4) = 11

$\det(A'A)$ = (17)(5)(21)+(8)(8)(16)+(8)(8)(16)–(16)(5)(16)–(8)(8)(17)–(8)(8)(21)
= 121

3.2

(a) $X'X = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $(X'X)^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$; $X'\boldsymbol{y} = \begin{bmatrix} 19 \\ 1 \\ 5 \end{bmatrix}$;

$(X'X)^{-1} X'\boldsymbol{y} = \begin{bmatrix} 4.75 \\ 0.25 \\ 1.25 \end{bmatrix}$

(b) Diagonal matrices; the diagonal elements are the same.

3.3

(a) $$X'X = \begin{bmatrix} 5 & \sum_{i=1}^{5} x_{i1} & \sum_{i=1}^{5} x_{i2} \\ \sum_{i=1}^{5} x_{i1} & \sum_{i=1}^{5} (x_{i1})^2 & \sum_{i=1}^{5} x_{i1}x_{i2} \\ \sum_{i=1}^{5} x_{i2} & \sum_{i=1}^{5} x_{i1}x_{i2} & \sum_{i=1}^{5} (x_{i2})^2 \end{bmatrix}$$

(b) These quantities are entries in the $X'X$ matrix; see (a).

3.4

(a) det(A) = (2)(2) - (-1)(-1) = 5

(b) The eigenvalues are the solutions of the equation
$|A-\lambda I| = |(2-\lambda)(2-\lambda)-(-1)^2| = \lambda^2 - 4\lambda + 3 = 0$; they are 3 and 1.
The eigenvector corresponding to the eigenvalue $\lambda = 3$ is the solution to

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $x_2 = -x_1$. Normalizing the length of the eigenvector to 1 leads to $2(x_1)^2 = 1$ and $x_1 = 1/\sqrt{2}$. Hence $x_2 = -1/\sqrt{2}$, and the eigenvector corresponding to the first eigenvector is given by $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Similarly, the eigenvector corresponding to the second eigenvector is given by
$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

The matrix of eigenvectors is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

(c) The spectral representation of the matrix A is given by

$$A = P\Lambda P' = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

(d) The eigenvalues of A are positive, hence the matrix A is positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 2/2 & -1/\sqrt{(2)(2)} \\ -1/\sqrt{(2)(2)} & 2/2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix}.$$

3.5

(a) det(A) = (3)(4)(2) + (1)(1)(2) + (1)(1)(2) – (1)(1)(4) –(1)(1)(2) –(2)(2)(3) = 10.

The inverse is given by $A^{-1} = \begin{bmatrix} 0.4 & 0 & -0.2 \\ 0 & 0.5 & -0.5 \\ -0.2 & -0.5 & 1.1 \end{bmatrix}$. Check that $AA^{-1} = A^{-1}A = I$.

You can use a computer program to determine the inverse and also to check your calculations.

(b) The three eigenvalues are the solutions to the cubic equation $|A - \lambda I| = 0$. They are given by 5.8951, 2.3973, and 0.7076. The corresponding eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} -0.4317 & 0.8857 & 0.1706 \\ -0.7526 & -0.4579 & 0.4732 \\ -0.4973 & -0.0759 & -0.8643 \end{bmatrix}$$

(c) The spectral representation of the matrix A is given by

$$A = P\Lambda P' = \begin{bmatrix} -0.4317 & 0.8857 & 0.1706 \\ -0.7526 & -0.4579 & 0.4732 \\ -0.4973 & -0.0759 & -0.8643 \end{bmatrix} \begin{bmatrix} 5.8951 & 0 & 0 \\ 0 & 2.3973 & 0 \\ 0 & 0 & 0.7076 \end{bmatrix} \begin{bmatrix} -0.4317 & 0.8857 & 0.1706 \\ -0.7526 & -0.4579 & 0.4732 \\ -0.4973 & -0.0759 & -0.8643 \end{bmatrix}'$$

(d) The eigenvalues of A are positive, hence the matrix A is positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 3/3 & 1/\sqrt{(3)(4)} & 1/\sqrt{(3)(2)} \\ 1/\sqrt{(3)(4)} & 4/4 & 2/\sqrt{(4)(2)} \\ 1/\sqrt{(3)(2)} & 2/\sqrt{(4)(2)} & 2/2 \end{bmatrix} = \begin{bmatrix} 1 & 0.289 & 0.408 \\ 0.289 & 1 & 0.707 \\ 0.408 & 0.707 & 1 \end{bmatrix}.$$

3.6

(a) det(A) = (2)(4)(1) + (1)(0)(1) + (1)(0)(1) – (1)(4)(1) –(0)(0)(2) –(1)(1)(1) = 3.

The inverse is given by $A^{-1} = \begin{bmatrix} 4/3 & -1/3 & -4/3 \\ -1/3 & 1/3 & 1/3 \\ -4/3 & 1/3 & 7/3 \end{bmatrix}$. Check that $AA^{-1} = A^{-1}A = I$.

You can also use a computer program to check the calculations.

(b) The three eigenvalues are the solutions to the cubic equation $|A - \lambda I| = 0$. They are given by 4.4605, 2.2391, and 0.3004. The corresponding eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix}$$

The spectral representation of the matrix A is given by

$$A = P\Lambda P' == \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix} \begin{bmatrix} 4.4605 & 0 & 0 \\ 0 & 2.2391 & 0 \\ 0 & 0 & 0.3004 \end{bmatrix} \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix}'$$

(c) The eigenvalues of A are positive, hence the matrix A is positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 2/2 & 1/\sqrt{(2)(4)} & 1/\sqrt{(2)(1)} \\ 1/\sqrt{(2)(4)} & 4/4 & 0/\sqrt{(4)(1)} \\ 1/\sqrt{(2)(1)} & 0/\sqrt{(4)(1)} & 1/1 \end{bmatrix} = \begin{bmatrix} 1 & 0.354 & 0.707 \\ 0.354 & 1 & 0 \\ 0.707 & 0 & 1 \end{bmatrix}.$$

3.7

(a) det(A) = (2)(2)(6) + (1)(3)(3) + (1)(3)(3) – (3)(2)(3) –(1)(1)(6) –(3)(3)(2) = 0.

(b) The three eigenvalues are the solutions to the cubic equation $|A - \lambda I| = 0$. They are given by 9, 1, and 0. The corresponding eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

(c) The eigenvalues are nonnegative; hence the matrix A is semi-positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 2/2 & 1/\sqrt{(2)(2)} & 3/\sqrt{(2)(6)} \\ 1/\sqrt{(2)(2)} & 2/2 & 3/\sqrt{(2)(6)} \\ 3/\sqrt{(2)(6)} & 3/\sqrt{(2)(6)} & 6/6 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.866 \\ 0.5 & 1 & 0.866 \\ 0.866 & 0.866 & 1 \end{bmatrix}$$

One eigenvalue is zero; hence there is a deterministic relationship among the three variables. The eigenvector corresponding to the eigenvalue 0 indicates the deterministic relationship. The linear combination $-y_1 - y_2 + y_3$ has variance zero.

3.8

(a) $AB = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 17 \\ 18 & 13 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 4 & 1 \\ 2 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 17 & 10 \\ 8 & 10 & 8 \\ 14 & 12 & 12 \end{bmatrix}$

3.9 An orthogonal matrix satisfies $PP' = P'P = I$. The eigenvectors of any symmetric matrix form an orthogonal matrix. Consider the matrix A in Exercise 3.6, for example. The eigenvectors are the columns in the matrix

$$P = \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix}.$$

Then $PP' = P'P = I$.

3.10

$$X = \begin{bmatrix} 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 40 \\ 1 & 40 \\ 1 & 40 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \end{bmatrix}; \quad \boldsymbol{y} = \begin{bmatrix} 55.8 \\ 59.1 \\ 54.8 \\ 54.6 \\ 43.1 \\ 42.2 \\ 45.2 \\ 31.6 \\ 30.9 \\ 30.8 \\ 17.5 \\ 20.5 \\ 17.2 \\ 16.9 \end{bmatrix}; \quad X'X = \begin{bmatrix} 14 & 630 \\ 630 & 30{,}300 \end{bmatrix};$$

$$(X'X)^{-1} = \begin{bmatrix} 1.10989 & -0.02308 \\ -0.02308 & 0.00051 \end{bmatrix} \qquad X'\boldsymbol{y} = \begin{bmatrix} 520.2 \\ 20{,}940.0 \end{bmatrix} \text{ and}$$

$$\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{y} = \begin{bmatrix} 1.10989 & -0.02308 \\ -0.02308 & 0.00051 \end{bmatrix} \begin{bmatrix} 520.2 \\ 20{,}940.0 \end{bmatrix} = \begin{bmatrix} 94.1341 \\ -1.2662 \end{bmatrix}$$

3.11

(a) The distribution of $(y_1, y_2)'$ is bivariate normal with mean vector $(2,6)'$ and covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) The conditional distribution of $(y_1, y_2)'$, given that $y_3 = 5$, is bivariate normal with mean vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (1/3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}(y_3 - 4) = \begin{bmatrix} (2/3) + (1/3)y_3 \\ (22/3) - (1/3)y_3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 17/3 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - (1/3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 5/3 \end{bmatrix}$.

3.12

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{E}(y_1) \\ \mathrm{E}(y_2) \\ \mathrm{E}(y_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{V}(\boldsymbol{y}) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 17 \\ 6 & 6 & 12 \\ 17 & 12 & 29 \end{bmatrix}$$

(b) $\mathrm{E}(\bar{y}) = (7 + 0 + 0)/3 = 7/3$

(c) $$\mathrm{V}(\bar{y}) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 11 & 6 & 17 \\ 6 & 6 & 12 \\ 17 & 12 & 29 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 12.8889$$

3.13

(a) H is a (nxn) symmetric matrix; $\mathrm{H}' = \mathrm{H}$

I - H is a (nxn) symmetric matrix; $(\mathrm{I} - \mathrm{H})' = \mathrm{I} - \mathrm{H}$

$\mathrm{HH} = \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}' = \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}' = \mathrm{H}$; H is idempotent

$(\mathrm{I} - \mathrm{H})(\mathrm{I} - \mathrm{H}) = \mathrm{I} - \mathrm{H} - \mathrm{H} + \mathrm{HH} = \mathrm{I} - \mathrm{H} - \mathrm{H} + \mathrm{H} = \mathrm{I} - \mathrm{H}$

$\mathrm{HX} = \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{X} = \mathrm{X}$

(b) $\mathrm{A}(\mathrm{I} - \mathrm{H}) = (\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}') = (\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}' - (\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}' = \mathrm{O}$,

a (pxn) matrix of zeros

$(\mathrm{I} - \mathrm{H})\mathrm{A}' = [\mathrm{A}(\mathrm{I} - \mathrm{H})]' = \mathrm{O}'$ a (nxp) matrix of zeros

$$H(I-H) = H - HH = H - H = O \quad \text{a (nxn) matrix of zeros}$$

$$(I-H)'H' = [H(I-H)]' = O \qquad \text{a (nxn) matrix of zeros}$$

3.14

(a) $V(A\boldsymbol{y}) = AV(\boldsymbol{y})A' = \sigma^2 AA' = \sigma^2 (X'X)^{-1}X'X(X'X)^{-1} = \sigma^2 (X'X)^{-1}$

(b) $V(H\boldsymbol{y}) = HV(\boldsymbol{y})H' = \sigma^2 HH' = \sigma^2 HH = \sigma^2 H = \sigma^2 X(X'X)^{-1}X'$

(c) $V[(I-H)\boldsymbol{y})] = (I-H)V(\boldsymbol{y})(I-H) = \sigma^2 (I-H) = \sigma^2 (I - X(X'X)^{-1}X')$

(d) $V\left(\begin{bmatrix} A \\ I-H \end{bmatrix}\boldsymbol{y}\right) = \sigma^2 \left(\begin{bmatrix} A \\ I-H \end{bmatrix}\begin{bmatrix} A' & I-H \end{bmatrix}\right) = \sigma^2 \begin{bmatrix} AA' & A(I-H) \\ (I-H)A' & I-H \end{bmatrix}$

$$= \sigma^2 \begin{bmatrix} (X'X)^{-1} & O \\ O' & I - X(X'X)^{-1}X' \end{bmatrix}$$

3.15

(a) The eigenvalues are the solutions to the quadratic equation
$|A - \lambda I| = |(1-\lambda)(1-\lambda) - \rho^2| = \lambda^2 - 2\lambda + (1-\rho^2) = 0$. They are $1+\rho$ and $1-\rho$.

(b) The eigenvector corresponding to the eigenvalue $1+\rho$ is the solution to the (vector) equation $(A - (1+\rho)I)\mathbf{p}_1 = \mathbf{0}$. The (normalized) solution is given by
$\mathbf{p}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.
Similarly, the solution to the (vector) equation $(A - (1-\rho)I)\mathbf{p}_2 = \mathbf{0}$ is given by
$\mathbf{p}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Hence $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

(c) Confirm the result by multiplication of the matrices.

(d) Experiment with several different values of ρ (-0.3, 0.3, -0.7, 0.7). Select a specific value of ρ. Use any computer software such as Minitab or SPSS to generate

20 independent random variables x_1 with variance $1+\rho$, and 20 independent random variables x_2 with variance $1-\rho$. This results in twenty independent pairs (x_1, x_2).

Apply the transformation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute the sample covariance matrix and check that it is close to the expected covariance matrix A.

3.16

(a) and (b) The steps in the derivations are spelled out in detail. Follow the algebra by substituting the relevant matrices.

(c) With correlation among the error and the regressor, the least squares estimate is no longer an unbiased estimate of β_1. This has important implications for regression modeling as standard least squares results in incorrect (biased) estimates.
Such a situation can arise if the regression model is missing an important variable, z, that is correlated with the regressor in the model, x (that is, $\rho_{\text{zx}} \neq 0$). Then the error in the incomplete original regression model can be written as $\varepsilon = \alpha\text{z} + \varepsilon^*$, where ε^* is an independent random error, and the correlation between the error and the regressor x in the model is $\rho_{\alpha\text{z}+\varepsilon^*,\text{x}} = \alpha\rho_{\text{zx}} \neq 0$.

(d) Follow the steps by using your computer software of choice for generating the random variables. For $\rho_{\varepsilon\text{x}} = 0.5$, the standard least squares estimate is estimating 2.5, and not the value $\beta_1 = 2$. For $\rho_{\varepsilon\text{x}} = -0.5$, the standard least squares estimate is estimating 1.5, and not the value $\beta_1 = 2$. The standard least squares estimate is an unbiased estimate of $\beta_1 = 2$ if $\rho_{\varepsilon\text{x}} = 0$.

3.17 The quadratic form can be written as $\boldsymbol{y}'A\boldsymbol{y}$ where the 3 x 3 symmetric matrix A is given as

$$\text{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$

The determinant of this matrix is 0. The rank of the matrix A is 2, as we can find a 2x2 submatrix with a nonzero determinant. Furthermore, the matrix A is idempotent; $\text{AA} = \text{A}$. Hence the distribution of the (normalized) quadratic form

$\left(y_1^2 + 0.5y_2^2 + 0.5y_3^2 + y_2 y_3\right)/\sigma^2$ follows a chi-square distribution with 2 degrees of freedom.

3.18 The matrices in the two quadratic forms are $A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

The product

$(1/\sigma^2)A_1A_2 = (1/\sigma^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence the two quadratic forms are independent.

CHAPTER 4

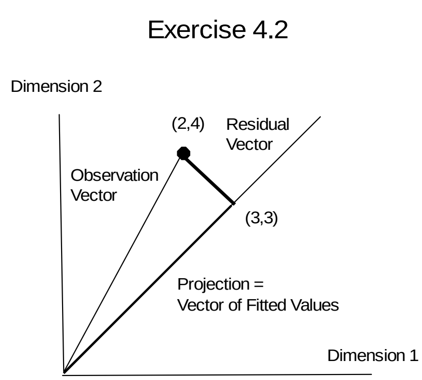
4.1

$$X'X = \begin{bmatrix} 10 & 55 \\ 55 & 385 \end{bmatrix};\ (X'X)^{-1} = \begin{bmatrix} 0.4667 & -0.0667 \\ -0.0667 & 0.0121 \end{bmatrix};$$

$$V(\hat{\boldsymbol{\beta}}) = \sigma^2 \begin{bmatrix} 0.4667 & -0.0667 \\ -0.0667 & 0.0121 \end{bmatrix}$$

$$V(\hat{\beta}_0) = (0.4667)\sigma^2;\ V(\hat{\beta}_1) = (0.0121)\sigma^2$$

4.2 L(**1**) represents the 45 degree line through the origin in two-dimensional space. Projecting the observation vector $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ onto the subspace L(**1**) results in the fitted values $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the least squares estimate $\hat{\beta}_0 = 3$. The residual vector $\boldsymbol{e} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the projection $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ are orthogonal. The picture is given below.



4.3 L(X) is the two-dimensional subspace in three-dimensional space that is described by all linear combinations of the two vectors, $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. You need to visualize this as a plane in three-dimensional space. The orthogonal projection of the observation vector $\boldsymbol{y} = \begin{bmatrix} 2.2 \\ 3.9 \\ 3.1 \end{bmatrix}$ onto this plane results in the vector of fitted values (the projection) $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 2.21667 \\ 3.91667 \\ 3.06667 \end{bmatrix}$ and the least squares estimates $\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 1.36667 \\ 0.85000 \end{bmatrix}$, satisfying $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 2.21667 \\ 3.91667 \\ 3.06667 \end{bmatrix} = \mathrm{X}\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1.36667 \\ 0.85000 \end{bmatrix}$. The residual vector $\boldsymbol{e} = \begin{bmatrix} 2.1 - 2.21667 \\ 3.9 - 3.91667 \\ 3.1 - 3.06667 \end{bmatrix} = \begin{bmatrix} -0.01667 \\ -0.01667 \\ 0.03333 \end{bmatrix}$ and the projection $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 2.21667 \\ 3.91667 \\ 3.06667 \end{bmatrix}$ are orthogonal.

The difference of the data vector $\boldsymbol{y}$ and the projection $\hat{\boldsymbol{\mu}}$ is quite small, indicating that the data vector is almost in the plane spanned by the matrix X.

4.4 L(X) is the two-dimensional subspace in three-dimensional space that is described by all linear combinations of the two vectors, $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. You need to visualize this as a plane in three-dimensional space. The orthogonal projection of the observation vector $\boldsymbol{y} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ onto this plane results in the vector of fitted values (the projection) $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$ and the least squares estimates $\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, satisfying

$\hat{\boldsymbol{\mu}} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = X\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The residual vector $\boldsymbol{e} = \begin{bmatrix} 2-3 \\ 4-5 \\ 6-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ and the projection $\hat{\boldsymbol{\mu}} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$ are orthogonal.

Note that the difference of the data vector and the projection is larger here than in Exercise 4.3. The data vector is not close to the space spanned by the matrix X.

4.5

(a) $V(\hat{\beta}_1) = 18$

(b) $\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = 1.2$

(c) $\text{Corr}(\hat{\beta}_1, \hat{\beta}_3) = 0.0943$

(d) $V(\hat{\beta}_1 - \hat{\beta}_3) = V(\hat{\beta}_1) + V(\hat{\beta}_3) - 2\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = 24.6$

4.6

(a) $V(\hat{\beta}_2) = 4$; $\text{s.e.}(\hat{\beta}_2) = 2$

(b) $t(\hat{\beta}_2) = \hat{\beta}_2 / \text{s.e.}(\hat{\beta}_2) = 15/2 = 7.5$; p-value = $2P(t(12) \geq 7.5) < 0.001$; reject $\beta_2 = 0$ in favor of $\beta_2 \neq 0$

(c) $\text{Cov}(\hat{\beta}_1, \hat{\beta}_2) = -0.05$

(d) Test $\beta_1 - \beta_2 = 0$; $V(\hat{\beta}_1 - \hat{\beta}_2) = V(\hat{\beta}_1) + V(\hat{\beta}_2) - 2\text{Cov}(\hat{\beta}_1, \hat{\beta}_2) = 1 + 4 + 1 = 6$;
$\hat{\beta}_1 - \hat{\beta}_2 / \text{s.e.}(\hat{\beta}_1 - \hat{\beta}_2) = -3/\sqrt{6} = -1.22$; p-value = $2P(t(12) \leq -1.22) = 0.136$;
conclude $\beta_1 - \beta_2 = 0$, or $\beta_1 = \beta_2$.
95% confidence interval for $\beta_1 - \beta_2$: $-3 \pm (2.179)\sqrt{6}$ or (-8.34, 2.34) covers zero.

(e) SST = 120, SSE = 2(15-3) = 24, and SSR = 96; F = (96/2)/(24/12) = 24; very small p-value; reject $\beta_1 - \beta_2 = 0$.

4.7

(a) $R^2 = 0.9324$

(b) F-statistic = 110.35; p-value = 0.000; reject $\beta_1 = \beta_2 = \beta_3 = 0$

(c) 95% confidence interval for β_{taxes}: (0.074, 0.306); reject $\beta_{taxes} = 0$; cannot simplify model
95% confidence Interval for β_{baths}: (-16.83, 180.57); can not reject $\beta_{baths} = 0$; can simplify model by dropping “baths”

4.8

(a) R^2 = 500074/541119 = 0.9241

(b) F-statistic = 152.29; p-value = 0.000; reject $\beta_1 = \beta_2 = 0$

(c) t-ratio for taxes $= 0.24237/0.04884 = 4.96$; p-value = $P(t(25) \geq 4.96) = 0.0000$

reject $\beta_{\text{taxes}} = 0$; response is related to taxes.

4.9

(a) From $\hat{\beta} = (X'X)^{-1}X'y$, we obtain $\hat{\beta}_0 = 885.161, \hat{\beta}_1 = -6.571, \hat{\beta}_2 = -1.374$;

s = 36.49;

From $V(\hat{\beta}) = s^2(X'X)^{-1}$: $\text{s.e.}(\hat{\beta}_0) = 61.75, \text{s.e.}(\hat{\beta}_1) = 0.5832, \text{s.e.}(\hat{\beta}_2) = 0.1943$

(b) $t(\hat{\beta}_0) = 14.33, t(\hat{\beta}_1) = -11.27, t(\hat{\beta}_2) = -7.07$; 97.5th percentile: t(27,0.975) = 2.052;

can reject $\beta_1 = 0$; can reject $\beta_2 = 0$.

(c) R^2 = 0.841

4.10

(a) Estimated equation: $\hat{\mu} = 3.453 + 0.496x_1 + 0.0092x_2$; s^2 = 4.7403;

s.e.($\hat{\beta}_0$) = 2.431, s.e.($\hat{\beta}_1$) = 0.00605, s.e.($\hat{\beta}_2$) = 0.00097

(b) $t(\hat{\beta}_1) = 0.496/0.00605 = 81.89$; p-value (2-sided) = $2\,P(t(12) > 81.89)$ = 0.000, which is very small. We reject the null hypothesis $\beta_1 = 0$.

$t(\hat{\beta}_2) = 0.009191/0.00097 = 9.49$; p-value (2-sided) = 0.000, which is very small. We reject the null hypothesis $\beta_2 = 0$.

Neither of the two explanatory variables can be omitted from the model.

4.11

(a) Minitab output:

```
The regression equation is
Y = 295 - 481 X1 - 829 X2 + 0.00794 X3 + 2.36 X4

Predictor        Coef     SE Coef         T        P
Constant       295.33       40.18      7.35    0.000
X1             -480.8       150.4     -3.20    0.006
X2             -829.4       196.5     -4.22    0.001
X3           0.007936    0.003554      2.23    0.041
X4             2.3603      0.7616      3.10    0.007

S = 46.77       R-Sq = 88.3%     R-Sq(adj) = 85.1%
```

```
Analysis of Variance

Source            DF        SS         MS        F        P
Regression         4    246538      61635    28.18    0.000
Residual Error    15     32807       2187
Total             19    279345
```

(b) Test $\beta_1 = 0 : \mathrm{t}(\hat{\beta}_1) = -3.20$; p-value = 0.006; reject $\beta_1 = 0$; the number of beds in for profit hospitals is important.

(c) The observations (for the two time periods for each state) look very similar and, most likely, they are correlated. If the correlation is very high, it is reasonable to discard one of them or average the two observations, and reanalyze the data.

(d) Instead of selecting ten states at random, one could classify the states into three groups according to population size - small, medium, and large - and select three or four hospitals at random from each category.

4.12 The output from R software, using the function lm(formula = usage ~ TEMP + PROD + DAYS + PAYR + HOUR) is given below:

```
Coefficients:
              Estimate    Std. Error    t value      Pr (>|t|)
(Intercept)  39.437054     12.110986      3.256      0.00765
TEMP          0.084067      0.060469      1.390      0.19194
PROD          0.001876      0.000607      3.091      0.01027
DAYS          0.131704      0.289800      0.454      0.65833
PAYS         -0.215677      0.098810     -2.183      0.05162
HOUR         -0.014475      0.030052     -0.482      0.63949

Residual standard error: 3.213 on 11 degrees of freedom
Multiple R-Squared: 0.6446, Adjusted R-squared: 0.4831
F-statistic: 3.991 on 5 and 11 DF, p-value: 0.02607
```

R^2 = 0.6446, and the regression model is significant at 2.6% level. The output indicates that PROD is significant at the 1% level, even if other variables are present in the model. PAYS is also marginally significant (p-value = 0.051). All other variables are not significant when added last to the model. The model can be simplified

(b) In order to test $\beta_1 = \beta_3 = \beta_5 = 0$, we need to fit a reduced model that includes just x_2 and x_4. The R output for the reduced model with lm(formula = USAGE ~ PROD + PAYR) is listed below

```
Coefficients:
                 Estimate  Std. Error t value  Pr (>|t|)
(Intercept)    46.0177241  10.1085905   4.552  0.000452
PROD            0.0020353   0.0005587   3.643  0.002663
PAYR           -0.2157919   0.0895867  -2.409  0.030356

Residual standard error: 3.117 on 14 degrees of freedom
Multiple R-Squared: 0.5743, Adjusted R-squared: 0.5135
F-statistic: 9.442 on 2 and 14 DF, p-value: 0.002535
```

The additional sum of squares = ResidualSS (reduced model) – ResidualSS (full model) = SSR(full model) – SSR(reduced model) = 205.956 –183.48 and
F = [(205.956 –183.48)/3]/$(3.213)^2$ = 0.73; p-value = P(F(3,11) > 0.73) = 0.56; we can not reject $\beta_1 = \beta_3 = \beta_5 = 0$.

(c) We prefer the reduced model $\hat{\mu} = 46.02 + 0.00204\text{PROD} - 0.216\text{PAYR}$; R^2 = 0.574 (only slightly smaller than the R^2 of the full model = 0.6446).

(d) Production has the smallest p-value.

(e) Water usage as linear function of PROD and PAYR. For fixed value of PAYR, each unit increase in production increases water use by 0.0020353 (gallons/100). Similarly, for a fixed value of PROD, a unit increase in PAYR decreases water usage by 0.2157919 (gallons/100).

4.13

(a) Minitab output:

```
The regression equation is
Y = 177 + 2.17 X1 + 3.54 X2 - 22.2 X3 + 0.204 X4

Predictor        Coef     SE Coef          T        P
Constant      177.229       8.787      20.17    0.000
X1             2.1702      0.6737       3.22    0.009
X2             3.5380      0.1092      32.41    0.000
X3           -22.1583      0.5454     -40.63    0.000
X4             0.2035      0.3189       0.64    0.538

S = 5.119       R-Sq = 99.7%     R-Sq(adj) = 99.6%

Analysis of Variance

Source           DF          SS          MS         F        P
Regression        4       89285       22321    851.72    0.000
Residual Error   10         262          26
Total            14       89547
```

(b) R^2 = 0.997; estimates are part of the output given above

(c)

(i) t-ratio = 0.64; p-value = 0.538; conclude $\beta_4 = 0$

(ii) F = [(43968 – 262)/2]/[262/10] = 834.1; p-value = P(F(2,10) > 834.1) = 0.0000; reject $\beta_3 = \beta_4 = 0$

(iii) F = (58575 – 262)/(262/10) = 2,225.7; p-value = P(F(2,10) > 2,225.7) = 0.0000; reject $\beta_2 = \beta_3$

```
The regression equation is
Y = - 61 + 4.61 X1 + 3.05 X2+X3 + 2.54 X4
```

Predictor	Coef	SE Coef	T	P
Constant	-61.4	102.4	-0.60	0.561
X1	4.613	9.575	0.48	0.639
X2+X3	3.051	1.549	1.97	0.075
X4	2.541	4.490	0.57	0.583

```
S = 72.97       R-Sq = 34.6%     R-Sq(adj) = 16.7%
```

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	3	30972	10324	1.94	0.182
Residual Error	11	58575	5325		
Total	14	89547			

(iv) F = 851.72; p-value = 0.0000; reject $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$

(d) Minitab output:

```
The regression equation is
Y = 179 + 2.11 X1 + 3.56 X2 - 22.2 X3
```

Predictor	Coef	SE Coef	T	P
Constant	178.521	8.318	21.46	0.000
X1	2.1055	0.6479	3.25	0.008
X2	3.56240	0.09945	35.82	0.000
X3	-22.1880	0.5286	-41.98	0.000

```
S = 4.980       R-Sq = 99.7%     R-Sq(adj) = 99.6%
```

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	3	89274	29758	1200.14	0.000
Residual Error	11	273	25		
Total	14	89547			

(e) 95% prediction interval for sales when $x_1 = 3, x_2 = 45, x_3 = 10$: (111.46, 135.08)

4.14

(a)

$$X'X = \begin{bmatrix} 13.00 & 59.43 & 81.82 & 115.40 \\ 59.43 & 394.73 & 360.66 & 522.08 \\ 81.82 & 360.66 & 576.73 & 728.31 \\ 115.40 & 522.08 & 728.31 & 1035.96 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 8.06479464 & -0.082592705 & -0.094195115 & -0.790526876 \\ -0.08259271 & 0.008479816 & 0.001716687 & 0.003720020 \\ -0.09419511 & 0.001716687 & 0.016629424 & -0.002063308 \\ -0.79052688 & 0.003720020 & -0.002063308 & 0.088601286 \end{bmatrix}$$

$$X'\boldsymbol{y} = \begin{bmatrix} 377.700 \\ 1877.911 \\ 2247.285 \\ 3339.300 \end{bmatrix}$$

(c) Estimated equation: $\hat{\mu} = 39.482 + 1.0092x_1 - 1.873x_2 - 0.367x_3$

(d) (i) (22.802, 25.653); 90% confidence interval for the mean value of y when $x_1 = 3$, $x_2 = 8$ and $x_2 = 9$ can be obtained with the software R directly using the function “predict”.

Mean value	Lower limit	Upper limit
24.22764	22.80225	25.65302

There is also an option in Minitab.

(ii) (20.109, 28.346); 90% prediction interval for an individual value of y when $x_1 = 3$, $x_2 = 8$ and $x_2 = 9$ can also be obtained from the software R directly using the function “predict”.

(e) F-statistic = 30.08; p-value = 0.000; reject $\beta_1 = \beta_2 = \beta_3 = 0$.

4.15

(a) Linear relationship between y and x_1; perhaps some curvature in the scatter plot of y against x_2 (see part (e))

(b) $\hat{\beta}_0 = 2.59; \hat{\beta}_1 = -0.378; \hat{\beta}_2 = 0.877$

Fitted equation: $\hat{\mu} = 2.59 - 0.378x_1 + 0.877x_2$

(c) Significant relationship between y and the variables x_1 and x_2

```
The regression equation is
Y = 2.59 - 0.378 X1 + 0.877 X2
```

Predictor	Coef	SE Coef	T	P
Constant	2.58810	0.08349	31.00	0.000
X1	-0.37802	0.06630	-5.70	0.000
X2	0.8768	0.1723	5.09	0.000

S = 0.06263   R-Sq = 90.8%   R-Sq(adj) = 89.3%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	0.46419	0.23210	59.16	0.000
Residual Error	12	0.04707	0.00392		
Total	14	0.51127			

(d) Model with x_1: $R^2 = 0.709$. Model with x_2: $R^2 = 0.659$. Prefer model with x_2

(e) Prefer model with both x_1 and x_2 as neither variable can be omitted from the model (see t-ratios in (c)).

No need to add $(x_2)^2$ to the model; t-ratio = 0.24; p-value = 0.815

4.16

$$(X'X)^{-1} = \begin{bmatrix} 9.61093203 & 0.008587789 & -0.27914754 & -0.04452169 \\ 0.00858779 & 0.509964070 & -0.25886359 & 0.00077654 \\ -0.27914754 & -0.25886359 & 0.13949996 & 0.00073956 \\ -0.04452169 & 0.00077654 & 0.00073956 & 0.00036978 \end{bmatrix}$$

Correction Factor = $45^2/9 = 225$

SST = $y'y$ – CF = 285 – 225 = 60

SSR = $\hat{\beta}X'y$ – CF = 282.9725 –225 = 57.9725

SSE = SST - SSR = 60 – 57.9725 = 2.0275

ANOVA table:

Source	DF	SS	MS	F	P
Regression	3	57.9725	19.3242	47.66	2.129815e-05
Residual	5	2.0275	0.4055		
Total	8	60.0000			

F-statistic = 47.66; reject $\beta_1 = \beta_2 = \beta_3 = 0$

(b) Estimated equation: $\hat{\mu} = -1.16346 + 0.13527x_1 + 0.01995x_2 + 0.12195x_3$;

$s^2 = 0.4055$;

s.e.($\hat{\beta}_0$) = 1.974; s.e.($\hat{\beta}_1$) = 0.45474; s.e.($\hat{\beta}_2$) = 0.23784; s.e.($\hat{\beta}_3$) = 0.01225

$t(\hat{\beta}_1) = 0.295$; p-value = 0.78; can not reject $\beta_1 = 0$

$\mathrm{t}(\hat{\beta}_2) = 0.084$; p-value = 0.94; can not reject $\beta_2 = 0$
$\mathrm{t}(\hat{\beta}_3) = 9.955$; p-value = 0.000; reject $\beta_3 = 0$

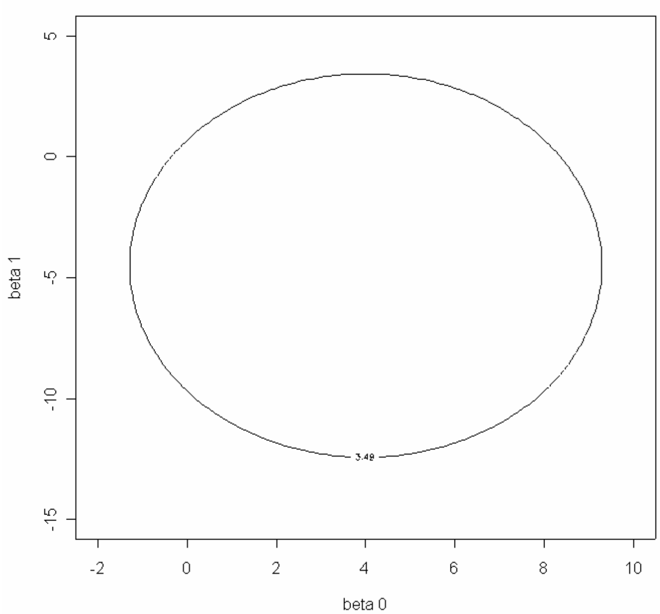
4.17

(a) $t(0.975;20) = 2.086$

95% confidence interval for $\beta_0 : 4 \pm (2.086)(2)$ or (-0.17, 8.17); covers $\beta_0 = 0$, but just barely.

95% confidence interval for $\beta_1 : -4.5 \pm (2.086)(3)$ or (-10.76, 1.76); covers $\beta_1 = 0$

(b) 95% confidence region is given below. The point $(\beta_0 = 0, \beta_1 = 0)$ is very close to the 95% contour (it is just barely within the 95% confidence region). This indicates that neither model A and B are particularly worthwhile.



(c) There is no conflict between the results in (a) and (b). In general there could have been a conflict if $\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1)$ was not zero.

4.18 $\text{Cov}(\boldsymbol{e}, \hat{\boldsymbol{\mu}}) = \text{Cov}((\text{I} - \text{H})\boldsymbol{y}, \text{H}\boldsymbol{y}) = \sigma^2 (\text{I} - \text{H})\text{H} = \text{O}$, a (n x n) matrix of zeros. Vectors $\boldsymbol{e}$ and $\hat{\boldsymbol{\mu}}$ are linear functions of $\boldsymbol{y}$ and are normal. Hence $\boldsymbol{e}$ and $\hat{\boldsymbol{\mu}}$ are statistically independent.

4.19

(a) $\hat{\beta}^{\text{WLS}} = \sum(\text{y}_\text{i}\text{x}_\text{i} / \text{x}_\text{i}^2) \big/ \sum(\text{x}_\text{i}^2 / \text{x}_\text{i}^2) = \sum(\text{y}_\text{i} / \text{x}_\text{i}) \big/ \text{n}$; $\text{V}(\hat{\beta}^{\text{WLS}}) = \sigma^2 \big/ \text{n}$

(b) $\hat{\beta}^{\text{WLS}} = 30/12 = 2.5$; $\text{V}(\hat{\beta}^{\text{WLS}}) = \sigma^2 / 12$

4.20

(a) $\hat{\beta}^{\text{WLS}} = \sum \text{y}_\text{i} \big/ \sum \text{x}_\text{i}$; $\text{V}(\hat{\beta}^{\text{WLS}}) = \sigma^2 \big/ \sum \text{x}_\text{i}$

(b) $\hat{\beta}^{\text{WLS}} = 30/2 = 15$; $\text{V}(\hat{\beta}^{\text{WLS}}) = \sigma^2 / 150$

4.21 See Exercise 4.9. Minitab output:

```
The regression equation is
y = 885 - 6.57 x1 - 1.37 x2

Predictor        Coef     SE Coef          T        P
Constant       885.16       61.75      14.33    0.000
x1            -6.5708      0.5832     -11.27    0.000
x2            -1.3743      0.1943      -7.07    0.000

S = 36.49       R-Sq = 84.0%     R-Sq(adj) = 82.8%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      189062       94531     71.00    0.000
Residual Error    27       35950        1331
Total             29      225011
```

95 percent confidence interval for the mean abrasion loss for rubber with hardness 70 and tensile strength 200: (134.65, 166.03)

4.22

(a) Linear model not appropriate.

(b) Fitted equation:

TensileStrength = -6.674 + 11.764 Hardwood - 0.635 $(\text{Hardwood})^2$

Source	DF	SS	MS	F	P
Regression	2	3104.2	1552.1	79.43	0.000
Residual Error	16	312.6	19.5		
Total	18	3416.9			

Model adequate; quadratic term needed; increases R^2 from 0.305 to 0.909.
95% confidence interval for mean response when hardwood 6 percent: (38.14, 44.00)
Prediction intervals are for individual observations while confidence intervals are for the mean value. Confidence intervals are shorter than the corresponding prediction intervals. 95% prediction interval for tensile strength for a batch of paper with 6 percent hardwood concentration: (31.25, 50.88).
The maximum hardwood concentration in the data set used to fit the model is 7 percent, which is very low compared to 20 percent. It is not advisable to use the fitted model to predict the mean tensile strength of paper for 20 percent hardwood concentration.

4.23

Quadratic model. Estimated equation: $\hat{\mu} = 82.385 - 38.310\mathrm{x} + 4.703\mathrm{x}^2$
Regression significant; adequate fit.
Stars with ln(surface temperature) < 4 appear different and should be investigated separately. Without these stars, a linear model is appropriate.

Predictor	Coef	SE Coef	T	P
Constant	82.385	9.581	8.60	0.000
x	-38.310	4.790	-8.00	0.000
x2	4.7025	0.5939	7.92	0.000

```
S = 0.3667      R-Sq = 60.6%     R-Sq(adj) = 58.8%

Analysis of Variance
```

Source	DF	SS	MS	F	P
Regression	2	9.0945	4.5472	33.82	0.000
Residual Error	44	5.9165	0.1345		
Total	46	15.01			

4.24

(a) Minitab regression output

```
The regression equation is
Y = 31.4 + 9.31 UFFI + 2.85 Tight
```

Predictor	Coef	SE Coef	T	P
Constant	31.373	2.461	12.75	0.000
UFFI	9.312	2.133	4.37	0.000
Tight	2.8545	0.3764	7.58	0.000

```
S = 5.223       R-Sq = 78.3%     R-Sq(adj) = 76.2%
```

```
Analysis of Variance

Source            DF        SS        MS       F       P
Regression         2    2063.3    1031.6   37.82   0.000
Residual Error    21     572.9      27.3
Total             23    2636.1
```

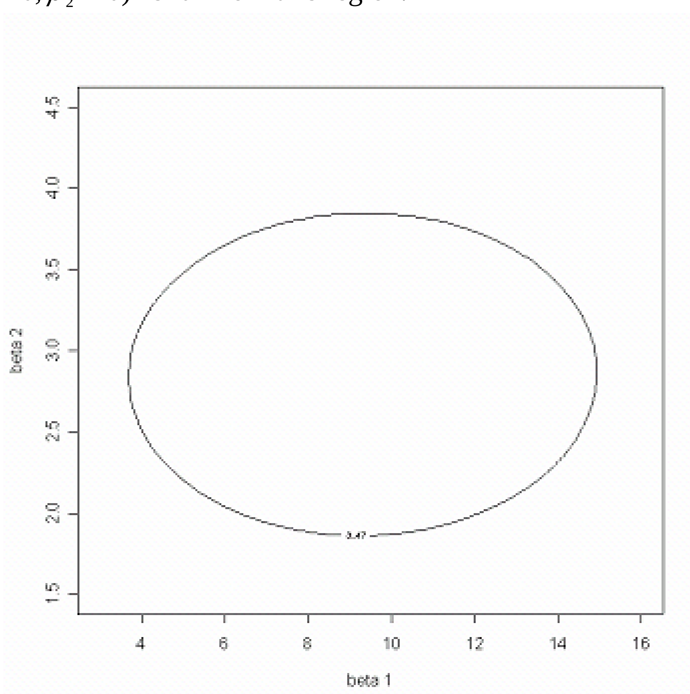
(b)

$$(X'X) = \begin{bmatrix} 24 & 12 & 123 \\ 12 & 12 & 61 \\ 123 & 61 & 823 \end{bmatrix};\ (X'X)^{-1} = \begin{bmatrix} 0.221946 & -0.085569 & -0.026828 \\ -0.085569 & 0.166703 & 0.000433 \\ -0.026828 & 0.000433 & 0.005193 \end{bmatrix};$$

$$V(\hat{\boldsymbol{\beta}}) = s^2 (X'X)^{-1} = \begin{bmatrix} 6.05462 & -2.33430 & -0.73187 \\ -2.33430 & 4.54761 & 0.01180 \\ -0.73187 & 0.01180 & 0.14165 \end{bmatrix}$$

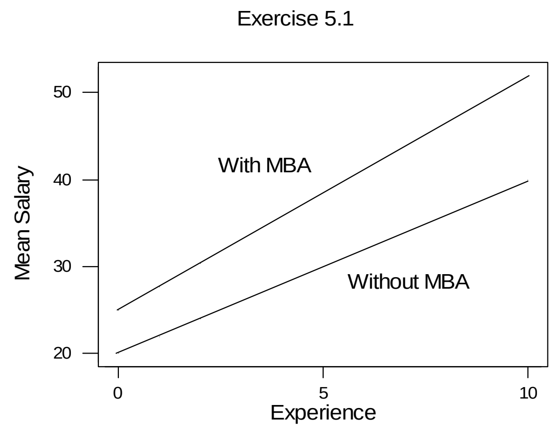
$\text{s.e.}(\hat{\beta}_0) = \sqrt{6.05462} = 2.461$; $\text{s.e.}(\hat{\beta}_1) = \sqrt{4.54761} = 2.133$;
$\text{s.e.}(\hat{\beta}_2) = \sqrt{0.14165} = 0.376$

(c) The 95 percent confidence region for (β_1, β_2) is shown below. The point $(\beta_1 = 0, \beta_2 = 0)$ is far from this region.



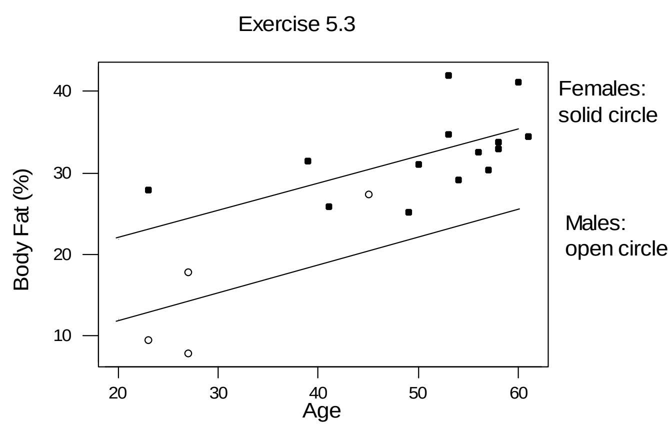
CHAPTER 5

5.1 Interaction; bonus for having a MBA; furthermore, salary increases faster for MBAs.



5.2 (a) $ 3,000; (b) $ 900

5.3



Minitab regression output. Significant age and gender effects; body fat of males is 9.79 percent lower than that of females. However, very few data for males.

```
The regression equation is
bodyfat = 15.1 + 0.339 age - 9.79 gender

Predictor        Coef     SE Coef          T        P
Constant       15.071       6.224       2.42    0.029
age            0.3392      0.1196       2.84    0.013
gender         -9.791       3.697      -2.65    0.018

S = 4.905       R-Sq = 74.6%     R-Sq(adj) = 71.2%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2     1060.66      530.33     22.04    0.000
Residual Error    15      360.88       24.06
Total             17     1421.54
```

Regression with an interaction component: Interaction component is not needed.

```
The regression equation is
bodyfat = 20.1 + 0.240 age - 29.3 gender + 0.572 age*gen

Predictor        Coef     SE Coef          T        P
Constant       20.112       6.239       3.22    0.006
age            0.2401      0.1204       1.99    0.066
gender         -29.27       10.41      -2.81    0.014
age*gen        0.5725      0.2893       1.98    0.068

S = 4.488       R-Sq = 80.2%     R-Sq(adj) = 75.9%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3     1139.51      379.84     18.86    0.000
Residual Error    14      282.02       20.14
Total             17     1421.54
```

5.4 $\text{VIF}_1 = 1/(1-R_1^2) = 2.5$; $\text{VIF}_2 = 1/(1-R_2^2) = 5$; $\text{VIF}_3 = 1/(1-R_3^2) = 10$; evidence of multicollinearity since variance inflation factors are large (10 or larger).

5.5 (e)

5.6 Define two indicator variables x_1 and x_2 such that $x_1 = 0$ and $x_2 = 0$ represent the group Sparrow, $x_1 = 1$, $x_2 = 0$ represent Robin, and $x_1 = 0$ and $x_2 = 1$ represent Wren. Then the model can be expressed as $E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ in which $\beta_1 = \mu(\text{Robin}) - \mu(\text{Sparrow})$ and $\beta_2 = \mu(\text{Wren}) - \mu(\text{Sparrow})$.

```
Analysis of Variance
                                 Sum of          Mean
 Source                  DF      Squares        Square  F Value  Pr > F
 Model                    2     31.11193      15.55596    22.33  <.0001
 Error                   42     29.26052       0.69668
 Corrected Total         44     60.37244
```

F-statistic = 22.33 tests whether there are differences among the three group means; p-value < 0.0001; reject H_0: $\mu_1 = \mu_2 = \mu_3$ (or $\beta_1 = \beta_2 = 0$)

5.7 Minitab output for regression with averages

```
The regression equation is
yield = 78.4 - 3.55 fac1 - 1.45 fac2 + 3.20 fac3

Predictor        Coef     SE Coef          T        P
Constant       78.375       1.022      76.65    0.000
fac1           -3.550       1.022      -3.47    0.026
fac2           -1.450       1.022      -1.42    0.229
fac3            3.200       1.022       3.13    0.035

S = 2.892       R-Sq = 85.6%     R-Sq(adj) = 74.9%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3     199.560      66.520      7.95    0.037
Residual Error     4      33.455       8.364
Total              7     233.015
```

$V(\bar{y}_i) = s^2/5 = 40/5 = 8$; $s(\bar{y}_i) = \sqrt{8} = 2.83$ (calculated from the pure error sum of squares) is very similar to s = 2.892 that is calculated from the residuals. Hence there is no lack of fit. However, in general this must not be the same, and should be checked.

$$V(\hat{\boldsymbol{\beta}}) = (X'X)^{-1}X'\boldsymbol{y} = (s^2/5)(X'X)^{-1} = 8\begin{bmatrix} 0.125 & 0 & 0 & 0 \\ 0 & 0.125 & 0 & 0 \\ 0 & 0 & 0.125 & 0 \\ 0 & 0 & 0 & 0.125 \end{bmatrix}$$

s.e.$(\hat{\beta}_i) = 1$; $t(\hat{\beta}_1) = -3.55$; $t(\hat{\beta}_2) = -1.45$; $t(\hat{\beta}_3) = 3.20$; the effect of factor 2 is not significant.

5.8

(a) Expected difference in systolic blood pressure for females versus males who drink the same number of cups of coffee, excercise the same, and are of the same age

(b) Represents variation due to measurement error and omitted factors

(c) Association, but not causation

(d) Represents interaction between gender and coffee consumption

5.9

(a) $E(y_t) = \begin{cases} \beta_0 + \beta_1 t, \ t = 1,2,...,7 \\ \beta_2 + \beta_3 t, \ t = 8,9,...,14 \end{cases}$

Intersecting lines at t = 8: $\beta_2 = \beta_0 + 8(\beta_1 - \beta_3)$, and

E (y_t) = $\begin{cases} \beta_0 + \beta_1 t, \ t = 1,2,...,7 \\ \beta_0 + \beta_1 8 + \beta_3 (t-8), \ t = 8,9,...,14 \end{cases}$

In matrix form, $E(\boldsymbol{y}) = X\boldsymbol{\beta}$ where

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ . & . & . \\ 1 & 7 & 0 \\ 1 & 8 & 0 \\ 1 & 8 & 1 \\ . & . & . \\ 1 & 8 & 6 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_3 \end{bmatrix}$$

(b) $E(y_t) = \beta_0 + \beta_1 t, \ t = 1,2,...,14$

(c) F = 55.95; p-value = P(F(1,11) > 55.95) = 0.0000; model in (a) is preferable.

5.10

(a) $E(y_t) = \beta_0 + \beta_1 t, \ t = 1,2,...,12$

(b) $E(y_t) = \beta_0 + \beta_1 t + \beta_2 t^2, \ t = 1,2,...,12$

(c) $E(y_t) = \beta_0 + \beta_1 t + \beta_2 x_t, \ t = 1,2,...,12$ where $x_t = 0$ for $t = 1,2...,6$, and $x_t = 1$ for t = 7,8,...,12

(d) $E(y_t) = \begin{cases} \beta_0 + \beta_1 t, \ t = 1,2,...,7 \\ \beta_2 + \beta_3 t, \ t = 8,9,...,14 \end{cases}$

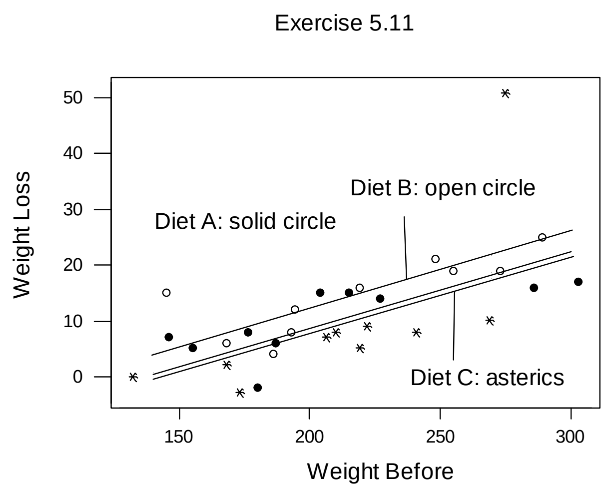
Intersecting lines at t = 7: $\beta_2 = \beta_0 + 7(\beta_1 - \beta_3)$, and

$$E(y_t) = \begin{cases} \beta_0 + \beta_1 t, & t = 1,2,...,6 \\ \beta_0 + \beta_1 7 + \beta_3 (t-7), & t = 7,8,...,12 \end{cases}$$

In matrix form, $E(\boldsymbol{y}) = X\boldsymbol{\beta}$ where

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ . & . & . \\ 1 & 6 & 0 \\ 1 & 7 & 0 \\ 1 & 7 & 1 \\ . & . & . \\ 1 & 7 & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_3 \end{bmatrix}$$

5.11



Note the unusual observation for one subject on diet C (x = 275, y = 51). We define indicators for the three diets: IndA = 1 if diet A and = 0 otherwise; IndB = 1 if diet B and = 0 otherwise; IndC = 1 if diet C and = 0 otherwise.

Minitab output from the estimation of the model $y = \beta_0 + \beta_1 x + \beta_2 \text{IndB} + \beta_3 \text{IndC} + \varepsilon$ is shown below.

Using all n = 30 cases we find not much difference between the three diets. F-statistic for testing $\beta_2 = \beta_3 = 0$: F = (1740.1 - 1650.12)/2] / (1650.12/26) = 0.71; p-value = P(F(2,26) > 0.71) = 0.50; conclude $\beta_2 = \beta_3 = 0$.

Models with all 30 cases:

```
The regression equation is
y = - 18.4 + 0.137 x + 3.15 IndB - 0.89 IndC

Predictor        Coef     SE Coef          T        P
Constant      -18.388       7.067      -2.60    0.015
x             0.13703     0.03176       4.31    0.000
IndB            3.153       3.574       0.88    0.386
IndC           -0.893       3.565      -0.25    0.804

S = 7.967       R-Sq = 44.5%     R-Sq(adj) = 38.1%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3     1323.25      441.08      6.95    0.001
Residual Error    26     1650.12       63.47
Total             29     2973.37


The regression equation is
y = - 18.2 + 0.140 x

Predictor     Coef  SE Coef      T      P
Constant   -18.167    6.799  -2.67  0.012
x          0.13954  0.03132   4.45  0.000

S = 7.88328   R-Sq = 41.5%   R-Sq(adj) = 39.4%

Analysis of Variance

Source          DF      SS      MS      F      P
Regression       1  1233.3  1233.3  19.84  0.000
Residual Error  28  1740.1    62.1
Total           29  2973.4
```

The observation (diet C; x = 275, y = 51) is highly unusual. Omitting this case, leads to the results given below. In the next chapter (Chapter 6) you will learn about diagnostic measures that allow you to quantify the effects of outliers. After reading Chapter 6, you may want to confirm that this case leads to the standardized residual = 4.48 and Cook's distance = 0.98.

Models with outlying case omitted:

```
The regression equation is
y = - 10.2 + 0.0977 x + 3.51 IndB - 4.65 IndC

Predictor     Coef  SE Coef      T      P
Constant   -10.205    3.567  -2.86  0.008
x          0.09767  0.01610   6.07  0.000
IndB         3.511    1.747   2.01  0.055
IndC        -4.651    1.789  -2.60  0.015

S = 3.89272   R-Sq = 72.0%   R-Sq(adj) = 68.7%

Analysis of Variance

Source          DF       SS      MS      F      P
Regression       3   975.03  325.01  21.45  0.000
Residual Error  25   378.83   15.15
Total           28  1353.86


The regression equation is
y = - 12.1 + 0.106 x

Predictor     Coef  SE Coef      T      P
Constant   -12.132    4.465  -2.72  0.011
x          0.10574  0.02079   5.09  0.000

S = 5.06040   R-Sq = 48.9%   R-Sq(adj) = 47.0%

Analysis of Variance

Source          DF       SS      MS      F      P
Regression       1   662.45  662.45  25.87  0.000
Residual Error  27   691.41   25.61
Total           28  1353.86
```

F-statistic for testing $\beta_2 = \beta_3 = 0$: F = (691.41 – 378.83)/2] / (378.83/25) = 10.31; p-value = P(F(2,25) > 10.31) = 0.001; reject $\beta_2 = \beta_3 = 0$.

(b) There are differences among the three diets in terms of their effectiveness on weight reduction. Diet C has the largest benefit.

5.12

```
Analysis of Variance
                             Sum of          Mean
Source              DF      Squares       Squares    F Value  Pr > F
Model                4     39.37694       9.84423      14.07  <.0001
Error               25     17.49506       0.69980
Corrected Total     29     56.87200
```

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-0.91221	0.87548	-1.04	0.3074
x1	1	0.16073	0.06617	2.43	0.0227
x2	1	0.21978	0.03406	6.45	<.0001
x3	1	0.01123	0.00497	2.26	0.0330
x4	1	0.10197	0.05874	1.74	0.0948

(b) $\hat{\mu} = -0.9122 + 0.1607x_1 + 0.2198x_2 + 0.0112x_3 + 0.1020x_4$; R^2 = 0.692; s = 0.8365;

(i) $t(\hat{\beta}_1) = 2.43$; p-value = 0.023; reject $\beta_1 = 0$

(ii) F = (5.45747/2)/(0.69980) = 3.90 (use of additional SS); p-value = 0.034; reject the null hypothesis $\beta_3 = \beta_4 = 0$

(iii) F=14.07; p-value <.0001; reject hypothesis $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$.

(c)
$\hat{\mu} = -1.462 + 0.1536x_1 + 0.3221x_2 + 0.0166x_3 + 0.0571x_4 - 0.00087x_2x_3 + 0.00599x_2x_4$

$H_0 : \beta_5 = \beta_6 = 0$: F = 0.40; p-value = 0.67; interactions not important.

(d) (i) Since all coefficients are positive: Lower wrinkle resistance for lower x_1, x_2, x_3, and x_4.

(ii) Increased wrinkle resistance for higher x_1, x_2, x_3, and x_4.

(e) It is difficult to generalize the conclusions from this study since the values of x_1, x_2, x_3, and x_4 were not controlled. One suggestion for improvement is to conduct an experiment in which the values of x_1, x_2, x_3, and x_4 are controlled and the resulting response *y* measured.

5.13

(b) z = 0 (protein-rich); z = 1 (protein-poor): $\hat{\mu} = 50.324 + 16.009x + 0.918z - 7.329xz$

H_0: $\beta_2 = \beta_3 = 0$. Test whether the linear relationship between height (y) and age (x) is the same for the two diets. Additional SS = ResidualSS (reduced model) – ResidualSS (full model) = 1120.22, and F = (1120.22/2)/(5.22290) = 107.24; p-value < 0.0001; reject $\beta_2 = \beta_3 = 0$; linear relationships between height and age not the same for the two diets.

5.14

(a) Since the columns of X are orthogonal, X'X is a diagonal matrix. Let $X'X = \Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{p+1})$. We have seen that $\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{y}$. Also $V(\hat{\boldsymbol{\beta}}) = (X'X)^{-1}\sigma^2 = \Lambda^{-1}\sigma^2 = \sigma^2\text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_{p+1}^{-1})$. Since the off diagonal elements are zero, $\text{Cov}(\hat{\beta}_i, \hat{\beta}_j) = 0$, for all $i \neq j$. In addition, $\hat{\beta}_i$ and $\hat{\beta}_j$ are normally distributed. Hence $\hat{\beta}_i$ and $\hat{\beta}_j$ are statistically independent.

(b) $\boldsymbol{y} = X\boldsymbol{\beta} + \gamma\, \boldsymbol{z} + \boldsymbol{\varepsilon}$, where $\boldsymbol{z}$ is orthogonal to the columns of X; that is, $X'\boldsymbol{z} = \boldsymbol{0}$ and $\boldsymbol{z}'X = \boldsymbol{0}'$. Let $X_1 = [X\ \ \boldsymbol{z}]$ be a new matrix containing the columns of X and $\boldsymbol{z}$. Then

$$\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\gamma} \end{pmatrix} = (X_1'X_1)^{-1}X_1'\boldsymbol{y} = \left[\begin{pmatrix} X' \\ \boldsymbol{z}' \end{pmatrix}(X\ \ \boldsymbol{z})\right]^{-1}\begin{bmatrix} X' \\ \boldsymbol{z}' \end{bmatrix}\boldsymbol{y} = \begin{bmatrix} X'X & X'\boldsymbol{z} \\ \boldsymbol{z}'X & \boldsymbol{z}'\boldsymbol{z} \end{bmatrix}^{-1}\begin{bmatrix} X' \\ \boldsymbol{z}' \end{bmatrix}\boldsymbol{y}$$

$$= \begin{bmatrix} X'X & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{z}'\boldsymbol{z} \end{bmatrix}^{-1}\begin{bmatrix} X'\boldsymbol{y} \\ \boldsymbol{z}'\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} (X'X)^{-1} & \boldsymbol{0} \\ \boldsymbol{0}' & (\boldsymbol{z}'\boldsymbol{z})^{-1} \end{bmatrix}\begin{bmatrix} X'\boldsymbol{y} \\ \boldsymbol{z}'\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} (X'X)^{-1}X'\boldsymbol{y} \\ (\boldsymbol{z}'\boldsymbol{z})^{-1}\boldsymbol{z}'\boldsymbol{y} \end{bmatrix} = \begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \tilde{\gamma} \end{pmatrix}.$$

Note that $\tilde{\boldsymbol{\beta}}$ is exactly the same as $\hat{\boldsymbol{\beta}}$, and hence they have the same distribution.

(c) Let us first explain the phrase "columns are centered about their means". Let $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, …, $\boldsymbol{w}_p$ be column vectors of the matrix $W = [\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_p]$. Let $\overline{w}_i$ be the average of column vector $\boldsymbol{w}_i$. Define $\boldsymbol{x}_i = \boldsymbol{w}_i - \boldsymbol{1}\overline{w}_i$ where $\boldsymbol{1}$ is a column vector with n ones. Then $X_1 = [\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_p]$ has columns that are centered about their means. This implies that the sum of the elements in each column of the matrix X_1 is zero; that is, $\boldsymbol{1}'\boldsymbol{x}_i = 0$, for each i.
Defining the matrix $X = [\boldsymbol{1}, X_1]$ leads to the estimates

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\boldsymbol{\beta}}_* \end{bmatrix} = (X'X)^{-1}X'\boldsymbol{y}$$

$$= \begin{bmatrix} n & \boldsymbol{0}' \\ \boldsymbol{0} & X_1'X_1 \end{bmatrix}^{-1}\begin{pmatrix} \boldsymbol{1}' \\ X_1' \end{pmatrix}\boldsymbol{y} = \begin{bmatrix} n^{-1} & \boldsymbol{0}' \\ \boldsymbol{0} & (X_1'X_1)^{-1} \end{bmatrix}\begin{pmatrix} \boldsymbol{1}'\boldsymbol{y} \\ X_1'\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \overline{y} \\ (X_1'X_1)^{-1}X_1'\boldsymbol{y} \end{pmatrix}$$

This shows that $\hat{\beta}_0 = \overline{y}$.

Furthermore, $V(\hat{\boldsymbol{\beta}}) = (X'X)^{-1}\sigma^2 = \begin{bmatrix} n^{-1} & \boldsymbol{0}' \\ \boldsymbol{0} & (X_1'X_1)^{-1} \end{bmatrix}\sigma^2$ implies that the covariance between $\hat{\beta}_0$ and $\hat{\beta}_j$, for j = 1, 2, …, p, is zero. In addition, $\hat{\boldsymbol{\beta}}$ is normally distributed. Hence $\hat{\beta}_0$ is distributed independently of all other $\hat{\beta}_j$, for j = 1, 2, …, p.

5.15 Weight (x_1); $x_2 = 0$ (type A engine); $x_2 = 1$ (type B engine);
(a) $\mu = \beta_0 + \beta_1 x_1 + \beta_2 x_2$; (b) $\mu = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$

5.16

(a) β_3 represents the change in expected yield of catalyst 2 over catalyst 1 when temperature is held fixed.

(b) Test of $\beta_3 = 0$: $t(\hat{\beta}_3) = -0.32/0.36 = -0.89$; p-value = $2P(t(26) \le -0.89) = 0.38$; conclude $\beta_3 = 0$; no evidence to suggest a difference in catalysts.

95% confidence interval for β_2 : $\hat{\beta}_2 \pm (0.975;26)\text{s.e.}(\hat{\beta}_2)$, $0.41 \pm (2.065)(0.11)$ or (0.18, 0.64).

(c) (i) $\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = 0$. Since $\hat{\boldsymbol{\beta}}$ is normally distributed, $\text{Cov}(\hat{\beta}_1, \hat{\beta}_3) = 0$ implies that $\hat{\beta}_1$ and $\hat{\beta}_3$ are independent.

(ii) 95% confidence interval for E(y) when x = 0 and z =1. Let $\theta = E(y) = \beta_0 + \beta_3$.

Estimate: $\hat{\theta} = \hat{\beta}_0 + \hat{\beta}_3 = 29.51$

$$V(\hat{\theta}) = V(\hat{\beta}_0) + V(\hat{\beta}_3) + 2\text{Cov}(\hat{\beta}_0, \hat{\beta}_3) = s^2[0.114 + 0.133 + 2(-0.0671)]$$
$$= (25.05/26)[0.114 + 0.133 + 2(-0.0671)] = 0.1087$$

$\hat{\theta} \pm (0.975;26)\sqrt{V(\hat{\theta})}$, $29.51 \pm (2.065)\sqrt{0.1087}$, or (28.83, 30.19).

(iii) 95% prediction interval

$\hat{\theta} \pm (0.975;26)\sqrt{s^2 + V(\hat{\theta})}$, $29.51 \pm (2.065)\sqrt{(25.05/26) + (0.1087)}$, or (27.37, 31.65)

(d) Model equation for catalyst 1: $E(y) = \beta_0 + \beta_1 x + \beta_2 x^2$

Model equation for catalyst 2: $E(y) = (\beta_0 + \beta_3) + (\beta_1 + \beta_4)x + (\beta_2 + \beta_5)x^2$

Test $\beta_3 = \beta_4 = 0$: Additional SS = 25.05-19.70 = 5.35. Thus F = (5.35/2)/(19.70/24) = 3.26; p-value = 0.056. There is some weak evidence that the effect of temperature changes with the catalysts.

5.17

(a) Minitab output is given below. It helps to include the square of poverty as an explanatory variable (t-ratio = 2.72 and p-value = 0.007).

On Poverty only:

```
The regression equation is
test = 74.6 - 0.536 pov

Predictor       Coef  SE Coef       T      P
Constant      74.606    1.613   46.25  0.000
pov         -0.53578  0.03262  -16.43  0.000

S = 8.76595   R-Sq = 67.3%   R-Sq(adj) = 67.1%
```

```
Analysis of Variance
Source           DF     SS     MS       F      P
Regression        1  20731  20731  269.79  0.000
Residual Error  131  10066     77
Total           132  30798
```

On Poverty and $(Poverty)^2$:

```
The regression equation is
test = 79.9 - 0.850 pov + 0.00343 pov**2

Predictor      Coef   SE Coef      T      P
Constant     79.950     2.520  31.72  0.000
pov         -0.8504    0.1201  -7.08  0.000
pov**2     0.003427  0.001261   2.72  0.007

S = 8.56001   R-Sq = 69.1%   R-Sq(adj) = 68.6%

Analysis of Variance

Source           DF     SS     MS       F      P
Regression        2  21272  10636  145.16  0.000
Residual Error  130   9526     73
Total           132  30798
```

(c) It is not necessary to include an indicator for students in the college community Iowa City (t-ratio = 0.73 and p-value = 0.467).

On Poverty, $(Poverty)^2$, and Indicator for Iowa City:

```
The regression equation is
test = 79.2 - 0.832 pov + 0.00332 pov**2 + 1.73 IowaCity

Predictor      Coef   SE Coef      T      P
Constant     79.197     2.728  29.03  0.000
pov         -0.8322    0.1229  -6.77  0.000
pov**2     0.003319  0.001272   2.61  0.010
IowaCity      1.735     2.380   0.73  0.467

S = 8.57548   R-Sq = 69.2%   R-Sq(adj) = 68.5%

Analysis of Variance

Source           DF       SS      MS      F      P
Regression        3  21311.3  7103.8  96.60  0.000
Residual Error  129   9486.5    73.5
Total           132  30797.8
```

CHAPTER 6

6.1 (a) The Minitab output of the three regressions is shown below.
In the model involving x_1 alone, the hypothesis $\beta_1 = 0$ can not be rejected. This indicates that x_1 by itself is not important.
Similarly, in the model involving x_2 alone, x_2 by itself is not significant ($\beta_2 = 0$ can not be rejected).
The model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$ leads to a large R^2 = 0.794, and the partial t-tests for $\beta_1 = 0$ and $\beta_2 = 0$ are significant. This indicates that x_1 helps explain y at fixed levels of x_2; and x_2 helps explain y at fixed levels of x_1.
This example is instructive as it shows that regressors may be insignificant when studied alone, but taken jointly they may help explain a large part of the variability. It provides an example where stepwise procedures lead to different solutions. Forward selection and stepwise regression would not include any variables, whereas backward elimination would select the model with both regressors. This shows that it is preferable to look at all possible regressions. Note that x_1 and x_2 are correlated (r = 0.734).

```
The regression equation is
Y = 889 - 6.52 X1

Predictor        Coef     SE Coef          T        P
Constant        889.3       268.9       3.31    0.011
X1             -6.519       8.289      -0.79    0.454

S = 123.2       R-Sq = 7.2%      R-Sq(adj) = 0.0%


The regression equation is
Y = 387 + 1.55 X2

Predictor        Coef     SE Coef          T        P
Constant        387.4       287.4       1.35    0.214
X2              1.550       1.509       1.03    0.334

S = 120.2       R-Sq = 11.7%     R-Sq(adj) = 0.6%


The regression equation is
Y = 547 - 31.1 X1 + 6.00 X2

Predictor        Coef     SE Coef          T        P
Constant        547.1       152.0       3.60    0.009
X1            -31.147       6.491      -4.80    0.002
X2              6.003       1.212       4.95    0.002

S = 62.04       R-Sq = 79.4%     R-Sq(adj) = 73.5%
```

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      103859       51930     13.49    0.004
Residual Error     7       26941        3849
Total              9      130800
```

(b) Observation #2 (with $x_1 = 43$, x_2 = 223 and y = 480) is unusual and somewhat different than the rest. We remove this observation and refit the three models. The results are similar, with the model with both x_1 and x_2 leading to the best representation.

```
The regression equation is
Y = 287 - 17.6 X1 + 5.18 X2

Predictor        Coef     SE Coef          T        P
Constant        286.8       155.1       1.85    0.114
X1            -17.557       7.323      -2.40    0.053
X2             5.1801      0.9733       5.32    0.002

S = 46.90       R-Sq = 84.7%     R-Sq(adj) = 79.6%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2       73159       36579     16.63    0.004
Residual Error     6       13197        2199
Total              8       86356
```

6.2

(a) Linear model: $\hat{\mu} = 23.35 + 1.045x$; R^2 = 0.955; s = 0.737;
F(lack of fit) = 10.01; p-value = 0.002; lack of fit.

Source	d.f	S.S	M.S	F	Prob$\geq$F
Model	1	195.2428	195.2428	359.3	0.0001
Error	17	9.2382	0.5434		
Lack of Fit	9	8.4849	0.9427	10.01	<0.01
Pure Error	8	0.7533	0.0942		

(b) Quadratic model: $\hat{\mu} = 22.56 + 1.67x - 0.068x^2$; R^2 = 0.988; s = 0.394;
$t(\hat{\beta}_2) = -0.06796/0.01031 = -6.59$; reject $\beta_2 = 0$;
F(lack-of-fit) = 2.30; p-value = 0.13; no lack of fit.

Source	d.f	S.S	M.S	F	Prob>F
Model	2	201.9944	100.9972	649.86	0.0001
Error	16	2.4866	0.1554		
Lack of Fit	8	1.7333	0.2166	2.3	>.10
Pure Error	8	0.7533	0.0947		

6.3 Vector of fitted values and residuals: $\hat{\boldsymbol{\mu}} = \mathrm{H}\boldsymbol{y}$; $\boldsymbol{e} = (\mathrm{I} - \mathrm{H})\boldsymbol{y} = (\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\boldsymbol{y}$, where $\mathrm{X} = [\mathbf{1}, \boldsymbol{x}]$ is the n x 2 matrix, and $\boldsymbol{\beta} = (\beta_0, \beta_1)'$.

True model : $\boldsymbol{y} = \beta_0 \mathbf{1} + \beta_1 \boldsymbol{x}_1 + \beta_2 \boldsymbol{x}_2 + \boldsymbol{\varepsilon}$ where $\boldsymbol{x}_2' = (\mathrm{x}_1^2, ..., \mathrm{x}_n^2)$

$$\mathrm{E}(\boldsymbol{e}) = (\mathrm{I} - \mathrm{X}'(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\mathrm{E}(\boldsymbol{y}) = (\mathrm{I} - \mathrm{H})[\mathrm{X}\boldsymbol{\beta} + \beta_2 \boldsymbol{x}_2 + \mathrm{E}(\boldsymbol{\varepsilon})] = (\mathrm{I} - \mathrm{H})\mathrm{X}\boldsymbol{\beta} + \beta_2(\mathrm{I} - \mathrm{H})\boldsymbol{x}_2$$
$$= \beta_2(\mathrm{I} - \mathrm{H})\boldsymbol{x}_2 \quad \text{since} \quad (\mathrm{I} - \mathrm{H})\mathrm{X} = \mathrm{O}$$

6.4

(a) $\mathrm{E}(\hat{\boldsymbol{\mu}}) = \mathrm{E}(\mathrm{X}\hat{\boldsymbol{\beta}}) = \mathrm{X}\mathrm{E}(\hat{\boldsymbol{\beta}}) = \mathrm{X}\boldsymbol{\beta}$

$\mathrm{V}(\hat{\boldsymbol{\mu}}) = \mathrm{V}(\mathrm{X}\hat{\boldsymbol{\beta}}) = \mathrm{X}\mathrm{V}(\hat{\boldsymbol{\beta}})\mathrm{X}' = \mathrm{X}(\sigma^2(\mathrm{X}'\mathrm{X})^{-1})\mathrm{X}' = \sigma^2\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'$

(b) $\sum_{i=1}^{n} \mathrm{V}(\hat{\mu}_i) = \sigma^2 \,\mathrm{tr}\,(\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}') = \sigma^2 \,\mathrm{tr}\,((\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{X}) = \sigma^2 \,\mathrm{tr}\,(\mathrm{I}) = \sigma^2(\mathrm{p}+1)$

Hence $\frac{1}{n}\sum_{i=1}^{n} \mathrm{V}(\hat{\mu}_i) = \frac{(\mathrm{p}+1)}{n}\sigma^2$

(c) $\boldsymbol{a}_i'\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\boldsymbol{a}_i = \boldsymbol{a}_i'\mathrm{H}\boldsymbol{a}_i \geq 0$ because $(\mathrm{X}'\mathrm{X})^{-1}$ is a positive semidefinite matrix.
Select $\boldsymbol{a}_i$ as the vector with all components 0 except for a "1" in the ith element.
Thus $\mathrm{h}_{ii} \geq 0$.

H is symmetric and idempotent. $\mathrm{H} = \mathrm{HH}$ implies $\mathrm{h}_{ii} = \mathrm{h}_{ii}^2 + \sum_{j \neq i}^{n} \mathrm{h}_{ij}^2 \geq 0$ and

$\sum_{j \neq i}^{n} \mathrm{h}_{ij}^2 = \mathrm{h}_{ii}(1 - \mathrm{h}_{ii}) \geq 0$. Since $\mathrm{h}_{ii} \geq 0$, we find that $(1 - \mathrm{h}_{ii}) \geq 0$ and $\mathrm{h}_{ii} \leq 1$.

(d) We can parameterize the model as $\boldsymbol{y} = \mathbf{1}\alpha + \mathrm{V}\boldsymbol{\beta}_* + \boldsymbol{\varepsilon}$ where
$\alpha = \beta_0 + \beta_1\bar{\mathrm{x}}_1 + ... + \beta_p\bar{\mathrm{x}}_p$, $\mathrm{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, ...\boldsymbol{v}_p]$ contains the mean corrected regressors $\boldsymbol{v}_j = \boldsymbol{x}_j - \bar{\mathrm{x}}_j\mathbf{1}$, $\bar{\mathrm{x}}_j$ is the average of the elements of the vector $\boldsymbol{x}_j$, and $\boldsymbol{\beta}_*$ is the vector $\boldsymbol{\beta}$ without the element β_0.
Note that $\mathrm{X} = [\mathbf{1}, \mathrm{V}]$ and $\mathbf{1}'\boldsymbol{v}_j = 0$, for j = 1, 2, …, p. Hence

$$X'X = \begin{bmatrix} n & 0 \\ 0 & V'V \end{bmatrix}, \quad (X'X)^{-1} = \begin{bmatrix} n^{-1} & 0 \\ 0 & (V'V)^{-1} \end{bmatrix}, \text{ and}$$

$$H = [\mathbf{1} \quad V] \begin{bmatrix} n^{-1} & 0 \\ 0 & (V'V)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1}' \\ V' \end{bmatrix} = [n^{-1}\mathbf{1}\mathbf{1}' + V(V'V)^{-1}V'].$$

The matrix $H^* = V(V'V)^{-1}V'$ is symmetric and idempotent; we have shown in 6.4(c) that its diagonal elements h^*_{ii} are between 0 and 1. Hence the ith diagonal element of H, $h_{ii} = n^{-1} + h^*_{ii} \ge n^{-1}$.

(e) Both $\hat{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\beta}}$ are solutions of $(X'X)\boldsymbol{\beta} = X'\boldsymbol{y}$. Hence $(X'X)\hat{\boldsymbol{\beta}} = X'\boldsymbol{y}$ and $(X'X)\tilde{\boldsymbol{\beta}} = X'\boldsymbol{y}$, and $(X'X)(\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}}) = \mathbf{0}$.

Let $\hat{\boldsymbol{\mu}} = X\hat{\boldsymbol{\beta}}$, $\tilde{\boldsymbol{\mu}} = X\tilde{\boldsymbol{\beta}}$, and $\hat{\boldsymbol{\mu}} - \tilde{\mu} = X(\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}})$.

$\sum_{i=1}^{n} (\hat{\mu}_i - \tilde{\mu}_i)^2 = (\hat{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}})'(\hat{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}) = (\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}})' X'X(\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}}) = (\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}})'\mathbf{0} = 0$

The sum of squares is zero if and only if $(\hat{\mu}_i - \tilde{\mu}_i) = 0$ for all i. Hence $\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}$.

6.5

(a) We need to show: $(I + \alpha \boldsymbol{v}\boldsymbol{w}') \left[I - \left(\frac{\alpha}{1 + \alpha \boldsymbol{v}'\boldsymbol{w}} \right) \boldsymbol{v}\boldsymbol{w}' \right] = I$

The left hand side is given by

$$\text{LHS} = I + \alpha \boldsymbol{v}\boldsymbol{w}' - \left[\frac{\alpha [\boldsymbol{v}\boldsymbol{w}' + \alpha \boldsymbol{v}\boldsymbol{w}'\boldsymbol{v}\boldsymbol{w}']}{1 + \alpha \boldsymbol{v}'\boldsymbol{w}} \right]$$

$$= I + \alpha \boldsymbol{v}\boldsymbol{w}' - \left[\frac{\alpha}{1 + \alpha \boldsymbol{v}'\boldsymbol{w}} \right] [1 + \alpha \boldsymbol{v}'\boldsymbol{w}] \boldsymbol{v}\boldsymbol{w}' = I + \alpha \boldsymbol{v}\boldsymbol{w}' - \alpha \boldsymbol{v}\boldsymbol{w}' = I$$

(b) For full rank matrices with the same dimension: $(CD)^{-1} = D^{-1}C^{-1}$. Hence $(A + \boldsymbol{w}\boldsymbol{w}')^{-1} = [A(I + A^{-1}\boldsymbol{w}\boldsymbol{w}')]^{-1} = (I + A^{-1}\boldsymbol{w}\boldsymbol{w}')^{-1} A^{-1}$.

Let $A^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $\alpha = 1$. Then

$$(A + \boldsymbol{w}\boldsymbol{w}')^{-1} = (I + \boldsymbol{v}\boldsymbol{w}')^{-1} A^{-1} = \left[I - \left(\frac{1}{1 + \boldsymbol{v}'\boldsymbol{w}} \right) \boldsymbol{v}\boldsymbol{w}' \right] A^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{w}\boldsymbol{w}'A^{-1}}{1 + \boldsymbol{w}'A^{-1}\boldsymbol{w}}.$$

(c) (i) Note that $X_1 = \begin{bmatrix} X \\ \boldsymbol{w}' \end{bmatrix}$; $(X_1'X_1)^{-1} = (X'X + \boldsymbol{w}\boldsymbol{w}')^{-1}$

Let $X'X = A$. Then

$$(X_1'X_1)^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{w}\boldsymbol{w}'A^{-1}}{1-\boldsymbol{w}'A^{-1}\boldsymbol{w}} = (X'X)^{-1} - \frac{(X'X)^{-1}\boldsymbol{w}\boldsymbol{w}'(X'X)^{-1}}{1-\boldsymbol{w}'(X'X)^{-1}\boldsymbol{w}}$$

(ii) $\hat{\boldsymbol{\beta}}_1 = (X_1'X_1)^{-1}X_1'\boldsymbol{y}_1 = (X'X + \boldsymbol{w}\boldsymbol{w}')^{-1}(X\boldsymbol{y} + \boldsymbol{w}y_{n+1}) =$

$$= \hat{\boldsymbol{\beta}} - \frac{(X'X)^{-1}\boldsymbol{w}\boldsymbol{w}'\hat{\boldsymbol{\beta}}}{1-\boldsymbol{w}'(X'X)^{-1}\boldsymbol{w}} + (X'X)^{-1}\boldsymbol{w}y_{n+1} - \frac{(X'X)^{-1}\boldsymbol{w}\boldsymbol{w}'(X'X)^{-1}\boldsymbol{w}y_{n+1}}{1-\boldsymbol{w}'(X'X)^{-1}\boldsymbol{w}}$$

Define the scalar $h = \boldsymbol{w}'(X'X)^{-1}\boldsymbol{w}$. Then

$$\hat{\boldsymbol{\beta}}_1 = \hat{\boldsymbol{\beta}} - \frac{(X'X)^{-1}\boldsymbol{w}\boldsymbol{w}'}{1-h}\hat{\boldsymbol{\beta}} + \frac{(X'X)^{-1}\boldsymbol{w}(1-h)y_{n+1}}{1-h}$$

$$= \hat{\boldsymbol{\beta}} + (X'X)^{-1}\boldsymbol{w}(y_{n+1} - \frac{1}{1-h}\boldsymbol{w}'\hat{\boldsymbol{\beta}})$$

6.6 The estimate of $\boldsymbol{\beta}$ in the model with all the x's, $\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, is

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\boldsymbol{\beta}}_{(K)} \\ \hat{\beta}_K \end{bmatrix} = \begin{bmatrix} \tilde{X}'\tilde{X} & \tilde{X}\boldsymbol{x}_K \\ \boldsymbol{x}_K'\tilde{X} & \boldsymbol{x}_K'\boldsymbol{x}_K \end{bmatrix}^{-1} \begin{bmatrix} \tilde{X}'\boldsymbol{y} \\ \boldsymbol{x}_K'\boldsymbol{y} \end{bmatrix}$$

where the n x (k-1) matrix $\tilde{X}$ is as defined in the hint and where $\hat{\boldsymbol{\beta}}_{(K)}$ denotes the vector of estimates $\hat{\boldsymbol{\beta}}$ without the element $\hat{\beta}_K$.

Using the results on the inverse of a partitioned matrix given in the appendix of Chapter 6, we obtain

$$\hat{\beta}_K = \frac{\boldsymbol{x}_K'(I-\tilde{H})\boldsymbol{y}}{\boldsymbol{x}_K'(I-\tilde{H})\boldsymbol{x}_K}$$ where $\tilde{H} = I - \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}'$ is an idempotent matrix; $\tilde{H}\tilde{H} = \tilde{H}$.

In step 1, when we regress $\boldsymbol{y}$ on $\tilde{X}$ we obtain the vector of residuals $\boldsymbol{r} = (I-\tilde{H})\boldsymbol{y}$.
In step 2, when we regress $\boldsymbol{x}_K'$ on $\tilde{X}$ we obtain the vector of residuals $\boldsymbol{u} = (I-\tilde{H})\boldsymbol{x}_K'$
Note that the means of the residual vectors $\boldsymbol{r}$ and $\boldsymbol{u}$ are zero. Hence the slope of the regression of $\boldsymbol{r}$ on $\boldsymbol{u}$ in step 3 is

$$\tilde{\beta}_K = \boldsymbol{u}'\boldsymbol{r}/\boldsymbol{u}'\boldsymbol{u} = \frac{\boldsymbol{x}_K'(I-\tilde{H})(I-\tilde{H})\boldsymbol{y}}{\boldsymbol{x}_K'(I-\tilde{H})(I-\tilde{H})\boldsymbol{x}_K} = \boldsymbol{x}_K'(I-\tilde{H})\boldsymbol{y}\big/\boldsymbol{x}_K'(I-\tilde{H})\boldsymbol{x}_K = \hat{\beta}_K$$

6.7
(a) True. For a correct model, $\text{Cov}(\boldsymbol{e}, \hat{\boldsymbol{\mu}}) = \text{O}$, and a plot of the residuals e_i against the fitted values $\hat{\mu}_i$ should show no association. However, $\text{Cov}(\boldsymbol{e}, \boldsymbol{y}) = \sigma^2(\text{I} - \text{H})$; the correlation makes the interpretation of the plot of e_i against y_i difficult.
(b) Not true. Outliers should be scrutinized, but not necessarily rejected.
(c) True

6.8 (a) 5; (b) 2; (c) 4; (d) 1

6.9 (a) True; (b) True; (c) False; (d) False; (e) False

6.10 (d) True. Linear regression of $\ln(y)$ on $\ln(x_1)$ and $\ln(x_2)$ to estimate β_1 and β_2.

6.11 (a) No; (b) No; (c) No; (d) No; (e) True

6.12 A (Palm Beach); B (Broward); C (Dade); D (Pasco)

6.13 Consider the stock price data **lenzing** and refer to Exercise 10.9

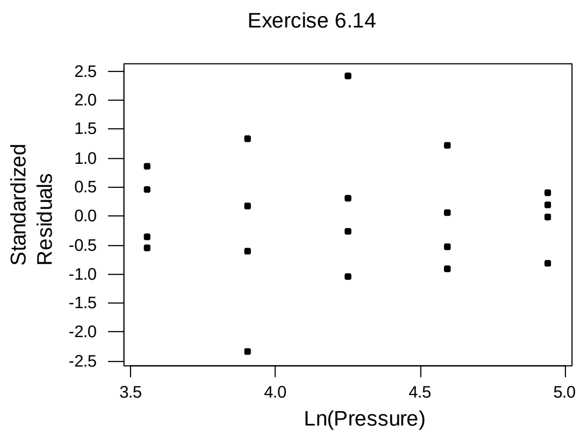
6.14 Note that the pressures are equally spaced on the logarithmic scale, suggesting that the investigator expected equal changes in the ratio of pressures to produce equal changes in the tearing factor. This suggests that a logarithmic transformation of pressure (x) may be appropriate.
Scatter plots of y against x, y against ln(x), ln(y) against x, and ln(y) against ln(x) were constructed. For a data set of such small size, the choice among the various transformations is difficult. Here we consider a model of y on ln(x).

R-output

```
              Estimate   Std. Error   t value    Pr(>|t|)
(Intercept)    152.451       10.493    14.529    2.19e-11
lnx            -10.604        2.453    -4.322    0.000411

Residual standard error: 5.378 on 18 degrees of freedom
Multiple R-Squared: 0.5093,     Adjusted R-squared: 0.482
F-statistic: 18.68 on 1 and 18 DF,  p-value: 0.0004105
```

Because of the replications it is possible to calculate a test for lack of fit. The F-statistic is small and no lack of fit is indicated. The residual plot suggests that the variability in the response may not be the same at all settings of pressure. However, this fact is difficult to assess with a small data set such as this.

```
Minitab output and test for lack of fit
The regression equation is
Y=Tear = 152 - 10.6 LnX

Predictor        Coef     SE Coef          T        P
Constant       152.45       10.49      14.53    0.000
LnX           -10.604       2.453      -4.32    0.000

S = 5.378       R-Sq = 50.9%     R-Sq(adj) = 48.2%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      540.23      540.23     18.68    0.000
Residual Error    18      520.57       28.92
  Lack of Fit      3       28.57        9.52      0.29    0.832
  Pure Error      15      492.00       32.80
Total             19     1060.80
```



6.15 Scatter plots of y , ln(y) and 1/y against x point to a log transformation. The estimate of the transformation parameter in Box-Cox family is $\hat{\lambda} \approx 0$, indicating a logarithmic transformation of the response y.
Regression of ln(y) on x: $\hat{\mu} = 2.436 + 0.000567x$; $R^2 = 0.986$; s = 0.0845.
The first case is quite influential (x = 574; y = 21.9; Cook = 0.585).

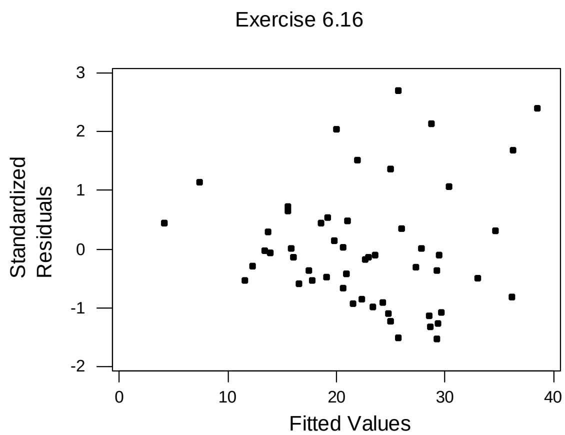
Box -Cox transformation

λ	$s(\lambda)$	R^2
-1.00	11.270	0.922
-0.75	8.569	0.948
-0.50	6.331	0.969
-0.25	4.690	0.982
-0.10	4.165	0.985
0.001 (ln)	4.082	0.986
0.10	4.232	0.985
0.25	4.849	0.980
0.50	6.629	0.965
0.75	9.033	0.942
1.00	11.960	0.912

$s(\lambda)$ is the residual standard error and R^2 is the coefficient of determination in the regression of $\frac{y^{\lambda}-1}{\lambda(\bar{y}_g)^{\lambda-1}}$ on x.

6.16 The regression shows that neither of the two variables can be omitted from the model. The residual plot indicates no major model violations. Also the scatter plots of the residuals against the two explanatory variables are unremarkable. The case with the largest Cook's distance is case # 48 with $x_1 = 2.35$, $x_2 = 56$ and $y = 72$ (Cook = 0.27)

```
The regression equation is
Y = 23.0 + 23.6 X1 - 0.715 X2

Predictor        Coef     SE Coef          T        P
Constant        23.01       18.28       1.26    0.214
X1             23.639       6.848       3.45    0.001
X2            -0.7147      0.3014      -2.37    0.022

S = 14.84       R-Sq = 20.2%     R-Sq(adj) = 17.0%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      2783.2      1391.6      6.32    0.004
Residual Error    50     11007.9       220.2
Total             52     13791.2
```



6.17 Scatter plots indicate that a linear regression of rigidity on elasticity and density is appropriate. Partial output from R is given below:

```
Coefficients:
             Estimate  Std. Error  t value   Pr(>|t|)
(Intercept)  -1.8300    121.1577   -0.015      0.988
x1            3.4179      0.7925    4.313   8.21e-05
x2           19.5830      3.2851    5.961   3.08e-07

Residual standard error: 185.9 on 47 degrees of freedom
Multiple R-Squared: 0.8119, Adjusted R-squared: 0.8039
F-statistic: 101.4 on 2 and 47 DF, p-value: < 2.2e-16
```

Residual diagnostics indicate that observation # 40 has large influence (Cook = 0.572). This observation should be scrutinized.
We remove this observation and refit the model on the reduced data set. The Minitab results are shown below. The residual plot is unremarkable, except perhaps for a large positive and a large negative residual. However, the Cook influence from the case with the large positive residual (original case # 46) is not particularly worrisome (Cook = 0.215).

```
The regression equation is
Y = - 9.2 + 4.21 X1 + 15.9 X2

Predictor        Coef     SE Coef          T        P
Constant        -9.17       94.51      -0.10    0.923
X1             4.2146      0.6344       6.64    0.000
X2             15.949       2.644       6.03    0.000

S = 145.0       R-Sq = 87.6%     R-Sq(adj) = 87.1%
```

```
Analysis of Variance

Source            DF        SS        MS        F       P
Regression         2   6843941   3421971   162.76   0.000
Residual Error    46    967129     21025
Total             48   7811070
```

6.18

(a) The correlation between liver weight (LW) and body weight (BW) is 0.5. This is also confirmed by the plot of LW versus BW.

(b) Pair-wise scatter plots of y against the three regressors show very little association. We regress y (dose in liver) on BW = body weight, LW = liver weight and DL = dose. The regression results indicate that BW and DL are significant, which is somewhat surprising as we have not seen strong associations in the pair-wise scatter plots.
Case # 3 (with BW = 190, LW = 9.0, Dose = 1.00, and y = 0.56) is a very influential observation (Cook = 0.930). This case should be scrutinized. Dropping this case from the data set, leads to the regression results shown below. Neither one of the three regressors is significant (F-statistic = 0.10), which supports the conclusion from the earlier scatter plots.

R output (all observations)

```
Coefficients:
             Estimate  Std. Error   t value    Pr(>|t|)
(Intercept)  0.265922    0.194585     1.367      0.1919
BW          -0.021246    0.007974    -2.664      0.0177
LW           0.014298    0.017217     0.830      0.4193
D            4.178111    1.522625     2.744      0.0151

Residual standard error: 0.07729 on 15 degrees of freedom
```

```
Multiple R-Squared: 0.3639,  Adjusted R-squared: 0.2367
F-statistic:  2.86 on 3 and 15 DF, p-value: 0.07197
```

Minitab output (case # 3 removed)

```
The regression equation is
Y = 0.311 - 0.0078 BW + 0.0090 LW + 1.48 Dose

Predictor        Coef     SE Coef          T        P
Constant       0.3114      0.2051       1.52    0.151
BW           -0.00778     0.01872      -0.42    0.684
LW            0.00899     0.01866       0.48    0.637
Dose            1.485       3.713       0.40    0.695

S = 0.07825     R-Sq = 2.1%      R-Sq(adj) = 0.0%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3    0.001844    0.000615      0.10    0.958
Residual Error    14    0.085717    0.006123
Total             17    0.087561
```

6.19

Pair-wise scatter plots of y against the two regressors show moderate association and an outlying case (case #17 with x_1 = 26.8, x_2 = 58 and y =168). The regression results shown below indicate a significant regressor x_1 and R^2 = 0.482. The influence of case #17 is large (Cook = 0.838). Removing this case from the data set leads to the revised estimates. Variable x_2 can be dropped from the model. Inorganic phosphorus explains about half of the variation in plant phosphorus (R^2 = 0.519).

Minitab output

```
The regression equation is
Y = 56.3 + 1.79 X1 + 0.087 X2

Predictor        Coef     SE Coef          T        P
Constant        56.25       16.31       3.45    0.004
X1             1.7898      0.5567       3.21    0.006
X2             0.0866      0.4149       0.21    0.837

S = 20.68       R-Sq = 48.2%     R-Sq(adj) = 41.3%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      5975.7      2987.8      6.99    0.007
Residual Error    15      6413.9       427.6
Total             17     12389.6
```

Minitab output (case #17 omitted)

```
The regression equation is
Y = 66.5 + 1.29 X1 - 0.111 X2

Predictor        Coef     SE Coef          T        P
Constant       66.465       9.850       6.75    0.000
X1             1.2902      0.3428       3.76    0.002
X2            -0.1110      0.2486      -0.45    0.662

S = 12.25       R-Sq = 52.5%     R-Sq(adj) = 45.7%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      2325.2      1162.6      7.75    0.005
Residual Error    14      2101.3       150.1
Total             16      4426.5
```

Minitab output (x1 only; case #17 omitted)

```
The regression equation is
Y = 62.6 + 1.23 X1

Predictor        Coef     SE Coef          T        P
Constant       62.569       4.452      14.05    0.000
X1             1.2291      0.3058       4.02    0.001

S = 11.92       R-Sq = 51.9%     R-Sq(adj) = 48.6%
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      2295.2      2295.2     16.15    0.001
Residual Error    15      2131.2       142.1
Total             16      4426.5
```

6.20

The scatter plot of vocabulary (y) against age (x) indicates an approximate linear relationship, with the exception of case #1 (Age = 1; Vocabulary = 3). Fitting the linear regression on age leads to the results shown below. The first case exerts large influence (Cook = 1.126). Omitting this observation leads to the revised estimates. The fit improves; the standard deviation of the residuals decreases from 116.7 to 81.45. Also the residual plots improve.

R output (all observations)

```
Coefficients:
               Estimate  Std. Error   t value    Pr(>|t|)
(Intercept)     -763.86       88.25    -8.656    2.47e-05
Age              561.93       24.29    23.134    1.29e-08

Residual standard error: 116.7 on 8 degrees of freedom
Multiple R-Squared: 0.9853,     Adjusted R-squared: 0.9834
F-statistic: 535.2 on 1 and 8 DF,  p-value: 1.294e-08
```

R output (after dropping case #1)

```
Coefficients:
               Estimate  Std. Error   t value    Pr(>|t|)
(Intercept)     -894.75       74.88    -11.95    6.54e-06
Age              592.34       19.63     30.18    1.13e-08

Residual standard error: 81.45 on 7 degrees of freedom
Multiple R-Squared: 0.9924,     Adjusted R-squared: 0.9913
F-statistic: 910.7 on 1 and 7 DF,  p-value: 1.131e-08
```

6.21
Scatter plot of ln(y) against ln(x) shows a linear association with three outlying observations (brachiosaurus, diplodocus, and triceratops). Omitting these three cases and fitting the linear model to the reduced data set leads to an adequate fit. Estimated equation: $\hat{\mu} = 2.15 + 0.752\ln(x)$; $R^2 = 0.922$; s = 0.726. The two observations with the largest positive residuals and the largest Cook influence are human (stand. residual = 2.72; Cook = 0.174) and Rhesus monkey (stand. residual = 2.25; Cook =0.119).

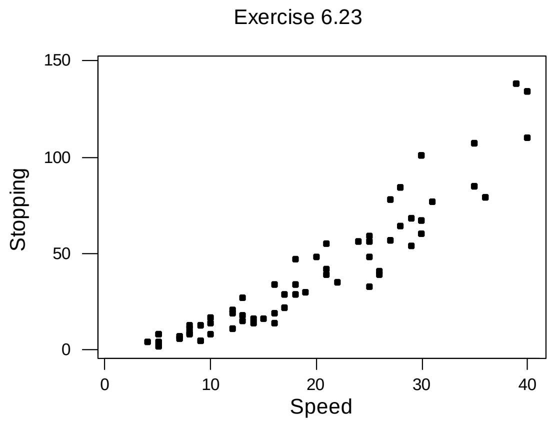
6.22
Estimated equation: $\hat{\mu} = 74.319 - 2.089\text{Conc} + 0.430\text{Ratio} - 0.372\text{Temp}$;
$R^2 = 0.939$; s = 0.74; F(lack of fit) = 7.44; p-value = 0.036; indication of lack of fit.

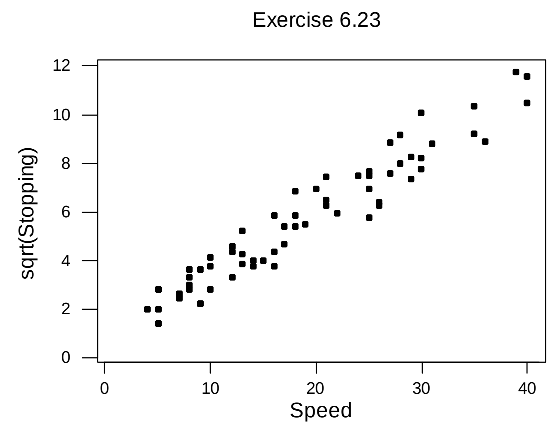
Analysis of Variance

Source	DF	SS	MS	F	P
Regression	3	92.304	30.768	56.17	0.000
Residual Error	11	6.026	0.548		
Lack of Fit	7	5.596	0.799	7.44	0.036
Pure Error	4	0.430	0.108		
Total	14	98.329			

Run #2 (Conc = 1, Ratio = -1,Temp = -1; Yield = 73.9) influential, with large Cook's distance. This run should be investigated. Without this run, no lack of fit.

6.23 Scatter plots of y, ln(y), $\sqrt{y}$, 1/y against x indicate that the square root transformation works best to (i) achieve a linear relationship, and (ii) stabilize the variance.

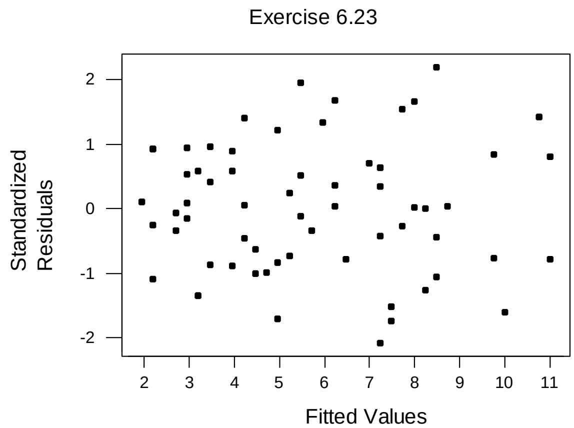


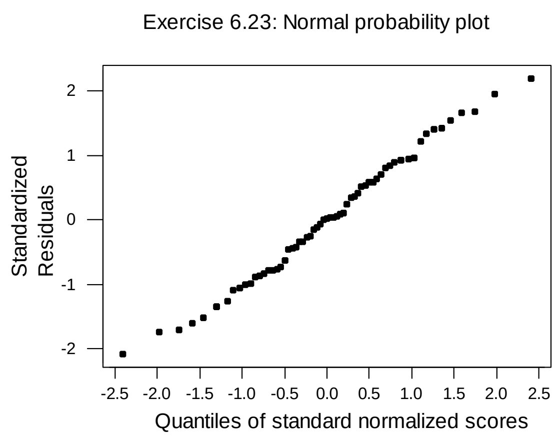


The regression results for the square root transformation of the response are shown below. The residual plot shows no remaining patterns. The normal probability plot of the residuals is adequate.

```
The regression equation is
sqrt(Stopping) = 0.918 + 0.253 Speed

Predictor         Coef      SE Coef          T        P
Constant        0.9183       0.1974       4.65    0.000
Speed         0.252568     0.009246      27.32    0.000

S = 0.7193      R-Sq = 92.4%     R-Sq(adj) = 92.3%
```

```
Analysis of Variance

Source           DF        SS        MS       F       P
Regression        1    386.06    386.06  746.22   0.000
Residual Error   61     31.56      0.52
Total            62    417.62
```



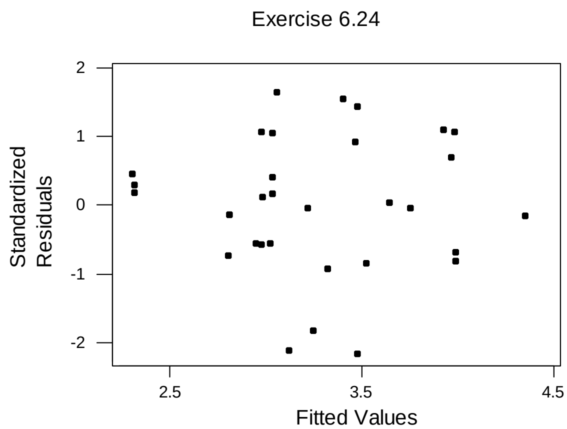


The transformation parameter of the Box-Cox family is estimated by regressing the transformed response $\frac{y^{\lambda}-1}{\lambda(\bar{y}_g)^{\lambda-1}}$ on x, and finding the λ that minimizes the error sum of squares or the residual standard error $s(\lambda)$. The results show that the square root transformation is the appropriate transformation to use.

λ	$s(\lambda)$
-1.00	40.90
-0.75	27.11
-0.50	18.49
-0.25	12.99
0.00 ln	9.49
0.25	7.61
0.50 sqrt	7.34
0.75	8.77
1.00	11.80

6.24 From the equation for the volume of a cylinder, one can expect a model of the form $V = \alpha(x_1)^2 x_2$, or after taking the logarithm, $\ln(V) = \beta_0 + \beta_1 \ln(x_1) + \beta_2 \ln(x_2)$. The fit of this model is quite good; R^2 = 0.626. The residual plot is adequate, and even the largest Cook's influence (0.224 for case #18) is not particularly worrisome.

```
The regression equation is
lny = - 6.63 + 1.98 lnx1 + 1.12 lnx2

Predictor        Coef     SE Coef         T        P
Constant      -6.6316      0.7998     -8.29    0.000
lnx1          1.98265     0.07501     26.43    0.000
lnx2           1.1171      0.2044      5.46    0.000

S = 0.08139     R-Sq = 97.8%     R-Sq(adj) = 97.6%

Analysis of Variance

Source            DF          SS          MS        F        P
Regression         2      8.1232      4.0616   613.19    0.000
Residual Error    28      0.1855      0.0066
Total             30      8.3087
```



6.25 The linear model is capable of approximating the relationship; $R^2 = 0.626$. Cases #6 and #10 have the largest influence on the results (Cook = 0.327 and 0.414). Models that include the squares and the product of x1 and x2 (which could be expected from the formula for the volume of an ellipsoid) do not fare better.

```
The regression equation is
Volume = - 8.63 + 1.90 Diameter + 5.45 CrossSection

Predictor        Coef     SE Coef          T        P
Constant       -8.634       3.694      -2.34    0.044
Diameter       1.9037      0.6867       2.77    0.022
CrossSec        5.446       1.624       3.35    0.008

S = 0.07831     R-Sq = 62.6%     R-Sq(adj) = 54.3%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2    0.092505    0.046253      7.54    0.012
Residual Error     9    0.055187    0.006132
Total             11    0.147692
```

6.26
Linear model: $\hat{\mu} = 0.131 + 0.241x$, with $R^2 = 0.874$, is not appropriate.
Quadratic model: $\hat{\mu} = -1.16 + 0.723x - 0.0381x^2$, with $R^2 = 0.968$, is a possibility.
90% confidence interval: (1.972, 2.102).
Reciprocal transformation on x: $\hat{\mu} = 2.98 - 6.93(1/x)$, with $R^2 = 0.980$, is better.
90% confidence interval: (1.951, 2.026).

CHAPTER 7

7.1

(a) Backward elimination: Drop x_3 (step 1); drop x_4 (step 2); next candidate x_2 for elimination can not be dropped. Model with x_1 and x_2.

(b) Forward selection: Enter x_4 (step1); enter x_1 (step 2); enter x_2 (step 3); next candidate x_3 for selection can not be entered. Model with x_1, x_2, and x_4.

(c) Stepwise Regression: Steps 1, 2 and 3 of forward selection; x_4 can be dropped from the model containing x_1, x_2, and x_4; no reason to add x_3 to the model with x_1 and x_2. Model with x_1 and x_2.

(d) Model with x_1 and x_2: C_p = 2.68, close to desired value 3. Full model: C_p = 5. Prefer model with x_1 and x_2.

(e) x_2 and x_4 are highly correlated.

(f) F = 68.6; p-value less than 0.001; reject $\beta_1 = \beta_3 = 0$.

7.2

(a) C_p: Model with x_1 and x_2 (C_p = 2.7)

R^2: Model with x_1 and x_2, or model with x_1 and x_4. Small gain by going to more complicated models.

(b) Backward elimination ($\alpha_{drop} = 0.1$): Model with x_1 and x_2.

Forward selection ($\alpha_{enter} = 0.1$): Model with x_1, x_2, and x_4.

Stepwise regression ($\alpha_{drop} = \alpha_{enter} = 0.1$): Model with x_1 and x_2.

7.3

Minitab Best Subset Regression results:

Response is Y_1

Vars	R-Sq	R-Sq(adj)	C-p	S	X1	X2	X3	X4
1	49.3	45.4	9.8	1470.5				X
1	34.0	29.0	16.1	1677.2			X	
2	63.3	57.2	6.1	1301.8			X	X
2	49.6	41.2	11.7	1526.3	X			X
3	66.8	57.8	6.6	1293.4	X		X	X
3	64.6	54.9	7.5	1335.8		X	X	X
4	75.6	65.9	5.0	1162.2	X	X	X	X

Response is Y_2

Vars	R-Sq	R-Sq(adj)	C-p	S	X1	X2	X3	X4
1	98.4	98.3	7.3	43.517		X		
1	97.8	97.6	14.6	51.392	X			
2	99.1	99.0	1.1	33.550	X	X		
2	98.5	98.2	8.5	44.288		X		X
3	99.1	98.9	3.0	34.965	X	X	X	
3	99.1	98.9	3.0	35.021	X	X		X
4	99.1	98.8	5.0	36.644	X	X	X	X

Response is Y_3

Vars	R-Sq	R-Sq(adj)	C-p	S	X1	X2	X3	X4
1	36.1	31.2	8.1	90.890			X	
1	5.6	0.0	17.2	110.45	X			
2	66.3	60.7	1.1	68.686	X		X	
2	65.1	59.3	1.4	69.938		X	X	
3	66.4	57.3	3.0	71.616	X		X	X
3	66.3	57.1	3.0	71.731	X	X	X	
4	66.5	53.1	5.0	75.051	X	X	X	X

Minitab Stepwise Regression results:

Response is Y_1

The regression equation is
Y1 = 7770 + 49.6 X3 + 45.1 X4

Predictor	Coef	SE Coef	T	P
Constant	7770	2349	3.31	0.006
X3	49.55	23.14	2.14	0.053
X4	45.07	14.56	3.10	0.009

S = 1302   R-Sq = 63.3%   R-Sq(adj) = 57.2%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	35115127	17557564	10.36	0.002
Residual Error	12	20335325	1694610		
Total	14	55450452			

Response is Y_2

The regression equation is
Y2 = - 67.4 + 5.66 X1 + 8.02 X2

Predictor	Coef	SE Coef	T	P
Constant	-67.40	41.20	-1.64	0.128

X1	5.662	1.802	3.14	0.009
X2	8.018	1.864	4.30	0.001

S = 33.55 R-Sq = 99.1% R-Sq(adj) = 99.0%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	1546691	773346	687.05	0.000
Residual Error	12	13507	1126		
Total	14	1560198			

Response is Y_3

The regression equation is
Y3 = 292 - 2.68 X1 + 5.94 X3

Predictor	Coef	SE Coef	T	P
Constant	292.4	122.2	2.39	0.034
X1	-2.6796	0.8168	-3.28	0.007
X3	5.943	1.278	4.65	0.001

S = 68.69 R-Sq = 66.3% R-Sq(adj) = 60.7%

Analysis of Variance

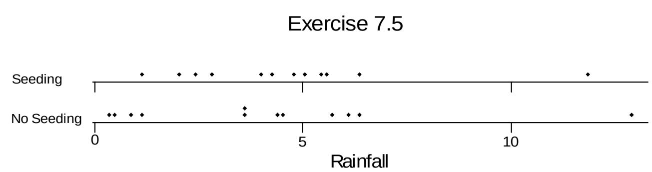
Source	DF	SS	MS	F	P
Regression	2	111462	55731	11.81	0.001
Residual Error	12	56613	4718		
Total	14	168075			

(a) For production overhead costs (y_1): x_3 and x_4 are important. For direct production costs (y_2): x_1 and x_2 are important. For marketing costs (y_3): x_1 and x_3 are important.
(b) For production overhead costs (y_1), the change in production from the last period (x_4) is the single most important variable. For direct production costs (y_2), the production quantity (x_2) is the single most important variable.

7.4
(a) False; different models may result if multicollinearity is present
(b) True
(c) False; can stay the same

7.5
Dot plots of rainfall for days with and without seeding are shown below. We see little difference between the two groups. The results of the two-sample t-test shown below indicate that the group difference is not significant.

```
Two-sample T for Rainfall
SA           N      Mean     StDev   SE Mean
0  (NO)     12      4.17      3.52       1.0
1  (YES)    12      4.63      2.78      0.80

Difference = mu (0) - mu (1)
Estimate for difference:  -0.46
95% CI for difference: (-3.16, 2.24)
T-Test of difference = 0 (vs not =):T-Value = -0.36 P-Value = 0.725 DF=20
```



The question now becomes whether the significance of the seeding action changes when other explanatory variables are included in the model. The results of the full model shown below are:

F = 1.77 for overall regression; p-value = 0.1647; the evidence for including any of the variables is quite weak;

t-values of the regression coefficients are small; their p-values are large, indicating that the variables are not important given that the other variables are in the model.

Seeding action is insignificant, indicating that it is difficult to justify cloud seeding.

Case diagnostics reveal that case 2 has a large studentized residual = -2.278, Cook's D = 4.748 and leverage = 0.865.

```
The regression equation is
y=Rainfall = 4.65 + 1.01 SA - 0.0321 Time - 0.911 SC + 0.006 EchoCov
           + 2.17 EchoMot + 1.84 PreWet
```

Predictor	Coef	SE Coef	T	P
Constant	4.654	3.337	1.39	0.181
SA	1.013	1.203	0.84	0.411
Time	-0.03212	0.02892	-1.11	0.282
SC	-0.9109	0.7512	-1.21	0.242
EchoCov	0.0057	0.1149	0.05	0.961
EchoMot	2.168	1.579	1.37	0.188
PreWet	1.844	2.758	0.67	0.513

```
S = 2.836       R-Sq = 38.5%     R-Sq(adj) = 16.8%
```

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	6	85.584	14.264	1.77	0.165
Residual Error	17	136.751	8.044		
Total	23	222.335			

We also investigate the effects of interaction effects between the seeding action (SA) and the other explanatory variables. Using stepwise regression leads to a model with SA, the interaction between SA and SC, and time.

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	6.27308	1.04889	5.98	<.0001
SA	1	7.81779	3.47088	2.25	0.0357
Time	1	-0.06076	0.02132	-2.85	0.0099
SA*SC	1	-2.18142	0.99308	-2.20	0.0400

The significant estimate of SA indicates that seeding action may be effective. However, the negative interaction SA*SC is difficult to explain; it indicates that the rainfall under cloud seeding decreases with increasing suitability. Also, there are two cases with relatively large Cook's distances (0.38 and 0.56). Omitting these two cases makes the effects of SA and SA*SC insignificant, leaving time (with a negative coefficient) as the only significant variable. In summary, this small data set is not particularly helpful in settling the issue whether cloud seeding is effective.

7.6 The Minitab Best Subset Regression procedure suggests a model with police expenditures (PE), the number of families per 1,000 earning below one half of the median income (IncInequ), the mean number of years of schooling x 10 of the population (Ed), and the number of males aged 14-24 per 1,000 of total state population (Age). Case #29 exhibits the largest leverage (0.471):

```
The regression equation is
Crime Rate = - 425 + 1.30 PE + 0.641 IncInequ + 1.66 Ed + 0.760 Age
```

Predictor	Coef	SE Coef	T	P
Constant	-424.92	85.85	-4.95	0.000
PE	1.2980	0.1438	9.03	0.000
IncInequ	0.6409	0.1527	4.20	0.000
Ed	1.6605	0.4580	3.63	0.001
Age	0.7602	0.3442	2.21	0.033

S = 22.15   R-Sq = 70.0%   R-Sq(adj) = 67.2%

```
Analysis of Variance

Source          DF       SS        MS       F       P
Regression       4    48196     12049   24.55   0.000
Residual Error  42    20614       491
Total           46    68809
```

7.7

$\hat{\mu} = -5.0359 + 0.0671\text{AirFlow} + 0.1295\text{CoolTemp}$; $R^2 = 0.909$; $C_p = 2.9$.

Last case (AirFlow = 70; CoolTemp = 20; StackLoss = 1.5) is an influential observation and should be scrutinized. Without this case:

$\hat{\mu} = -5.1076 + 0.0863\text{AirFlow} + 0.0803\text{CoolTemp}$; $R^2 = 0.946$

7.8

Stepwise regression ($\alpha_{\text{drop}} = \alpha_{\text{enter}} = 0.15$):

$\hat{\mu} = -62.60 + 7.427\%\text{ASurf} + 6.828\%\text{ABase} - 5.2685\text{Run}$;

$R^2 = 0.724$; $R^2_{\text{adj}} = 0.693$; $C_p = 1.3$.

Similar model: $\hat{\mu} = -23.00 + 5.975\%\text{ASurf} - 5.4058\text{Run}$;

$R^2 = 0.695$; $R^2_{\text{adj}} = 0.673$; $C_p = 1.9$.

Cases 13 and 15 with large Cook's influence. Second set of runs with considerably smaller change in rut depth.

7.9 Case 89 with age =197 should be omitted from the data set. The age of this child is very different from the ages of the other children. Results of the remaining n = 108 students are shown below:

Correlation among the variables:

	age	iq	math1	math2	read1
iq	-0.724				
math1	0.095	-0.024			
math2	-0.293	**0.542**	-0.418		
read1	-0.286	**0.474**	0.133	0.176	
read2	-0.071	-0.006	0.380	-0.357	0.314

Math problem solving and reading speed are positively correlated with IQ; IQ and age are correlated. Since we don't really know how students were selected into this study it is unclear what to make of this strong negative correlation between age and IQ.

Strongest results for Math2 (mathematics problem solving). No gender effect, rather weak age effect, but strong relationship with IQ.

The regression equation is
math2 = - 85.6 + 0.319 age + 0.623 iq + 0.33 gender

Predictor	Coef	SE Coef	T	P
Constant	-85.59	30.33	-2.82	0.006
age	0.3186	0.1804	1.77	0.080
iq	0.6230	0.1060	5.88	0.000
gender	0.327	2.575	0.13	0.899

S = 13.24 R-Sq = 31.4% R-Sq(adj) = 29.4%

The regression equation is
math2 = - 85.3 + 0.317 age + 0.623 iq

Predictor	Coef	SE Coef	T	P
Constant	-85.28	30.08	-2.84	0.005
age	0.3173	0.1793	1.77	0.080
iq	0.6227	0.1055	5.90	0.000

S = 13.18 R-Sq = 31.4% R-Sq(adj) = 30.1%

The regression equation is
math2 = - 34.0 + 0.488 iq

Predictor	Coef	SE Coef	T	P
Constant	-33.998	8.170	-4.16	0.000
iq	0.48754	0.07349	6.63	0.000

S = 13.31 R-Sq = 29.3% R-Sq(adj) = 28.7%

Similar results for Read1 (reading speed). No gender effect, rather weak age effect, but strong relationship with IQ.

The regression equation is
read1 = - 14.2 + 0.0921 age + 0.241 iq + 1.19 gender

Predictor	Coef	SE Coef	T	P
Constant	-14.19	15.13	-0.94	0.351
age	0.09211	0.09001	1.02	0.309
iq	0.24059	0.05290	4.55	0.000
gender	1.193	1.285	0.93	0.355

S = 6.609 R-Sq = 23.8% R-Sq(adj) = 21.6%

The regression equation is
read1 = - 13.0 + 0.0875 age + 0.240 iq

Predictor	Coef	SE Coef	T	P
Constant	-13.02	15.07	-0.86	0.390
age	0.08749	0.08981	0.97	0.332
iq	0.23953	0.05285	4.53	0.000

S = 6.604 R-Sq = 23.2% R-Sq(adj) = 21.7%

```
The regression equation is
read1 = 1.12 + 0.202 iq

Predictor        Coef     SE Coef          T        P
Constant        1.118       4.052       0.28    0.783
iq            0.20226     0.03645       5.55    0.000

S = 6.603       R-Sq = 22.5%     R-Sq(adj) = 21.8%
```

7.10

The stepwise procedure in SAS (with Alpha-to-Enter = Alpha-to-Drop = 0.15) includes the proportion of males (%Male), the proportion of males older than 18 (%Male18), the proportion of the population older than 65 (%Pop65), the proportion of the rural (nonmetro) population (%nonMetro) and the proportion of households earning more than 100 thousand dollars %Inc100).

```
The regression equation is
% Votes for Bush = - 717 + 59.6 %Male - 44.3 %Male18 - 0.893 %Pop65
           + 0.149 %NonMetro - 2.04 %Incom100

Predictor        Coef     SE Coef          T        P
Constant       -717.4       156.0      -4.60    0.000
%Male           59.57       12.78       4.66    0.000
%Male18       -44.347       9.994      -4.44    0.000
%Pop65        -0.8928      0.5187      -1.72    0.092
%NonMetro     0.14864     0.04455       3.34    0.002
%Incom100     -2.0361      0.5481      -3.72    0.001

S = 5.531       R-Sq = 74.6%     R-Sq(adj) = 71.7%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         5     4034.86      806.97     26.38    0.000
Residual Error    45     1376.56       30.59
Total             50     5411.42
```

States 2 (Alaska) and 9 (District of Columbia) have large Cook’s distance and leverage values. They have smaller population compared with other states. The proportion of votes for Bush was small (compared to other states) in the District of Columbia, and it was large (compared to other states) in Alaska.

CHAPTER 8

8.1

(a) The Minitab output of various regression models is given below. For each fitted model we list the estimated equation (with estimates, standard errors, and p-values), the coefficient of determination R^2, the root mean square error *s*, and the Durbin-Watson statistic. Minitab flags observations with unusually large standardized residuals ("R") and with unusually large leverage ("X"). The Lockerbie model is simplified by omitting insignificant variables.

Campbell (n = 13):

```
Incumbent Vote = 25.8 + 0.492 Sept Trial + 2.26 GDP Growth

Predictor        Coef     SE Coef          T        P
Constant       25.754       2.953       8.72    0.000
Sept Trial    0.49173     0.05716       8.60    0.000
GDP Growth     2.2571      0.4921       4.59    0.001

S = 1.827       R-Sq = 92.2%     R-Sq(adj) = 90.7%

Unusual Observations
Obs   Sept Tri   Incumbent        Fit      SE Fit    Residual    St Resid
  9       48.7     44.700      44.226       1.570       0.474        0.51 X

X denotes an observation whose X value gives it large influence.

Durbin-Watson statistic = 2.15
```

Abramowitz(n = 13):

```
Incumbent Vote = 45.1 - 4.69 Term + 0.179 Popularity + 2.14 GDP Growth

Predictor        Coef     SE Coef          T        P
Constant       45.059       2.865      15.73    0.000
Term           -4.691       1.337      -3.51    0.007
Popularity    0.17855     0.05567       3.21    0.011
GDP Growth     2.1389      0.6352       3.37    0.008

S = 1.984       R-Sq = 91.7%     R-Sq(adj) = 89.0%

Unusual Observations
Obs       Term   Incumbent        Fit      SE Fit    Residual    St Resid
 13       0.00     54.600      58.480       0.929      -3.880       -2.21R

R denotes an observation with a large standardized residual

Durbin-Watson statistic = 1.76
```

Holbrook (n = 13):

```
Incumbent Vote = 17.6 + 0.0998 PresPop + 0.296 PersFin - 4.00 Tenure

Predictor        Coef     SE Coef          T        P
Constant       17.606       3.865       4.56    0.001
PresPop       0.09982     0.04668       2.14    0.061
PersFin       0.29589     0.04112       7.20    0.000
Tenure         -3.995       1.002      -3.99    0.003

S = 1.505       R-Sq = 95.3%     R-Sq(adj) = 93.7%

Durbin-Watson statistic = 2.07
```

Lockerbie (n = 11):

```
The regression equation is
Incumbent Vote = 22.4 + 0.635 Inc1 - 0.184 Inc2 + 1.13 NextYearBetter
           - 1.45 Tenure

Predictor        Coef     SE Coef          T        P
Constant       22.351       7.231       3.09    0.021
Inc1           0.6352      0.5136       1.24    0.262
Inc2          -0.1836      0.4923      -0.37    0.722
NextYear       1.1251      0.2103       5.35    0.002
Tenure        -1.4488      0.2489      -5.82    0.001

S = 1.661       R-Sq = 95.4%     R-Sq(adj) = 92.3%

Durbin-Watson statistic = 1.17


The regression equation is
Incumbent Vote = 21.4 + 0.604 Inc1 + 1.13 NextYearBetter - 1.39 Tenure

Predictor        Coef     SE Coef          T        P
Constant       21.423       6.359       3.37    0.012
Inc1           0.6044      0.4747       1.27    0.244
NextYear       1.1340      0.1956       5.80    0.001
Tenure        -1.3894      0.1793      -7.75    0.000

S = 1.555       R-Sq = 95.3%     R-Sq(adj) = 93.2%

Durbin-Watson statistic = 1.32


The regression equation is
Incumbent Vote = 16.6 + 1.30 NextYearBetter - 1.37 Tenure

Predictor        Coef     SE Coef          T        P
Constant       16.646       5.329       3.12    0.014
NextYear       1.3029      0.1493       8.73    0.000
Tenure        -1.3726      0.1857      -7.39    0.000
```

```
S = 1.615       R-Sq = 94.2%     R-Sq(adj) = 92.7%

Durbin-Watson statistic = 1.26
```

(b) The sample sizes for estimating these models is extremely small (n = 13 and n = 11). Considering the extremely small sample sizes, we can not detect violations of the assumption of independent errors.

(c) The root mean square errors for most fitted models are in the range from 1.5 to 2 percentage points. They are similar to the ones in the Fair and Lewis-Beck/Tien models. The size of the root mean square error implies that the half widths of 95% prediction intervals are at least 3 - 4 percentage points. Incorporating the uncertainty from the estimation and considering that the sample size is very small makes the prediction intervals even wider. Furthermore, the predictions are "within-sample" predictions, which means that the case being predicted is part of the data that are used for estimation. Prediction errors for "out-of-sample" predictions (where the case being predicted is not part of the data used for the estimation) are usually larger; see (d).

(d) Leaving out case *i*, running the regression on the reduced data set, and predicting the response of the case that has been left out using the estimates from the reduced data set, leads to the PRESS residuals $e_{(i)}$ in equation (6.21) of Chapter 6. Equation (6.22) implies that the PRESS residuals can be calculated from the regular residuals and the leverages. That is,

$$e_{(i)} = y_{(i)} - \hat{y}_{(i)} = e_i / (1 - h_{ii})$$

For illustration we have calculated the residuals, leverages and PRESS residuals for the regression model considered by Campbell in the beginning of this exercise. The PRESS residuals are larger than the ordinary residuals. For example, the (out-of-sample) prediction error for 1996 is -3.76.

Year	Incumbent Vote	Sept Trial	GDP Growth	Residuals	Leverage	PRESS
1948	52.32	45.61	0.91	2.08441	0.126153	2.38533
1952	44.59	42.11	0.27	-2.48002	0.166349	-2.97488
1956	57.75	55.91	0.64	3.05900	0.093183	3.37334
1960	49.92	50.54	-0.26	-0.09906	0.134083	-0.11439
1964	61.34	69.15	0.81	-0.24520	0.361195	-0.38384
1968	49.60	41.89	1.63	-0.43144	0.280740	-0.59984
1972	61.79	62.89	1.73	1.20653	0.235919	1.57906
1976	48.95	40.00	1.17	0.88618	0.257420	1.19338
1980	44.70	48.72	-2.43	0.47371	0.738021	1.80821
1984	59.17	60.22	1.79	-0.23597	0.203862	-0.29640
1988	53.90	54.44	0.79	-0.40671	0.083538	-0.44379

1992	46.55	41.94	0.35	-0.61699	0.168430	-0.74195
1996	54.74	60.67	1.04	-3.19446	0.151107	-3.76308

(e) The four prediction models studied in this exercise are no better and no worse than the models by Fair and Lewis-Beck/Tien. While they give us some indication about the winner of presidential elections, their large uncertainty makes them only useful in the rather uninteresting situation when there is little doubt about the winner of the election.

8.2

Part 1(a): Modeling the height and the weight at referral (HeightR, WeightR) as a function of age at referral (AgeR)

Models with a linear component of Age provide an adequate representation of the relationships. Addition of Age**2 is not necessary. The models lead to an R-square of about 60 percent for height, and 45 percent for weight. Height at referral is easier to predict than weight. Birth weight is marginally significant (estimate 2.26, with p-value 0.064). Addition of birth weight to the regression of weight at referral on age at referral increases the R-square from 45.9 to 48.3 percent. Each extra pound at birth increases the weight at referral by 2.26 pounds. Average weight at referral is 73 pounds, with standard deviation 20 pounds.

Regression Analysis: HeightR versus AgeR, AgeR2**

```
The regression equation is
HeightR = 19.1 + 0.452 AgeR - 0.00120 AgeR**2

77 cases used 16 cases contain missing values

Predictor        Coef     SE Coef        T       P
Constant       19.095       9.434     2.02   0.047
AgeR           0.4523      0.1700     2.66   0.010
AgeR**2    -0.0012036   0.0007501    -1.60   0.113

S = 2.999       R-Sq = 60.4%     R-Sq(adj) = 59.3%
```

Regression Analysis: HeightR versus AgeR

```
The regression equation is
HeightR = 33.9 + 0.181 AgeR

77 cases used 16 cases contain missing values

Predictor        Coef     SE Coef        T       P
Constant       33.912       1.949    17.40   0.000
AgeR          0.18088     0.01741    10.39   0.000

S = 3.030       R-Sq = 59.0%     R-Sq(adj) = 58.5%
```

Regression Analysis: WeightR versus AgeR, AgeR2**

```
The regression equation is
WeightR = - 0.9 + 0.656 AgeR + 0.00009 AgeR**2

80 cases used 13 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant        -0.94       46.45      -0.02    0.984
AgeR           0.6555      0.8387       0.78    0.437
AgeR**2      0.000094    0.003704       0.03    0.980

S = 15.09       R-Sq = 45.9%     R-Sq(adj) = 44.5%
```

Note: Because of the multicollinearity between AgeR and AgeR**2, both regression coefficients are (partially) insignificant. However, this does not imply that both can be omitted from the model at the same time. The results of the model given below show that AgeR is significant if it is the only variable in the model.

Regression Analysis: WeightR versus AgeR

```
The regression equation is
WeightR = - 2.09 + 0.677 AgeR

80 cases used 13 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -2.090       9.341      -0.22    0.824
AgeR          0.67658     0.08321       8.13    0.000

S = 14.99       R-Sq = 45.9%     R-Sq(adj) = 45.2%
```

Regression Analysis: WeightR versus AgeR, BirthWeight

```
The regression equation is
WeightR = - 16.1 + 0.653 AgeR + 2.26 BirthWeight

80 cases used 13 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -16.15       11.85      -1.36    0.177
AgeR          0.65326     0.08282       7.89    0.000
BirthWeight     2.259       1.202       1.88    0.064

S = 14.75       R-Sq = 48.3%     R-Sq(adj) = 46.9%
```

<u>Part 1(b):</u> Modeling the height and the weight at follow-up (HeightF, WeightF) as a function of age at follow-up (AgeF)
Similar conclusions as in 1(a). Models with a linear component of Age provide an adequate representation of the relationships. Addition of Age**2 is not needed. The

models lead to an R-square of about 40 percent for both height and weight. Birth weight is significant (estimate 4.97 with p-value 0.01). Each extra pound at birth increases the weight at follow-up by 5 pounds. Average weight at follow-up is 124 pounds, with standard deviation 32 pounds.

Regression Analysis: HeightF versus AgeF, AgeF2**

```
The regression equation is
HeightF = 10.0 + 0.458 AgeF - 0.00080 AgeF**2

81 cases used 12 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant        10.02       34.71       0.29    0.774
AgeF           0.4581      0.3937       1.16    0.248
AgeF**2     -0.000795    0.001106      -0.72    0.474

S = 4.115       R-Sq = 41.8%     R-Sq(adj) = 40.3%
```

Regression Analysis: HeightF versus AgeF

```
The regression equation is
HeightF = 34.8 + 0.176 AgeF

81 cases used 12 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       34.801       4.090       8.51    0.000
AgeF          0.17553     0.02347       7.48    0.000

S = 4.103       R-Sq = 41.4%     R-Sq(adj) = 40.7%
```

Regression Analysis: WeightF versus AgeF, AgeF2**

```
The regression equation is
WeightF = - 158 + 2.23 AgeF - 0.00339 AgeF**2

85 cases used 8 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -158.2       206.4      -0.77    0.445
AgeF            2.227       2.349       0.95    0.346
AgeF**2     -0.003387    0.006620      -0.51    0.610

S = 25.24       R-Sq = 39.9%     R-Sq(adj) = 38.5%
```

Regression Analysis: WeightF versus AgeF

```
The regression equation is
WeightF = - 53.4 + 1.03 AgeF
```

```
85 cases used 8 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -53.37       24.17      -2.21    0.030
AgeF           1.0269      0.1388       7.40    0.000

S = 25.13       R-Sq = 39.7%     R-Sq(adj) = 39.0%
```

Regression Analysis: WeightF versus AgeF, BirthWeight

```
The regression equation is
WeightF = - 82.0 + 0.982 AgeF + 4.97 BirthWeight

85 cases used 8 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -82.04       25.84      -3.18    0.002
AgeF           0.9815      0.1353       7.25    0.000
BirthWeight     4.967       1.910       2.60    0.011

S = 24.30       R-Sq = 44.3%     R-Sq(adj) = 43.0%
```

Part 1(c): Modeling the combined data: HeightCo, WeightCo and AgeCo.

Models with a linear component of AgeCo provide an adequate representation of the relationship between HeightCo and AgeCo. For weight, the addition of the quadratic component AgeCo**2 becomes necessary. The scatter plot of weight against age suggests that the variability increases with the level. The scatter plot of the logarithm of weight against age indicates that the variability is stabilized by this transformation. The residuals from the regression of ln(WeightCo) on AgeCo are unremarkable. No major lack of fit can be detected.

Regression Analysis: HeightCo versus AgeCo, AgeCo2**

```
The regression equation is
HeightCo = 31.3 + 0.221 AgeCo -0.000144 AgeCo**2

158 cases used 28 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       31.334       4.180       7.50    0.000
AgeCo         0.22070     0.06099       3.62    0.000
AgeCo**2   -0.0001437   0.0002121      -0.68    0.499

S = 3.604       R-Sq = 77.7%     R-Sq(adj) = 77.4%
```

Regression Analysis: HeightCo versus AgeCo

```
The regression equation is
HeightCo = 34.1 + 0.180 AgeCo

158 cases used 28 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       34.060       1.136      29.97    0.000
AgeCo        0.179700    0.007717      23.29    0.000

S = 3.597       R-Sq = 77.7%     R-Sq(adj) = 77.5%
```

Regression Analysis: WeightCo versus AgeCo, AgeCo2**

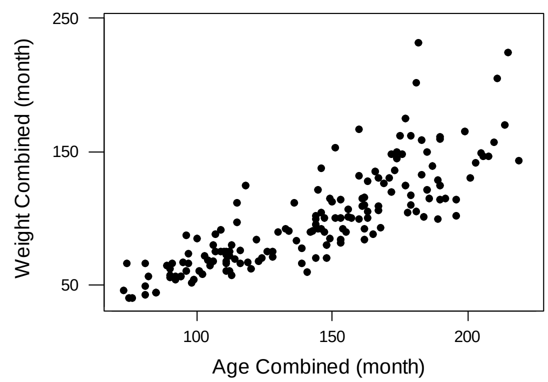
```
The regression equation is
WeightCo = 23.8 + 0.180 AgeCo + 0.00229 AgeCo**2

165 cases used 21 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant        23.78       23.55       1.01    0.314
AgeCo          0.1799      0.3435       0.52    0.601
AgeCo**2     0.002292    0.001195       1.92    0.057

S = 20.84       R-Sq = 69.3%     R-Sq(adj) = 68.9%
```

Regression Analysis: WeightCo versus AgeCo

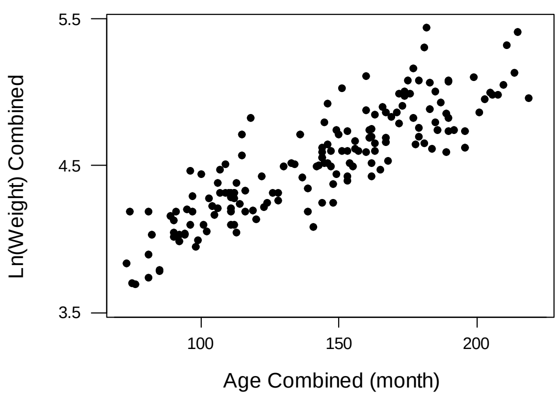
```
The regression equation is
WeightCo = - 19.7 + 0.833 AgeCo

165 cases used 21 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant      -19.659       6.507      -3.02    0.003
AgeCo         0.83340     0.04414      18.88    0.000

S = 21.01       R-Sq = 68.6%     R-Sq(adj) = 68.4%
```

Regression Analysis: ln(WeightCo) versus AgeCo

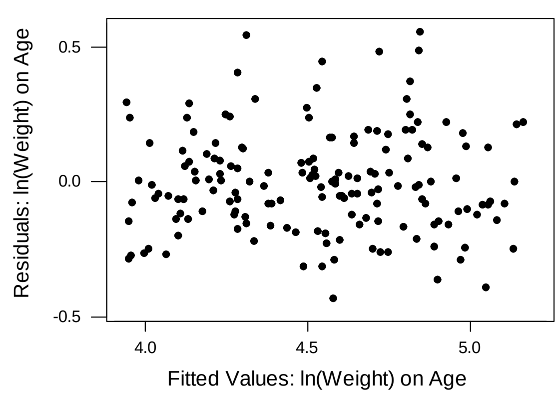
```
The regression equation is
ln(WeightCo) = 3.30 + 0.00864 AgeCo

165 cases used 21 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant      3.29653     0.05685      57.99    0.000
AgeCo       0.0086442   0.0003857      22.41    0.000

S = 0.1836      R-Sq = 75.5%     R-Sq(adj) = 75.4%
```



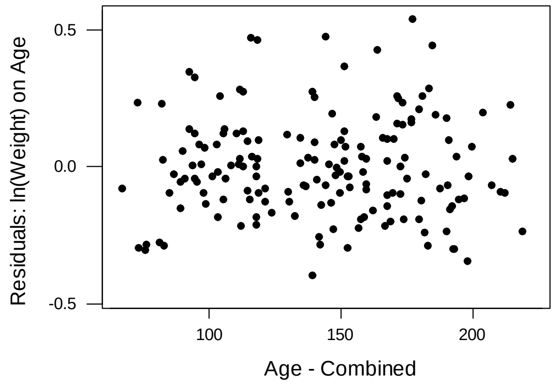
250
150
50
Weight Combined (month)
100
150
200
Age Combined (month)



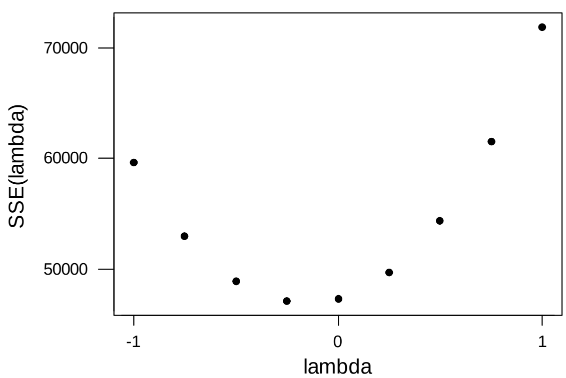
5.5
4.5
3.5
Ln(Weight) Combined
100
150
200
Age Combined (month)



0.5
0.0
-0.5
Residuals: ln(Weight) on Age
4.0
4.5
5.0
Fitted Values: ln(Weight) on Age



The Box-Cox transformation is applied to the response (see Section 6.5 in Chapter 6). For various values of λ we calculate the geometric mean $\bar{y}_g = (\Pi y_i)^{1/n}$ and the transformed response $(y^\lambda - 1)/\lambda(\bar{y}_g)^{\lambda-1}$, regress the transformed response on the explanatory variable AgeCo, and compute the error sum of squares $SSE(\lambda)$. The maximum likelihood estimate of λ minimizes $SSE(\lambda)$. The graph of $SSE(\lambda)$ against λ (given below) shows that the estimate of λ is close to 0. This confirms that the logarithmic transformation is appropriate.



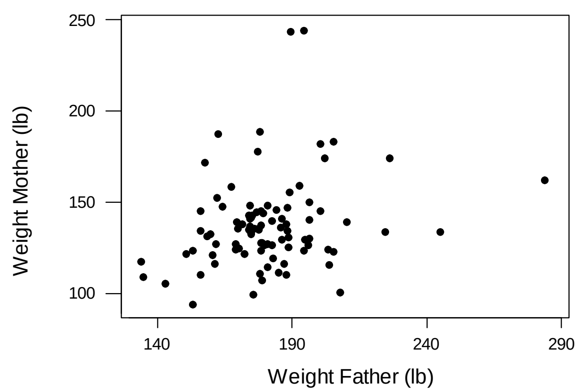
<u>Part 2(a):</u>
A plot of the weight against the height of mothers shows a relationship (correlation coefficient r = 0.336). A correlation coefficient of 0.34 implies that (only) about ten percent of the variability in weight is explained by height (because in simple linear

regression, $R^2 = r^2$). A similar conclusion can be reached for fathers. A plot of the weight against the height of fathers shows a similar-sized correlation (correlation coefficient r = 0.289).

Part 2(b):

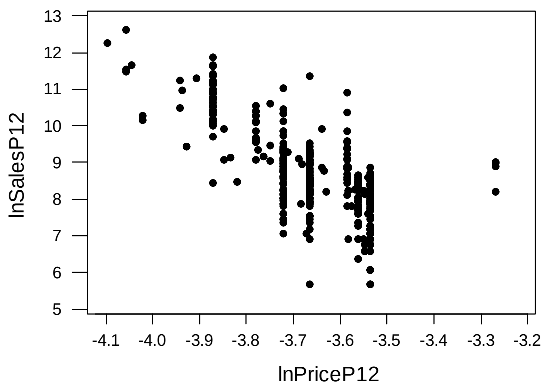
The correlation between the height of mothers and the height of fathers is small (r = 0.077).

The correlation between the weight of mothers and the weight of fathers is larger (0.242). There is some (but rather weak) evidence that both partners tend to be above or below the average weight. The scatter plot shows three unusual cases. In one case the father is quite heavy, while the mother is of average weight. In the other two cases the fathers are of average weight while the mothers have weights much above average. However, the omission of these three cases does not change the correlation coefficient (r = 0.243).



8.3

(a) A scatter plot of (weekly) logarithms of sales of 12-packs of brand P (lnSalesP12) against the logs of their prices (lnPriceP12) shows an expected negative relationship. As prices increase, sales decrease.



Regression Analysis: lnSalesP12 versus lnPriceP6, lnPriceP12, lnPriceP24

```
The regression equation is
lnSalesP12 = - 3.74 + 0.921 lnPriceP6 - 7.24 lnPriceP12 + 2.92
lnPriceP24

384 cases used 15 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -3.740       1.598      -2.34    0.020
lnPriceP6       0.9205      0.1603      5.74    0.000
lnPriceP12     -7.2420      0.3040    -23.82    0.000
lnPriceP24      2.9233      0.2895     10.10    0.000

S = 0.7338      R-Sq = 63.0%     R-Sq(adj) = 62.7%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3      347.92      115.97    215.40    0.000
Residual Error   380      204.59        0.54
Total            383      552.51
```

The results of fitting model M1 confirm a strong negative association with the product's own price. Each one percent increase in the price of 12-packs reduces the sales of 12-packs by 7.2 percent. The parameters in the model represent elasticities as the model regresses log sales on log prices; see Section 6.5.2. The elasticities of price changes in other pack-sizes of the same product (brand P) are positive and considerably smaller. Price increases in 6- and 24-packs increase the sales of 12-packs because buyers chose to buy 12-packs if the prices of other pack-sizes of their desired brand are raised. The response to price changes of 24-packs is stronger than the response to price changes of 6-packs (elasticity 2.92 as compared to 0.92).

The residuals of the regression model are stored in an additional column of the worksheet. Lagging the vector of residuals once and computing the correlation between residuals and lagged residuals results in the lag one autocorrelation of the residuals. Similar operations can be carried out to obtain higher lag autocorrelations. The lagging operation ignores missing observations in the time series. An alternative strategy is to omit all cases with missing entries, run the regression with the reduced data set (the regression estimates are unchanged), and calculate the autocorrelation function and Durbin-Watson test statistic from the resulting residuals. These latter autocorrelations are not exactly the same as the time spacing is changed by omitting missing cases. However, the differences are minor as there are relatively few missing observations. The autocorrelations shown below (calculated with the first approach) are consistently positive. In Chapter 10 we will revise the regression model by adding a time series component that takes account of this persistent positive autocorrelation.

$r_1 = 0.241$ Durbin-Watson $\approx 2(1-0.241) = 1.52$
$r_2 = 0.271$ $r_3 = 0.184$ $r_4 = 0.238$ $r_5 = 0.232$ $r_6 = 0.211$
$r_7 = 0.165$ $r_8 = 0.190$ $r_9 = 0.166$ $r_{10} = 0.114$

(b) Repeating the analysis for the other brand, brand C, leads to similar results. We find a strong negative elasticity for the price at the considered 12-pack size, and weaker and positive elasticities for prices of other pack-sizes. The response to price changes in 24-packs is stronger than the response to price changes in 6-packs (elasticity 2.08, as compared to 0.72).

Regression Analysis: lnSalesC12 versus lnPriceC6, lnPriceC12, lnPriceC24

```
The regression equation is
lnSalesC12 = - 4.32 + 0.718 lnPriceC6 - 6.31 lnPriceC12 + 2.08
lnPriceC24

384 cases used 15 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -4.320       1.494      -2.89    0.004
lnPriceC6       0.7176      0.1486       4.83    0.000
lnPriceC12     -6.3101      0.2606     -24.22    0.000
lnPriceC24      2.0808      0.2732       7.62    0.000

S = 0.7149      R-Sq = 64.4%     R-Sq(adj) = 64.1%


Analysis of Variance

Source            DF          SS          MS         F         P
Regression         3      351.47      117.16    229.22    0.000
Residual Error   380      194.22        0.51
Total            383      545.69
```

(c) The estimation results for model M3 show that the sales of 12-packs of brand P respond negatively to their own price changes (elasticity -6.99), and positively to price changes in other pack-sizes of brand P (elasticities 1.06 and 3.26 for 6- and 12-packs). Sales of 12-packs of brand P are not very sensitive to price changes (at all pack-sizes) of the other competing brand. Customers switch among different pack-sizes, but less among competing brands.

Regression Analysis: lnSalesP12 versus lnPriceP6, lnPriceP12, ...

```
The regression equation is
lnSalesP12 = - 5.10 + 1.06 lnPriceP6 - 6.99 lnPriceP12 + 3.26
lnPriceP24 - 0.178 lnPriceC6 - 0.349 lnPriceC12 - 0.567 lnPriceC24

383 cases used 16 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -5.098       1.738      -2.93    0.004
lnPriceP6       1.0578      0.2136      4.95    0.000
lnPriceP12     -6.9868      0.3606    -19.37    0.000
lnPriceP24      3.2575      0.3467      9.40    0.000
lnPriceC6      -0.1777      0.2034     -0.87    0.383
lnPriceC12     -0.3491      0.3189     -1.09    0.274
lnPriceC24     -0.5665      0.3391     -1.67    0.096

S = 0.7331      R-Sq = 63.3%     R-Sq(adj) = 62.7%

Analysis of Variance

Source            DF          SS          MS        F        P
Regression         6     348.311      58.052   108.02    0.000
Residual Error   376     202.068       0.537
Total            382     550.379
```

The results for sales of 12-packs of brand C are similar and are not shown.

(d) The estimation results for model M4 confirm that the elasticities have the expected signs. The brand P market share of 12-packs increases with decreasing 12-pack price of brand P, and increasing 12-pack price of brand C. The signs of the two price elasticities (-6.22 and 5.56) are different, but their magnitude is roughly the same. The elasticities for prices at other pack-sizes are smaller; the positive signs for brand P prices reflect a substitution effect for 12-packs when packs at other sizes of brand P become more expensive.

Regression Analysis: ln(SalesP12/SalesC12) versus lnPriceP6, lnPriceP12, ...

```
The regression equation is
ln(SalesP12/SalesC12) = 0.33 + 1.48 lnPriceP6 - 6.22 lnPriceP12
 + 2.97 lnPriceP24 - 1.19 lnPriceC6 + 5.56 lnPriceC12 - 2.54lnPriceC24
```

383 cases used 16 cases contain missing values

Predictor	Coef	SE Coef	T	P
Constant	0.331	1.685	0.20	0.844
lnPriceP6	1.4823	0.2071	7.16	0.000
lnPriceP12	-6.2159	0.3498	-17.77	0.000
lnPriceP24	2.9682	0.3362	8.83	0.000
lnPriceC6	-1.1860	0.1972	-6.01	0.000
lnPriceC12	5.5559	0.3092	17.97	0.000
lnPriceC24	-2.5388	0.3288	-7.72	0.000

S = 0.7109   R-Sq = 62.8%   R-Sq(adj) = 62.2%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	6	321.013	53.502	105.85	0.000
Residual Error	376	190.045	0.505		
Total	382	511.058			

Durbin-Watson statistic = 1.93

(e) The results for model M5 show that the coefficient of determination R^2 is hardly changed (0.620 versus 0.628), but the model is easier to interpret. The market share of brand P depends on the relative prices of the two brands. The market share of 12-packs increases with decreasing price ratios of 12-packs. The coefficients for other pack-sizes are considerably smaller and positive, indicating a substitution effect among the various pack-sizes.

Regression Analysis: ln(SalesP12/SalesC12) versus ln(PriceP6/PriceC6), ln(PriceP12/PriceC12), ...

The regression equation is
ln(SalesP12/SalesC12) = 0.126 + 1.27 ln(PriceP6/PriceC6)
           - 5.77 ln(PriceP12/PriceC12) + 2.70 ln(PriceP24/PriceC24)

383 cases used 16 cases contain missing values

Predictor	Coef	SE Coef	T	P
Constant	0.1258	0.0368	3.42	0.001
ln(PriceP6/PriceC6)	1.2657	0.1838	6.89	0.000
ln(PriceP12/PriceC12)	-5.7696	0.2868	-20.12	0.000
ln(PriceP24/PriceC24)	2.6998	0.2937	9.19	0.000

S = 0.7160   R-Sq = 62.0%   R-Sq(adj) = 61.7%

Durbin-Watson statistic = 1.93

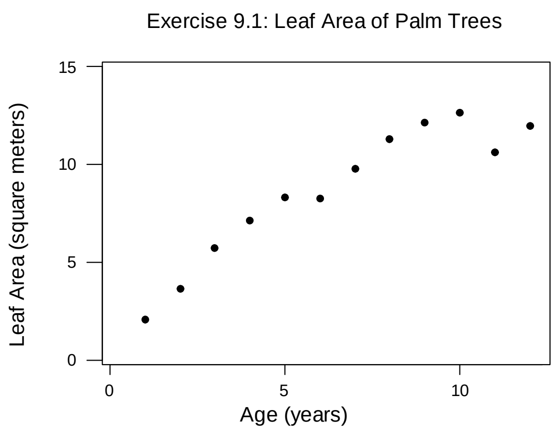
CHAPTER 9

A note on computing with SPSS (Version 11.5):

We use the SPSS software to fit the nonlinear regression models of Chapter 9. SPSS works like a spreadsheet program. We enter the data into the various columns of the spreadsheet and use the tabs: Analyze > Regression > Nonlinear. We write out the model equation and specify initial parameter values. We can save the fitted values and the residuals (also the derivatives of the objective function) into columns of the worksheet.

Several options for the iterative nonlinear estimation procedure are available. In the following examples we have used the Levenberg-Marquardt algorithm. Options for specifying the number of iterations and various convergence cutoffs are available. See the SPSS on-line help for further discussion and examples.

9.1 A graph of the leaf area against the age of the palm tree is given below.



Note that there is not an abundance of data points to determine the model. The graph indicates that the relationship between leaf area and age is not linear; a quadratic component needs to be added to the model. The estimation results for the quadratic model $y = \beta_0 + \beta_1 \text{Age} + \beta_2 \text{Age}^2 + \varepsilon$ (Minitab output) is shown below. The quadratic coefficient is clearly needed; the estimate of the coefficient for Age**2 is -0.09616, with a significant t-ratio of -4.95.

Regression Analysis: Area (square meters) versus Age, Age2**

The regression equation is
Area (square meters) = - 0.123 + 2.15 Age - 0.0962 Age**2

Predictor	Coef	SE Coef	T	P
Constant	-0.1234	0.7334	-0.17	0.870
Age	2.1496	0.2594	8.29	0.000
Age**2	-0.09616	0.01942	-4.95	0.001

S = 0.7096   R-Sq = 96.6%   R-Sq(adj) = 95.8%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	128.071	64.036	127.19	0.000
Residual Error	9	4.531	0.503		
Total	11	132.603			

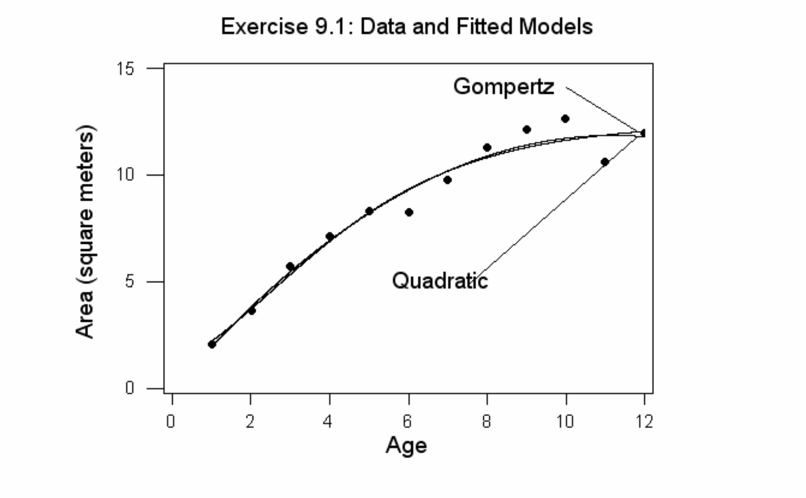
Rasch/Sedlacek use the Gompertz model $y = \mu + \varepsilon = \alpha \exp[-\beta \exp(-\gamma \text{Age})] + \varepsilon$ with parameters $\alpha > 0, \beta > 0, \gamma > 0$. Before fitting this model, we need to determine suitable starting values for the iterative nonlinear parameter estimation. The graph indicates that the saturation level for large values of Age is about 15. Hence a suitable starting value for α is given by 15. For Age = 1, the response is about 2; for Age = 5, the response is roughly 7. The model equation implies $-\beta \exp(-\gamma) = \ln(2/15)$ and $-\beta \exp(-5\gamma) = \ln(7/15)$. This implies $\exp(4\gamma) = [\ln(2/15)]/[\ln(7/15)]$ and $\gamma = \{\ln[\ln(2/15))/\ln(7/15)]\}/4 \approx 0.25$. Finally, $-\beta \exp(-\gamma) = \ln(2/15)$ and $\beta = -\ln(2/15)\exp(\gamma) \approx 2.6$. The starting values $\alpha = 15$, $\beta = 2.6$ and $\gamma = 0.25$ are used in the SPSS nonlinear regression routine. The (SPSS) outcome is given below:

Iteration	Residual SS	A	B	C
1	12.59000092	15.0000000	2.60000000	.250000000
1.1	15.64377972	11.3812687	2.34691685	.336739045
1.2	6.515778841	13.4641276	2.14122037	.271436482
2	6.515778841	13.4641276	2.14122037	.271436482
2.1	6.243186484	12.0109653	2.42204992	.359733910
3	6.243186484	12.0109653	2.42204992	.359733910
3.1	5.136619171	12.4921144	2.50012161	.359000316
4	5.136619171	12.4921144	2.50012161	.359000316
4.1	5.136518308	12.4937910	2.49764737	.358922047
5	5.136518308	12.4937910	2.49764737	.358922047
5.1	5.136518286	12.4936881	2.49773050	.358935226

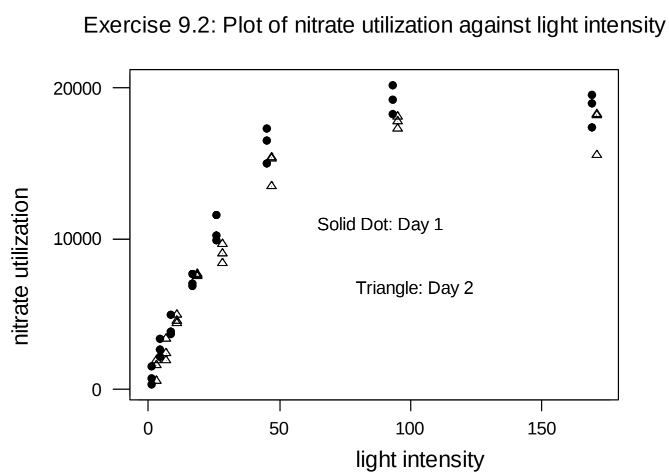
Run stopped after 11 model evaluations and 5 derivative evaluations.
Iterations have been stopped because the relative reduction between
successive residual sums of squares is at most SSCON = 1.000E-08

```
Nonlinear Regression Summary Statistics     Dependent Variable AREA

  Source                 DF  Sum of Squares  Mean Square

  Regression              3     1023.43418      341.14473
  Residual                9        5.13652         .57072
  Uncorrected Total      12     1028.57070
  (Corrected Total)      11      132.60269

  R squared = 1 - Residual SS / Corrected SS =     .96126

                                                  Asymptotic 95 %
                          Asymptotic     Confidence Interval
  Parameter   Estimate    Std. Error     Lower         Upper

  A (α)     12.493688057   .683789772 10.946848127 14.040527986
  B (β)      2.497730497   .440644079  1.500924338  3.494536656
  C (γ)       .358935226   .066769083   .207893067   .509977385

  Asymptotic Correlation Matrix of the Parameter Estimates

                    A         B         C

  A          1.0000    -.4983    -.8306
  B          -.4983    1.0000     .8339
  C          -.8306     .8339    1.0000
```

The estimate of α is 12.5; the estimate of β is 2.5, and the estimate of γ is 0.36. All estimates are statistically significant. There is a fair amount of correlation, especially between the estimates of γ and α (-0.83) and the estimates of γ and β (0.83). The coefficient of determination (0.961) is similar to the R^2 from the quadratic regression. There is little difference between the fits of the quadratic regression (which is linear in the parameters) and the Gompertz model (which is nonlinear in the parameters). Both models lead to similar fitted curves. One difference is that the fitted values for the Gompertz model increase with age to an asymptotic value, whereas the quadratic curve starts to decrease with age after having reached a maximum. However, over the observed age range the two fitted models are virtually indistinguishable.



9.2 A scatter plot of nitrate utilization versus light intensity is shown below. We use solid circles for day 1 observations, and triangles for day 2 observations. Furthermore, we have added some jitter to the light intensity in order to emphasize the differences between the measurements of day 1 and day 2. The day 2 measurements are slightly lower, especially at increasing light intensity.



Michaelis-Menton model: Nitrate utilization reaches an asymptote of about 20,000 for large light intensity. Letting x go to infinity in the model equation

$$\frac{\beta_1 \mathrm{x}}{\beta_2 + \mathrm{x}} = \frac{\beta_1}{1 + (\beta_2 / \mathrm{x})} \approx 20{,}000$$

leads to the starting value $\beta_1 \approx 20{,}000$. Furthermore, the average nitrate utilization at light intensity 2.2 is 1075. Solving the model equation with $\beta_1 = 20{,}000$ leads to the starting value $\beta_2 = 38.7$.

Using these starting values in the SPSS nonlinear regression routine results in the following estimation results:

```
Nonlinear Regression Summary Statistics     Dependent Variable NITRATE

  Source                 DF  Sum of Squares  Mean Square

  Regression              2  6467226758.31  3233613379.15
  Residual               46  96536195.6932  2098612.94985
  Uncorrected Total      48  6563762954.00
  (Corrected Total)      47  2076766799.92

  R squared = 1 - Residual SS / Corrected SS =     .95352

                                          Asymptotic 95 %
                         Asymptotic     Confidence Interval
  Parameter   Estimate   Std. Error     Lower         Upper

  B1        23582.527043 889.35646658 21792.345325 25372.708760
  B2        34.243774004  3.427314571 27.344947587 41.142600421

  Asymptotic Correlation Matrix of the Parameter Estimates

                  B1         B2

  B1          1.0000      .8785
  B2           .8785     1.0000
```

Exponential rise model: Nitrate utilization reaches an asymptote of about 20,000 for large light intensity. Letting x go to infinity in the equation for the exponential rise model leads to the starting value $\beta_1 \approx 20{,}000$. The average nitrate utilization at light intensity 2.2 is 1075. Solving the model equation with $\beta_1 = 20{,}000$ leads to the starting value $\beta_2 = -\frac{1}{2.2}\ln\left[1 - \frac{1075}{20{,}000}\right] = 0.025$. Using these starting values in the SPSS nonlinear regression program results in the estimation results:

```
Nonlinear Regression Summary Statistics     Dependent Variable NITRATE

  Source                 DF  Sum of Squares  Mean Square

  Regression              2  6504309173.87  3252154586.93
  Residual               46  59453780.1310  1292473.48111
  Uncorrected Total      48  6563762954.00
  (Corrected Total)      47  2076766799.92
```

```
  R squared = 1 - Residual SS / Corrected SS =     .97137

                                         Asymptotic 95 %
                         Asymptotic     Confidence Interval
  Parameter   Estimate    Std. Error     Lower         Upper

  B1        19014.305975 398.04663684 18213.079652 19815.532299
  B2          .030021624   .001629334   .026741945   .033301303

  Asymptotic Correlation Matrix of the Parameter Estimates

                  B1        B2

  B1          1.0000    -.7393
  B2          -.7393    1.0000
```

Quadratic Michaelis-Menton model: Starting with $\beta_1 = 20{,}000$ and $\beta_2 = 38.7$ (from the earlier Michaelis-Menton model) and a small value for the parameter in the quadratic component ($\beta_3 = 0.1$) leads to the following results:

```
Nonlinear Regression Summary Statistics     Dependent Variable NITRATE

  Source                 DF  Sum of Squares  Mean Square

  Regression              3  6520540397.33  2173513465.78
  Residual               45  43222556.6654   960501.25923
  Uncorrected Total      48  6563762954.00
  (Corrected Total)      47  2076766799.92

  R squared = 1 - Residual SS / Corrected SS =     .97919

                                         Asymptotic 95 %
                         Asymptotic     Confidence Interval
  Parameter   Estimate    Std. Error     Lower         Upper

  B1        66769.034924 17585.504714 31350.010284 102188.05956
  B2        137.82679758 43.735712594 49.738550634 225.91504453
  B3          .011281055   .004496402   .002224837   .020337274


 Asymptotic Correlation Matrix of the Parameter Estimates

                  B1        B2        B3

  B1          1.0000     .9964     .9941
  B2           .9964    1.0000     .9856
  B3           .9941     .9856    1.0000
```

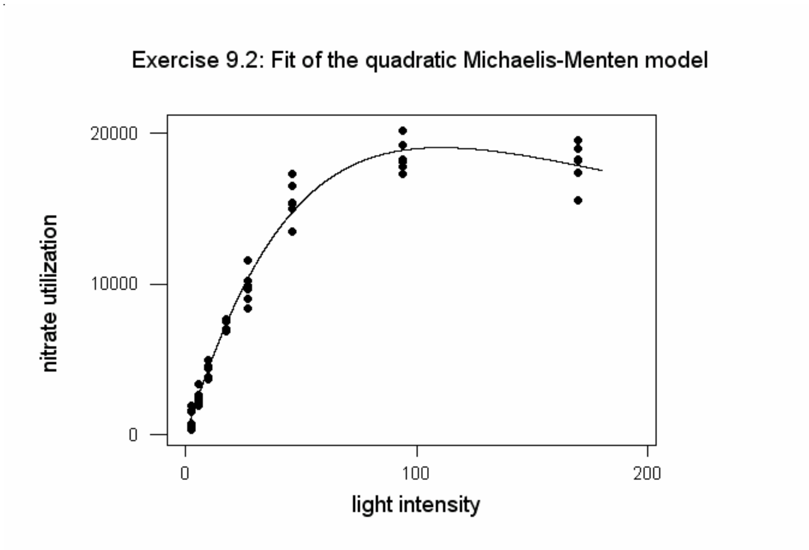
Modified exponential rise model: Using $\beta_1 = 20{,}000$ and $\beta_2 = 0.025$ from the earlier exponential rise model and a small value for $\beta_3 = 0.01$ leads to the following results:

```
Nonlinear Regression Summary Statistics     Dependent Variable NITRATE

  Source                 DF  Sum of Squares  Mean Square

  Regression              3  6519117089.28  2173039029.76
  Residual               45  44645864.7154   992130.32701
  Uncorrected Total      48  6563762954.00
  (Corrected Total)      47  2076766799.92

  R squared = 1 - Residual SS / Corrected SS =     .97850

                                               Asymptotic 95 %
                          Asymptotic     Confidence Interval
  Parameter   Estimate    Std. Error     Lower         Upper

  B1        33551.454219 9502.1687711 14413.103896 52689.804543
  B2          .018534079   .003572151   .011339397   .025728761
  B3          .003221159   .001338559   .000525162   .005917155

  Asymptotic Correlation Matrix of the Parameter Estimates

                  B1         B2         B3

  B1          1.0000    -.9898      .9948
  B2          -.9898    1.0000    -.9741
  B3           .9948    -.9741    1.0000
```

All four models lead to large R^2. The Michaelis-Menton and its quadratic extension lead to R^2 of 0.954 and 0.979, respectively. Carrying out an F-test for the significance of the quadratic component in the Michaelis-Menton model leads to the F-statistic $F = [96{,}536{,}195 - 43{,}222{,}556]/[43{,}222{,}556/45] = 55.5$, which is highly significant. This shows that the quadratic extension represents a significant improvement.

Similarly, the exponential rise model and its extension lead to R^2 of 0.971 and 0.979, respectively. The F-test for the significance of the extra component in the exponential rise model leads to the F-statistic
$F = [59{,}453{,}780 - 44{,}645{,}864]/[44{,}645{,}864/45] = 14.9$, which is also highly significant.

The extensions are beneficial. The modified Michaelis-Menton and the modified exponential rise models perform similarly. In the following graph we show the fit of the quadratic Michaelis-Menton model; the fitted values of the modified exponential rise model are virtually indistinguishable.



<u>Standard Michaelis-Menton model with an indicator for the change of day:</u> The final parameter estimates in the previous Michaelis-Menton model, $\hat{\beta}_1 = 23{,}500$ and $\hat{\beta}_2 = 34.2$, are taken as the starting values in the iterative nonlinear estimation. Small values for the day indicator $\alpha_1 = -1000, \alpha_2 = -1$ are used as the starting values for the two additional parameters. The estimation results are given below:

```
Nonlinear Regression Summary Statistics     Dependent Variable NITRATE

  Source                 DF  Sum of Squares  Mean Square

  Regression              4  6477253424.57  1619313356.14
  Residual               44  86509529.4274  1966125.66881
  Uncorrected Total      48  6563762954.00
  (Corrected Total)      47  2076766799.92

  R squared = 1 - Residual SS / Corrected SS =     .95834

                                             Asymptotic 95 %
                          Asymptotic     Confidence Interval
  Parameter   Estimate    Std. Error     Lower         Upper

  B1        24743.334444 1241.1211323 22242.019158 27244.649730
  B2        35.275400267  4.656586052 25.890667730 44.660132803
  A1        -2328.743446 1720.3472191 -5795.875448 1138.3885567
  A2        -2.172827290  6.626226364 -15.52710905 11.181454466
```

```
Asymptotic Correlation Matrix of the Parameter Estimates

                  B1        B2        A1        A2

  B1          1.0000     .8810    -.7214    -.6191
  B2           .8810    1.0000    -.6356    -.7028
  A1          -.7214    -.6356    1.0000     .8781
  A2          -.6191    -.7028     .8781    1.0000
```

The F-statistic for testing the null hypothesis $\alpha_1 = \alpha_2 = 0$ is $F = [(96{,}536{,}195 - 86{,}509{,}529)/2]/[86{,}509{,}529/44] = 2.55$. The probability value from the F(2,44) distribution is $P[F(2,44) \geq 2.55] = 1 - 0.91 = 0.09$. Hence there is only weak evidence for including a day effect. The individual confidence intervals for α_1 and α_2 cover zero, which makes the individual interpretation of the two day-effect parameters difficult. These estimates are also quite correlated.

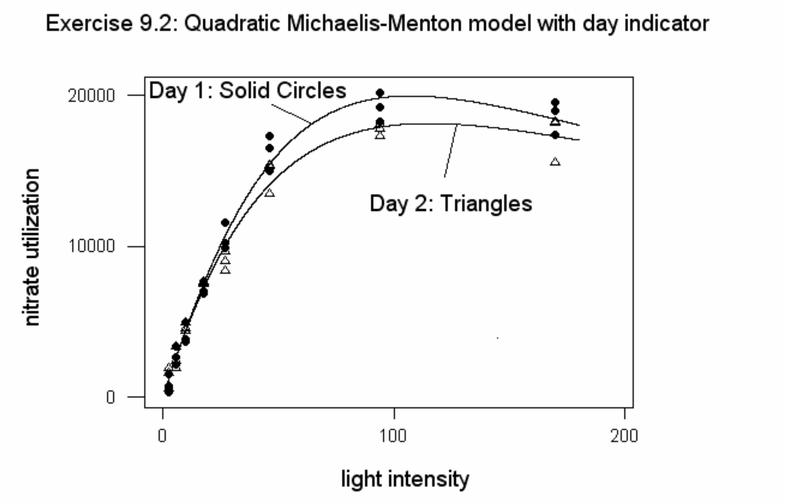
<u>Quadratic Michaelis-Menton model with an indicator for the change of day:</u> The final values from the earlier quadratic model $\hat{\beta}_1 = 66{,}700, \hat{\beta}_2 = 138, \hat{\beta}_3 = 0.01$ and small values for the three parameters associated with the day indicators, $\alpha_1 = -2000, \alpha_2 = -2, \alpha_3 = 0.001$, are used as the starting values in the iterative nonlinear SPSS estimation. The estimation results are given below:

```
Run stopped after 10 model evaluations and 5 derivative evaluations.
Iterations have been stopped because the relative reduction between
successive residual sums of squares is at most SSCON = 1.000E-08


Nonlinear Regression Summary Statistics     Dependent Variable NITRATE

  Source                 DF  Sum of Squares  Mean Square

  Regression              6  6531740362.05  1088623393.67
  Residual               42  32022591.9535   762442.66556
  Uncorrected Total      48  6563762954.00
  (Corrected Total)      47  2076766799.92

 R squared = 1 - Residual SS / Corrected SS =     .98458

                                            Asymptotic 95 %
                              Asymptotic     Confidence Interval
  Parameter   Estimate        Std. Error     Lower         Upper

  B1        89797.916970 37540.345749 14038.432096 165557.40184
  B2        186.61862445 89.984553967  5.022442558 368.21480635
  A1        -38897.78690 39982.748874 -119586.2408 41790.667033
  A2        -83.09078151 96.727346453 -278.2944695 112.11290653
  B3          .016252421   .009207288  -.002328638   .034833481
  A3         -.008449660   .009916211  -.028461384   .011562065
```

```
Asymptotic Correlation Matrix of the Parameter Estimates

              B1          B2          A1          A2          B3          A3

B1          1.0000       .9978      -.9389      -.9283       .9965      -.9252
B2           .9978      1.0000      -.9369      -.9303       .9913      -.9204
A1          -.9389      -.9369      1.0000       .9971      -.9356       .9953
A2          -.9283      -.9303       .9971      1.0000      -.9222       .9894
B3           .9965       .9913      -.9356      -.9222      1.0000      -.9285
A3          -.9252      -.9204       .9953       .9894      -.9285      1.0000
```

The F-statistic for testing the null hypothesis $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is given by $F = [(43{,}222{,}556 - 32{,}022{,}591)/3]/[32{,}022{,}591/42] = 4.90$. The probability value from the F(3,42) distribution is $P[F(3,42) \geq 4.90] = 1 - 0.995 = 0.005$, showing that the indicators for the day effect help explain the variation. Individually the three parameters are statistically insignificant and also highly correlated. This makes an individual interpretation of the estimates difficult.

The graph shown below compares the quadratic Michaelis-Menton model with and without the day indicator. The graph shows that the quadratic Michaelis-Menton model with a day indicator is capable of expressing the day differences.



9.3

Model 1: The logarithmic transformation of the first model leads to

$$\ln(y) = \ln(\beta_0) + \beta_1 \ln(x_1) + \beta_2 \ln(x_2) + \ln(\varepsilon)$$

A standard multiple linear regression of $\ln(y)$ on $\ln(x_1)$ and $\ln(x_2)$ leads to the estimates of $\alpha = \ln(\beta_0), \beta_1$, and β_2 The estimate of β_0 can be obtained

from $\beta_0 = \exp(\alpha)$. When carrying out the regression with the transformed variables we need to assume that the error $\ln(\varepsilon)$ satisfies the standard regression assumptions.
Model 2: Taking the reciprocal of the response in the second model leads to

$$1/y = \beta_0 + \beta_1 x + \varepsilon$$

A simple linear regression of (1/y) on x_1 leads to the estimates of β_0, β_1.
Model 3: The reciprocal of the response and a subsequent logarithmic transformation leads to the model

$$\ln[(1/y) - 1] = \beta_0 + \beta_1 x_1 + \ln(\varepsilon)$$

A simple linear regression of $\ln[(1/y) - 1]$ on x_1 leads to the estimates of β_0, β_1. We need to assume that the error $\ln(\varepsilon)$ satisfies the standard regression assumptions.

9.4 Search the literature.

CHAPTER 10

A note on computing in time series situations

The **Minitab** software is used here for calculating the autocorrelation function of time series observations and for fitting the autoregressive integrated moving average (ARIMA) models in Chapter 10. The class of ARIMA models includes the autoregressive, random walk, and noisy random walk models discussed in Chapter 10. The Minitab ARIMA routine also facilitates the computation of the predictions and prediction intervals.

Combined regression time series models can be estimated within the **SCA** software or within the econometrics software **EVIEWS**. Contact information for these two software providers are:

- SCA: Scientific Computing Associates Corp.,1410 N. Harlem Avenue, River Forest, IL 60305. www.scausa.com.
- EVIEWS: QMS (Quantitative Micro Software), 4521 Campus Drive, Irvine, CA, 92612. www.eviews.com

For SCA one needs to construct a text file macro which is then executed by the software. The output can be saved into a file. Here we list the text file macro for Exercise 10.13.

```
==MACRO
Input variables are year quarter FTEShare Car FTEComm.
      1952  3     112.7 105761     96.21
      1952  4     115.0 121874     93.74
      1953  1     121.4 126260     91.37
      …
      …
      1967  2     343.1 393808     79.90
      1967  3     360.8 375968     78.70
      1967  4     397.8 381692     81.50
end
print variables are year quarter FTEShare Car FTEComm.
Utsmodel name is m1. @
Model is FTEShare((1-B)) = (w1*B**6)Car((1-B)) @
+ (w2*B**7)FTEComm((1-B)) + (1-theta*B)noise.
```

Model m1 considers the differences of the response and the regressor variables. The regression model relates the differences of the response to the differences of Car (with lag 6) and the differences of FTECom (with lag 7). A first order moving average model is taken as the error model.

```
Uestim m1. Method is EXACT. Hold residuals(resid1).
```

```
Acf variable is resid1.
Utsmodel name is m2. @
Model is FTEShare((1-B)) = (w1*B**6)Car((1-B)) @
+ (w2*B**7)FTEComm((1-B)) + 1/(1-phi*B)noise.
```

Model m2 considers differences of the response and the regressor variables. A first order autoregressive model is used as the error model.

```
Uestim m2. Method is EXACT. Hold residuals(resid2).
Acf variable is resid2.
RETURN
```

Many options are available within SCA. See the SCA on-line help for further discussion and examples.

A short primer on the backshift operator

The <u>backshift operator</u> B simplifies the notation of time series models. When applied to a time series y_t, the backshift operator shifts the time index by one unit. That is,

$$By_t = y_{t-1}, B^2 y_t = y_{t-2}, B^3 y_t = y_{t-3}, \text{ and so on.}$$

Similarly,

$$Bx_t = x_{t-1}, B^2 x_t = x_{t-2}, B^3 x_t = x_{t-3}, \text{ and so on.}$$

<u>First differences</u> of a time series can be written as $y_t - y_{t-1} = y_t - By_t = (1-B)y_t$.

<u>Second differences</u> (the difference of differences) as

$$y_t - y_{t-1} - (y_{t-1} - y_{t-2}) = (1-B)y_t - (1-B)y_{t-1} = (1-B)(y_t - y_{t-1}) = (1-B)^2 y_t$$

The first order <u>moving average model</u> can be written as

$$\varepsilon_t = a_t - \theta a_{t-1} \quad \text{or} \quad \varepsilon_t = (1-\theta B)a_t .$$

The first order <u>autoregressive model</u> can be written as

$$\varepsilon_t = \phi\varepsilon_{t-1} + a_t \quad \text{or} \quad \varepsilon_t - \phi B\varepsilon_t = a_t \quad \text{or} \quad (1-\phi B)\varepsilon_t = a_t .$$

We can also write it as

$$\varepsilon_t = \frac{1}{1-\phi B} a_t = (1 + \phi B + \phi^2 B^2 + ...)a_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + ... \quad .$$

The <u>noisy random walk</u> also known as the ARIMA(0,1,1) model,

$\varepsilon_t - \varepsilon_{t-1} = a_t - \theta a_{t-1}$, can be written as $(1-B)\varepsilon_t = (1-\theta B)a_t$. Or, as $\varepsilon_t = \frac{1-\theta B}{1-B} a_t$.

<u>Regression models with (first-order) moving average errors</u>

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t \quad \text{with} \quad \varepsilon_t = (1-\theta B)a_t$$

can be combined as

$$y_t = \beta_0 + \beta_1 x_t + (1-\theta B)a_t .$$

<u>Regression models with (first-order) autoregressive errors</u>

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t \quad \text{with} \quad (1-\phi B)\varepsilon_t = a_t$$

can be combined as

$$y_t = \beta_0 + \beta_1 x_t + \frac{1}{1-\phi B} a_t .$$

<u>Regression models with noisy random walk errors</u>

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t \quad \text{with} \quad (1-B)\varepsilon_t = (1-\theta B)a_t$$

can be combined as

$$y_t = \beta_0 + \beta_1 x_t + \frac{1-\theta B}{1-B} a_t \,.$$

Alternatively, this model can be written as a regression of differences,

$$(1-B)y_t = \beta_1(1-B)x_t + (1-\theta B)a_t \,;$$

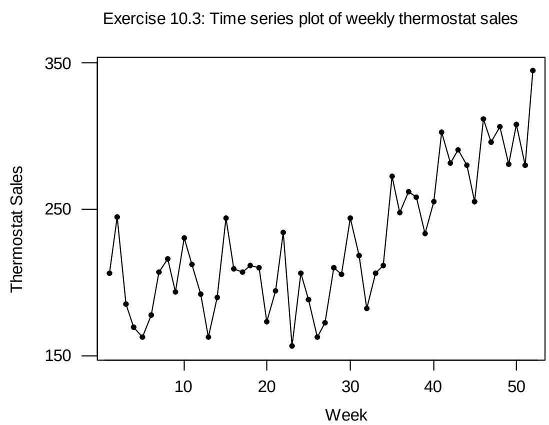
the constant disappears as $(1-B)\beta_0 = \beta_0 - \beta_0 = 0$.

10.1 $\ln(1+r_t) \approx \ln(1) + (r_t - 0)\frac{\partial \ln(1+r_t)}{\partial r_t}\Big|_{r_t=0} = r_t \frac{1}{1+r_t}\Big|_{r_t=0} = r_t$

$\ln(y_t) = \ln[y_{t-1}(1+r_t)] = \ln(y_{t-1}) + \ln(1+r_t) \approx \ln(y_{t-1}) + r_t$

10.2 Write out the matrices L' and L, form the matrix product $L'L$, and show that it equals $(1-\phi^2)V^{-1}$.

10.3 (a) The time series plot of the data is given below.



(b) The MINITAB output of the regression of sales on time, $y_t = \beta_0 + \beta_1 t + \varepsilon_t$, is shown below. The predictions and the 95 percent prediction intervals for the next three observations are calculated from the results in Section 4.3.2.

```
The regression equation is
Sales = 166 + 2.32 Time

Predictor        Coef      SE Coef          T        P
Constant      166.396        8.760      19.00    0.000
Time           2.3247       0.2876       8.08    0.000

S = 31.13       R-Sq = 56.6%     R-Sq(adj) = 55.8%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1       63299       63299     65.32    0.000
Residual Error    50       48451         969
Total             51      111750
```

Prediction for the next period (time = 53):

Prediction: $y_{52}(1) = 166.396 + (2.325)(53) = 289.60$

Prediction interval: 224.65, 354.56

Predictions and prediction intervals can be obtained with the Minitab option in the "regress" command. Alternatively, one can calculate them from the results in Chapter 2,

$$y_{52}(1) \pm 2.0086\sqrt{969}\sqrt{1 + \frac{1}{52} + \frac{(53 - 26.5)^2}{11{,}713}},$$

where 2.0086 is the 97.5th percentile of the t-distribution with 50 degrees of freedom, $26.5 = (1/52)\sum_{t=1}^{52} t$ and $11{,}713 = \sum_{t=1}^{52} (t - 26.5)^2$.

Prediction for two periods ahead (time = 54):

Prediction: $y_{52}(2) = 166.396 + (2.325)(54) = 291.93$

Prediction interval: 226.84, 357.02

$$y_{52}(2) \pm 2.0086\sqrt{969}\sqrt{1 + \frac{1}{52} + \frac{(54 - 26.5)^2}{11{,}713}}$$

Prediction for three periods ahead (time = 55):

Prediction: $y_{52}(3) = 166.396 + (2.325)(55) = 294.25$

Prediction interval: 229.02, 359.49

$$y_{52}(3) \pm 2.0086\sqrt{969}\sqrt{1 + \frac{1}{52} + \frac{(55 - 26.5)^2}{11{,}713}}$$

(c) The Durbin-Watson test statistic is 1.09, and far from the desired value 2. It is not acceptable. There is autocorrelation in the residuals. The first ten autocorrelations are given below (read across):

```
0.405962   0.257128   0.184543   0.192049    0.274287
0.401941   0.283462   0.172746   0.091004   -0.070815
```

The approximate standard error of an autocorrelation is given by $1/\sqrt{52} = 0.14$. Several of the autocorrelations exceed twice the standard error. The autocorrelations tend to be positive with a slow decay, indicating an autocorrelation problem and possible nonstationarity. A regression of sales on time, $y_t = \beta_0 + \beta_1 t + \varepsilon_t$, is definitely not an appropriate forecasting model. The plot of the residuals against time (given below) shows patterns.

(d) The mean of the first differences is 2.7255. This becomes the estimate of β_1 in the model $\Delta y_t = \beta_1 + a_t$. The standard deviation of the first differences is 32.51; this becomes the estimate of σ_a.

The forecasts for the next three observations are:

$$y_{52}(1) = y_{52} + \hat{\beta}_1 = 345 + 2.73 = 347.73$$
$$y_{52}(2) = y_{52}(1) + \hat{\beta}_1 = 347.73 + 2.73 = 350.46$$
$$y_{52}(3) = y_{52}(2) + \hat{\beta}_1 = 350.46 + 2.73 = 353.19$$

The prediction intervals are given by

$y_{52}(1) \pm (1.96)(32.51)$	or	347.73 ± 63.72
$y_{52}(2) \pm (1.96)(\sqrt{2})(32.51)$	or	350.46 ± 90.11
$y_{52}(3) \pm (1.96)(\sqrt{3})(32.51)$	or	353.19 ± 110.37

The first ten autocorrelations of the differenced series are given below (read across):

```
 -0.365082  -0.059187  -0.033625  -0.093252  -0.041308
  0.186040   0.048240  -0.038622   0.034502  -0.169835
```

The lag one autocorrelation exceeds twice its approximate standard error $1/\sqrt{51} = 0.14$. Hence this is not an appropriate forecasting model.

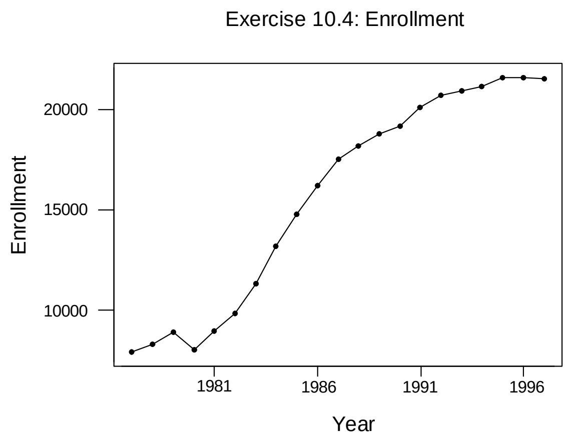
(e) The ARIMA time series procedure in MINITAB is used to estimate the noisy random walk model $\Delta y_t = y_t - y_{t-1} = \beta_1 + a_t - \theta a_{t-1}$. Using the MINITAB ARIMA command, we find

```
Estimates at each iteration
Iteration        SSE     Parameters
    0         49361.5    0.100    2.825
    1         45310.4    0.250    2.496
    2         42249.3    0.400    2.245
    3         39884.7    0.550    2.106
    4         38533.0    0.687    2.124
    5         38448.9    0.717    2.220
    6         38447.7    0.719    2.248
    7         38447.7    0.720    2.251
    8         38447.7    0.720    2.252
Relative change in each estimate less than  0.0010

Final Estimates of Parameters
Type          Coef     SE Coef         T        P
MA   1      0.7198      0.1010      7.13    0.000
Constant     2.252       1.127      2.00    0.051

Differencing: 1 regular difference
Number of observations:  Original series 52, after differencing 51
Residuals:    SS =  38356.2  (backforecasts excluded)
              MS =   782.8  DF = 49

Forecasts from period 52
                             95 Percent Limits
Period      Forecast        Lower         Upper
  53         313.544      258.696      368.392
  54         315.796      258.836      372.756
  55         318.048      259.052      377.045
```

The estimates are $\hat{\beta}_1 = 2.252$ and $\hat{\theta} = 0.72$. The forecasts and the 95 percent prediction intervals are part of the MINITAB output. The first ten autocorrelations of the residuals from this model are shown below. They are small (most of them smaller than their standard error), indicating that we have found an acceptable model.

0.066442	-0.067055	-0.127384	-0.104795	0.045999
0.283976	0.172438	0.061706	-0.010849	-0.161526

10.4 (a) The time series plot shows that the linear trend is not globally stable. The trend shifts over time. Hence a regression on time, $y_t = \beta_0 + \beta_1 t + \varepsilon_t$, is not appropriate. The residuals from the (incorrect) regression on time show (positive) autocorrelations and an unacceptable Durbin-Watson test statistic (0.26) that is considerably smaller than 2.




```
The regression equation is
enrollment = 6527 + 830 time

Predictor        Coef     SE Coef          T        P
Constant       6527.2       599.6      10.89    0.000
time           830.08       47.75      17.38    0.000

S = 1325        R-Sq = 94.1%     R-Sq(adj) = 93.8%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1   530560905   530560905    302.14    0.000
Residual Error    19    33363694     1755984
Total             20   563924600

Durbin-Watson statistic = 0.26

First four autocorrelations of residuals

   0.779040   0.504676   0.191752  -0.088873
```

The predictions and 95% prediction intervals for the next three periods are given below. Because of the residual problems with this model, these predictions should not be used:

For the next period (time = 22): 24,789 and (21,745 to 27,833)
For two periods ahead (time = 23): 25,619 and (22,537 to 28,701)
For three periods ahead (time = 24): 26,449 and (23,327 to 29,571)

(b) The mean of the first differences is 682. This becomes the estimate of β_1. The standard deviation of the first differences is 654; this becomes the estimate of σ_ε .

The forecasts for the next three observations are:

$y_{21}(1) = y_{21} + \hat{\beta}_1 = 21{,}531 + 682 = 22{,}213$

$y_{21}(2) = y_{21}(1) + \hat{\beta}_1 = y_{21} + 2\hat{\beta}_1 = 22{,}213 + 682 = 22{,}895$

$y_{21}(3) = y_{21}(2) + \hat{\beta}_1 = y_{21} + 3\hat{\beta}_1 = 22{,}895 + 682 = 23{,}577$

The prediction intervals are given by

$y_{21}(1) \pm (1.96)(654)$	or	$22{,}213 \pm 1{,}282$
$y_{21}(2) \pm (1.96)(\sqrt{2})(654)$	or	$22{,}895 \pm 1{,}813$
$y_{21}(3) \pm (1.96)(\sqrt{3})(654)$	or	$23{,}577 \pm 2{,}220$

The first four autocorrelations of the differenced series are given below (read across):

```
      0.491156   0.393677   0.114746  -0.074641
```

The lag one autocorrelation exceeds twice its approximate standard error $1/\sqrt{20} = 0.22$.
This forecasting model is not appropriate.

(c) The regression of enrollment on the previous two enrollments (lag one and two), $y_t = \beta_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$, is given below. The Durbin-Watson statistic is much better; it is close to the desired value 2. Also, the autocorrelations of the residuals are small. This model provides an appropriate forecasting method.

```
The regression equation is
enroll = 914 + 1.47 enroll-1 - 0.506 enroll-2

19 cases used 2 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant        914.4       477.6       1.91    0.074
enroll-1       1.4691      0.2147       6.84    0.000
enroll-2      -0.5061      0.2108      -2.40    0.029

S = 575.2       R-Sq = 98.8%     R-Sq(adj) = 98.6%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2   431791259   215895629    652.54    0.000
Residual Error    16     5293676      330855
Total             18   437084935

Durbin-Watson statistic = 2.32

First four autocorrelations of the residuals:
```

```
-0.168526   0.104467  -0.054733  -0.121096
```

The root mean square error from the second-order autoregression, $\sqrt{330{,}855} = 575$, is considerably smaller than the root mean square error of the regression on time in (a), $\sqrt{1{,}755{,}984} = 1{,}325$. The AR(2) model is preferable.

The forecasts can be obtained from:

$$y_{21}(1) = 914 + 1.47 y_{21} - 0.51 y_{20} = 914 + 1.47(21{,}531) - 0.51(21{,}624) = 21{,}536$$
$$y_{21}(2) = 914 + 1.47 y_{21}(1) - 0.51 y_{21} = 914 + 1.47(21{,}536) - 0.51(21{,}531) = 21{,}592$$
$$y_{21}(3) = 914 + 1.47 y_{21}(2) - 0.51 y_{21}(1) = 914 + 1.47(21{,}592) - 0.51(21{,}536) = 21{,}670$$

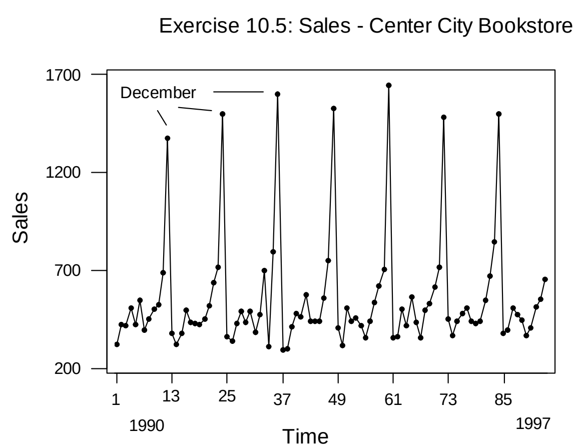
Another reasonable model for these data is the second difference model,

$$(y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) = y_t - 2y_{t-1} + y_{t-2} = \varepsilon_t \quad .$$

It is a special case of the AR(2) model with $\phi_1 = 2$ and $\phi_2 = -1$. The forecasts are

$$y_{21}(1) = 2y_{21} - y_{20} = 2(21{,}531) - 21{,}624 = 21{,}438$$
$$y_{22}(2) = 2y_{21}(1) - y_{21} = 2(21{,}438) - 21{,}531 = 21{,}345$$
$$y_{21}(3) = 2y_{21}(2) - y_{21}(1) = 2(21{,}345) - 21{,}438 = 21{,}252$$

10.5 (a) A time series graph of the observations shows the high sales activity during December months. The question whether or not the data exhibit a trend component is difficult to answer from just the graph alone.

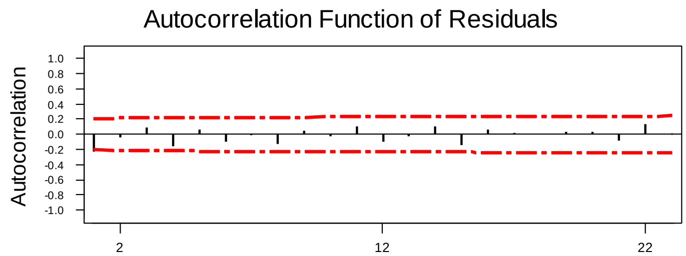


We consider a model with a linear time trend and monthly indicators that account for the seasonal pattern,

$$\text{Sales}_t = \beta_0 + \beta_1 t + \beta_2 \text{IndJan}_t + \beta_3 \text{IndFeb}_t + ... + \beta_{12} \text{IndNov}_t + \varepsilon_t \ .$$

The estimation results indicate a positive trend component. The probability value of the trend coefficient is 0.058, which indicates weak statistical significance. The magnitude of the trend coefficient, a 0.45 EURO increase per month, is of no practical importance. The coefficients of the indicators express differences in average sales for the various months and their base of comparison (December). For example, the value for January (-1,154) indicates that sales in January are on average 1,154 EUROs lower than those in December. The residuals from the regression are still autocorrelated, especially at lag 1; the lag one autocorrelation -0.23 exceeds twice its standard error, $1/\sqrt{94} = 0.10$. The Durbin-Watson statistic (2.45) is larger than 2, reflecting a negative lag one autocorrelation.

```
The regression equation is
Sales = 1500 + 0.449 Time - 1154 IndJan - 1169 IndFeb - 1073 IndMar
           - 1049 IndApr - 1057 IndMay - 1061 IndJun - 1126 IndJul
           - 1062 IndAug - 984 IndSep - 951 IndOct - 776 IndNov

Predictor        Coef     SE Coef          T        P
Constant      1500.18       25.68      58.42    0.000
Time           0.4487      0.2335       1.92    0.058
IndJan       -1154.47       31.66     -36.47    0.000
IndFeb       -1169.04       31.65     -36.94    0.000
IndMar       -1073.12       31.64     -33.91    0.000
IndApr       -1048.82       31.64     -33.15    0.000
IndMay       -1057.27       31.64     -33.42    0.000
IndJun       -1060.96       31.64     -33.54    0.000
IndJul       -1125.91       31.64     -35.59    0.000
IndAug       -1061.74       31.64     -33.56    0.000
IndSep        -983.94       31.64     -31.09    0.000
IndOct        -951.13       31.65     -30.05    0.000
IndNov        -776.41       32.67     -23.76    0.000

S = 61.13       R-Sq = 96.4%     R-Sq(adj) = 95.8%

Analysis of Variance

Source           DF         SS          MS         F        P
Regression       12    7992176      666015    178.25    0.000
Residual Error   81     302649        3736
Total            93    8294825

Durbin-Watson statistic = 2.45
```



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.23	-2.26	5.27	8	-0.13	-1.16	12.15	15	-0.14	-1.22	18.16	22	0.13	1.03	21.81
2	-0.05	-0.47	5.53	9	0.05	0.42	12.39	16	0.06	0.53	18.63	23	-0.00	-0.02	21.81
3	0.09	0.83	6.34	10	-0.03	-0.23	12.47	17	0.01	0.09	18.64				
4	-0.16	-1.45	8.85	11	0.10	0.88	13.59	18	-0.00	-0.04	18.65				
5	0.05	0.46	9.12	12	-0.10	-0.84	14.64	19	0.03	0.25	18.76				
6	-0.11	-0.96	10.31	13	-0.03	-0.25	14.74	20	0.03	0.26	18.88				
7	-0.02	-0.13	10.34	14	0.10	0.83	15.81	21	-0.09	-0.74	19.85				

(b) The autocorrelation function of the residuals from the model in (a) has a spike at lag one. This suggests a first-order moving average model for the errors. Alternatively, one could consider a first-order autoregressive model. We study both error models and show that the results for these two error models are very similar.

$$\text{MA(1): Sales}_t = \beta_0 + \beta_1 t + \beta_2 \text{IndJan}_t + \beta_3 \text{IndFeb}_t + \ldots + \beta_{12} \text{IndNov}_t + (1-\theta B)a_t$$

or,

$$\text{AR(1): Sales}_t = \beta_0 + \beta_1 t + \beta_2 \text{IndJan}_t + \beta_3 \text{IndFeb}_t + \ldots + \beta_{12} \text{IndNov}_t + \frac{1}{1-\phi B} a_t$$

We use SCA to estimate the models (alternatively, one could use Eviews). The results for MA(1) errors are shown first. The residuals from the revised model are uncorrelated. The lag one autocorrelation of the residuals is 0.10, and is well within one standard error. The trend coefficient is small and can be neglected for practical purposes. The seasonal component is very strong.

PARAMETER	LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	CNST		CNST	1	0	NONE	1500.8551	22.7676	65.92
2	B1	TIME	NUM.	1	0	NONE	.4363	.1545	2.82
3	B2	INDJAN	NUM.	1	0	NONE	-1154.6164	32.6848	-35.33
4	B3	INDFEB	NUM.	1	0	NONE	-1169.1776	29.3843	-39.79
5	B4	INDMAR	NUM.	1	0	NONE	-1073.2389	29.3796	-36.53
6	B5	INDAPR	NUM.	1	0	NONE	-1048.9253	29.3758	-35.71
7	B6	INDMAY	NUM.	1	0	NONE	-1057.3616	29.3727	-36.00
8	B7	INDJUN	NUM.	1	0	NONE	-1061.0479	29.3705	-36.13
9	B8	INDJUL	NUM.	1	0	NONE	-1125.9842	29.3691	-38.34
10	B9	INDAUG	NUM.	1	0	NONE	-1061.7955	29.3685	-36.15
11	B10	INDSEP	NUM.	1	0	NONE	-983.9818	29.3687	-33.50
12	B11	INDOCT	NUM.	1	0	NONE	-951.1681	29.3698	-32.39
13	B12	INDNOV	NUM.	1	0	NONE	-778.9498	33.9353	-22.95
14	THETA	SALES	MA	1	1	NONE	.2721	.0995	2.74

```
EFFECTIVE NUMBER OF OBSERVATIONS . .            94
R-SQUARE . . . . . . . . . . . . . .         0.966
RESIDUAL STANDARD ERROR. . . . . . .  0.548881E+02

AUTOCORRELATIONS OF RESIDUALS

  1- 12     .01 -.04  .04 -.15 -.01 -.13 -.09 -.15  .01 -.00  .08 -.09
  ST.E.     .10  .10  .10  .10  .11  .11  .11  .11  .11  .11  .11  .11

 13- 24    -.04  .06 -.12  .04  .03  .01  .04  .03 -.05  .11  .02 -.02
  ST.E.     .11  .11  .11  .11  .11  .11  .11  .11  .11  .11  .12  .12
```

The results for AR(1) errors (shown below) are similar:

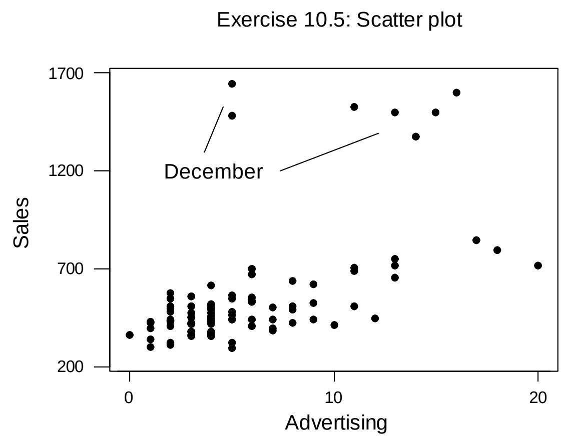
PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	CNST		CNST	1	0	NONE	1501.8372	23.1090	64.99
2	B1	TIME	NUM.	1	0	NONE	.4248	.1740	2.44
3	B2	INDJAN	NUM.	1	0	NONE	-1151.3433	33.8071	-34.06
4	B3	INDFEB	NUM.	1	0	NONE	-1170.5413	28.6817	-40.81
5	B4	INDMAR	NUM.	1	0	NONE	-1073.4953	29.6779	-36.17
6	B5	INDAPR	NUM.	1	0	NONE	-1049.4245	29.4279	-35.66
7	B6	INDMAY	NUM.	1	0	NONE	-1057.7898	29.4837	-35.88
8	B7	INDJUN	NUM.	1	0	NONE	-1061.4784	29.4667	-36.02
9	B8	INDJUL	NUM.	1	0	NONE	-1126.4000	29.4818	-38.21
10	B9	INDAUG	NUM.	1	0	NONE	-1062.2005	29.4317	-36.09
11	B10	INDSEP	NUM.	1	0	NONE	-984.3751	29.6497	-33.20
12	B11	INDOCT	NUM.	1	0	NONE	-951.5499	28.7184	-33.13
13	B12	INDNOV	NUM.	1	0	NONE	-779.1028	33.7655	-23.07
14	PHI	SALES	D-AR	1	1	NONE	-.2369	.1014	-2.34

```
EFFECTIVE NUMBER OF OBSERVATIONS . .            93
R-SQUARE . . . . . . . . . . . . . .         0.965
RESIDUAL STANDARD ERROR. . . . . . .  0.553693E+02

AUTOCORRELATIONS OF RESIDUALS

  1- 12    -.02 -.09  .05 -.14  .01 -.10 -.08 -.14  .01  .01  .07 -.10
  ST.E.     .10  .10  .10  .10  .11  .11  .11  .11  .11  .11  .11  .11

 13- 24    -.04  .06 -.11  .04  .03  .01  .04  .02 -.05  .11  .02 -.03
  ST.E.     .11  .11  .11  .11  .11  .11  .11  .11  .11  .11  .12  .12
```

(c) A scatter plot of sales against advertising is shown below. Adding advertising expenditures to our earlier specification, we consider the model

$$\text{Sales}_t = \beta_0 + \beta_1 t + \beta_2 \text{IndJan}_t + \beta_3 \text{IndFeb}_t + ... + \beta_{12} \text{IndNov}_t + \beta_{13} \text{Adv}_t + (1-\theta B) a_t$$

The estimation results are given below. We find little evidence that advertising provides additional information. This finding can be explained by the fact that advertising is (partially) confounded with the seasonal pattern represented by the seasonal indicators.



The results show that

PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	CNST		CNST	1	0	NONE	1478.9299	33.7541	43.81
2	W1	TIME	NUM.	1	0	NONE	.4167	.1598	2.61
3	W2	INDJAN	NUM.	1	0	NONE	-1140.0198	36.2740	-31.43
4	W3	INDFEB	NUM.	1	0	NONE	-1149.4535	36.8407	-31.20
5	W4	INDMAR	NUM.	1	0	NONE	-1064.2115	30.9192	-34.42
6	W5	INDAPR	NUM.	1	0	NONE	-1034.2650	33.6153	-30.77
7	W6	INDMAY	NUM.	1	0	NONE	-1043.4471	33.1970	-31.43
8	W7	INDJUN	NUM.	1	0	NONE	-1045.3278	34.2302	-30.54
9	W8	INDJUL	NUM.	1	0	NONE	-1112.7960	32.8056	-33.92
10	W9	INDAUG	NUM.	1	0	NONE	-1046.2914	34.1008	-30.68
11	W10	INDSEP	NUM.	1	0	NONE	-971.7751	32.3066	-30.08
12	W11	INDOCT	NUM.	1	0	NONE	-942.0039	30.9651	-30.42
13	W12	INDNOV	NUM.	1	0	NONE	-785.5350	34.2710	-22.92
14	W13	ADV	NUM.	1	0	NONE	2.0413	2.3240	.88
15	THETA	SALES	MA	1	1	NONE	.2522	.0999	2.52

```
EFFECTIVE NUMBER OF OBSERVATIONS . .            94
R-SQUARE . . . . . . . . . . . . . .         0.966
RESIDUAL STANDARD ERROR. . . . . . .  0.546787E+02

AUTOCORRELATIONS OF RESIDUALS

  1- 12     .00 -.03  .05 -.15 -.02 -.13 -.09 -.18  .01  .01  .07 -.09
  ST.E.     .10  .10  .10  .10  .11  .11  .11  .11  .11  .11  .11  .11

 13- 24    -.04  .06 -.11  .04  .03  .01  .05  .03 -.06  .11  .02 -.02
  ST.E.     .11  .11  .11  .11  .11  .11  .11  .11  .11  .12  .12  .12
```

10.6 We generated $\{a_t\}$ and $\{b_t\}$ as independent N(0,1) random variables. The random walks were calculated recursively, starting with $y_1 = a_1$ and $x_1 = b_1$; the first 500 realizations were omitted in order to exclude any effect of the starting values. The results for series of length n = 50 are shown below. In 60 percent of the cases (6 out of 10), the regression slope was significant at the 0.05 level; the average R^2 was 0.14.

Estimate	Std.Error	t-ratio	prob-value	R**2
0.4572	0.2584	1.77	0.083	0.061
-0.15913	0.05370	-2.96	0.005	0.155
-0.47148	0.09568	-4.93	0.000	0.336
0.04544	0.05243	0.87	0.390	0.015
0.0509	0.1119	0.45	0.651	0.004
0.3334	0.1002	3.33	0.002	0.187
-0.4025	0.1223	-3.29	0.002	0.184
0.3952	0.1197	3.30	0.002	0.184
-0.18640	0.08463	-2.20	0.032	0.185
-0.1219	0.1221	-1.00	0.323	0.092

Different random variables were used in the simulation for the series of length n = 100. We find a significant relationship in 50 percent of the cases (5 of 10), even though such a relationship should occur in only 5 percent (significance level) of the cases. The average R^2 was 0.08.

Estimate	Std.Error	t-ratio	prob-value	R**2
0.01853	0.08113	0.23	0.820	0.001
-0.08713	0.04945	-1.76	0.081	0.031
0.46433	0.08420	5.51	0.000	0.237
-0.2079	0.1764	-1.18	0.241	0.014
-0.1564	0.1118	-1.40	0.165	0.020
0.13184	0.03684	3.58	0.001	0.116
0.10329	0.04375	2.36	0.020	0.054
0.55219	0.08888	6.21	0.000	0.283
-0.1201	0.1369	-0.88	0.383	0.008
0.1919	0.1299	1.48	0.143	0.022

These results show the problem of spurious relationships when regressing two independent autocorrelated series.

10.7 (a) Regression results for each of the four products are shown below. The coefficients of determination are larger than 50 percent. For some products one or the other regressor can be omitted. The independence assumption of the errors is violated in the regressions for products 2 and 4. In these cases the Durbin-Watson statistics are considerably smaller than 2, indicating positive lag 1 autocorrelation.

<u>Product 1:</u>

```
The regression equation is
Product1 = 26.7 + 3.87 Chemicals(Index) - 0.097 Industrial Equipment
(Index)

Predictor        Coef     SE Coef          T        P
Constant        26.67       71.45       0.37    0.711
Chemical       3.8689      0.9406       4.11    0.000
Industrial    -0.0970      0.5528      -0.18    0.862

S = 21.79       R-Sq = 50.7%     R-Sq(adj) = 47.7%

Durbin-Watson statistic = 2.17

The regression equation is
Product1 = 27.0 + 3.75 Chemicals(Index)

Predictor        Coef     SE Coef          T        P
Constant        26.97       70.38       0.38    0.704
Chemical       3.7502      0.6438       5.82    0.000

S = 21.47       R-Sq = 50.7%     R-Sq(adj) = 49.2%

Durbin-Watson statistic = 2.18
```

<u>Product 2:</u>

```
The regression equation is
Product2 = - 44.6 + 0.217 Chemicals(Index) + 0.281 Industrial
Equipment (Index)

Predictor        Coef     SE Coef          T        P
Constant       -44.55       11.23      -3.97    0.000
Chemical       0.2171      0.1479       1.47    0.152
Industrial    0.28123     0.08691       3.24    0.003

S = 3.426       R-Sq = 55.7%     R-Sq(adj) = 53.0%

Durbin-Watson statistic = 1.09


The regression equation is
Product2 = - 32.8 + 0.373 Industrial Equipment (Index)

Predictor        Coef     SE Coef          T        P
Constant      -32.828       8.036      -4.09    0.000
Industrial    0.37300     0.06143       6.07    0.000

S = 3.485       R-Sq = 52.8%     R-Sq(adj) = 51.3%

Durbin-Watson statistic = 1.03
```

Product 3:

```
The regression equation is
Product3 = - 315 + 2.06 Chemicals(Index) + 2.69 Industrial Equipment
(Index)

Predictor        Coef     SE Coef          T        P
Constant      -315.02       58.32      -5.40    0.000
Chemical       2.0556      0.7678       2.68    0.012
Industrial     2.6905      0.4513       5.96    0.000

S = 17.79       R-Sq = 81.0%     R-Sq(adj) = 79.8%

Durbin-Watson statistic = 1.51
```

Product 4:

```
The regression equation is
Product4 = - 61.1 + 0.669 Chemicals(Index) + 0.178 Industrial
Equipment (Index)

Predictor        Coef     SE Coef          T        P
Constant       -61.09       16.17      -3.78    0.001
Chemical       0.6695      0.2129       3.14    0.004
Industrial     0.1783      0.1251       1.42    0.164

S = 4.932       R-Sq = 54.3%     R-Sq(adj) = 51.5%

Durbin-Watson statistic = 0.83


The regression equation is
Product4 = - 61.7 + 0.888 Chemicals(Index)

Predictor        Coef     SE Coef          T        P
Constant       -61.65       16.42      -3.76    0.001
Chemical       0.8876      0.1502       5.91    0.000

S = 5.008       R-Sq = 51.4%     R-Sq(adj) = 49.9%

Durbin-Watson statistic = 0.84

Autocorrelations of the residuals
   0.526111   0.545286   0.317164   0.321774   0.213099  -0.025736
```

(b) None of the contemporaneous regressions in (a) are suitable for prediction purposes, as the indexes of future chemical and industrial production are not available. For prediction purposes one must find models that explain current sales as functions of previous values of the regressors.

We use the first four lags of each of the two explanatory variables (we believe that higher lags are probably not justified), and start our model search with the following

eight regressors: $Chem_{t-1}, Chem_{t-2}, Chem_{t-3}, Chem_{t-4}$ and $Ind_{t-1}, Ind_{t-2}, Ind_{t-3}, Ind_{t-4}$. Stepwise regression (see Chapter 7) is used to decide on the significant regressors. The results are shown below. The R-square from these regressions are quite similar to those from the contemporaneous regressions (the R-square of the lag regression for product 1 is lower), and we still have problems with autocorrelation, mostly for product 4. The 95 percent margins for the prediction error are at least $\pm 2s$. For product 2, for example, this amounts to $\pm 2(2.944) \approx \pm 6$. Judging from the past sales history of product 2, this indicates considerable uncertainty.
Lagged values of sales could also be incorporated into the regressions.

Product 1:

```
The regression equation is
Product1 = 296 + 3.26 ChemLag1 - 1.95 ChemLag4

31 cases used 4 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant        295.7       104.1       2.84    0.008
ChemLag1       3.2578      0.9092       3.58    0.001
ChemLag4      -1.9503      0.9061      -2.15    0.040

S = 25.71       R-Sq = 32.0%     R-Sq(adj) = 27.2%

Durbin-Watson statistic = 1.95
```

Product 2:

```
The regression equation is
Product2 = - 33.4 + 0.218 ChemLag1 + 0.600 ChemLag2 - 0.301 IndLag4

31 cases used 4 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant       -33.44       11.56      -2.89    0.007
ChemLag1       0.2183      0.1999       1.09    0.284
ChemLag2       0.5995      0.2120       2.83    0.009
IndLag4      -0.30113     0.07209      -4.18    0.000

S = 2.944       R-Sq = 70.0%     R-Sq(adj) = 66.7%

Durbin-Watson statistic = 1.46
```

Product 3:

```
The regression equation is
Product3 = - 283 + 2.47 ChemLag1 + 2.12 IndLag1

34 cases used 1 cases contain missing values
```

Predictor	Coef	SE Coef	T	P
Constant	-283.45	67.15	-4.22	0.000
ChemLag1	2.4658	0.9234	2.67	0.012
IndLag1	2.1152	0.5621	3.76	0.001

```
S = 20.48       R-Sq = 72.6%     R-Sq(adj) = 70.8%

Durbin-Watson statistic = 1.48
```

Product 4:

```
The regression equation is
Product3 = - 290 + 5.06 ChemLag1

34 cases used 1 cases contain missing values
```

Predictor	Coef	SE Coef	T	P
Constant	-289.87	79.75	-3.63	0.001
ChemLag1	5.0608	0.7296	6.94	0.000

```
S = 24.33       R-Sq = 60.1%     R-Sq(adj) = 58.8%

Durbin-Watson statistic = 1.07
```

10.8 The autocorrelation function of the residuals in the regression model (brand P)

$$\ln \text{SalesP12}_t = \beta_0 + \beta_1 \ln \text{PriceP6}_t + \beta_2 \ln \text{PriceP12}_t + \beta_3 \ln \text{PriceP24}_t + \varepsilon_t$$

is shown below. When calculating the autocorrelations we had to omit a few weeks with missing observations. This affected the spacing of the observations, but this issue is ignored here. The standard error of the autocorrelations is about 0.05. The autocorrelations decay very slowly and indicate nonstationarity. The Durbin-Watson statistic (DW = 1.49) indicates unacceptable positive lag 1 autocorrelation.

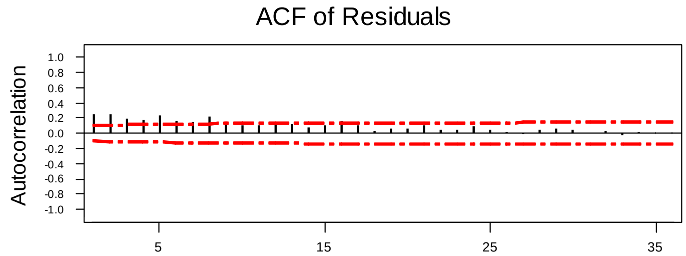
```
The regression equation is
lnsalesP12 = - 3.74 + 0.921 lnPriceP6 - 7.24 lnPriceP12 + 2.92
lnPriceP24
```

Predictor	Coef	SE Coef	T	P
Constant	-3.740	1.598	-2.34	0.020
LnPriceP6	0.9205	0.1603	5.74	0.000
LnPriceP12	-7.2420	0.3040	-23.82	0.000
LnPriceP24	2.9233	0.2895	10.10	0.000

```
S = 0.7338      R-Sq = 63.0%     R-Sq(adj) = 62.7%

Analysis of Variance
```

Source	DF	SS	MS	F	P
Regression	3	347.92	115.97	215.40	0.000
Residual Error	380	204.59	0.54		
Total	383	552.51			

```
Durbin-Watson statistic = 1.49
```



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.24	4.75	22.74	10	0.11	1.58	144.32	19	0.06	0.88	181.90	28	0.04	0.52	194.31
2	0.25	4.68	47.48	11	0.10	1.51	148.42	20	0.06	0.84	183.33	29	0.06	0.77	195.57
3	0.19	3.33	61.44	12	0.11	1.64	153.35	21	0.10	1.35	187.07	30	0.04	0.52	196.17
4	0.18	3.09	74.20	13	0.12	1.74	158.94	22	0.05	0.69	188.07	31	0.01	0.08	196.18
5	0.23	3.88	95.44	14	0.08	1.17	161.52	23	0.04	0.63	188.89	32	0.03	0.45	196.62
6	0.17	2.69	106.44	15	0.10	1.42	165.34	24	0.10	1.33	192.62	33	-0.03	-0.35	196.88
7	0.14	2.22	114.27	16	0.16	2.36	176.08	25	0.05	0.65	193.53	34	0.01	0.13	196.92
8	0.21	3.32	132.27	17	0.10	1.37	179.84	26	0.02	0.24	193.66	35	-0.00	-0.02	196.92
9	0.14	2.10	139.89	18	0.03	0.49	180.32	27	-0.01	-0.18	193.73	36	0.00	0.03	196.92

First differences of the residuals (not shown here) are stationary, with an autocorrelation function that shows a single large spike at lag 1. This suggests the noisy random walk (or ARIMA(0,1,1)), $(1-\mathrm{B})\varepsilon_t = (1-\theta \mathrm{B})a_t$, as an appropriate error model. Combining this with the previous regression leads to the model

$$\ln \mathrm{SalesP12}_t = \beta_1 \ln \mathrm{PriceP6}_t + \beta_2 \ln \mathrm{PriceP12}_t + \beta_3 \ln \mathrm{PriceP24}_t + \frac{1-\theta \mathrm{B}}{1-\mathrm{B}} a_t$$

or,

$$(1-\mathrm{B})[\ln \mathrm{SalesP12}_t] = \beta_1 (1-\mathrm{B})[\ln \mathrm{PriceP6}_t] + \beta_2 (1-\mathrm{B})[\ln \mathrm{PriceP12}_t] + \beta_3 (1-\mathrm{B})[\ln \mathrm{PriceP24}_t] + (1-\theta \mathrm{B})a_t \ .$$

Because of differencing we lose the ability to estimate the intercept β_0. The SCA estimation results are given below:

```
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --     M1
 ----------------------------------------------------------------------
 VARIABLE    TYPE OF    ORIGINAL     DIFFERENCING
           VARIABLE   OR CENTERED
                                            1
 LNSalesP12     RANDOM     ORIGINAL     (1-B  )
                                            1
  LNPriceP6     RANDOM     ORIGINAL     (1-B  )
                                            1
 LNPriceP12     RANDOM     ORIGINAL     (1-B  )
                                            1
 LNPriceP24     RANDOM     ORIGINAL     (1-B  )
 ----------------------------------------------------------------------

  PARAMETER    VARIABLE  NUM./  FACTOR  ORDER  CONS-      VALUE       STD     T
    LABEL        NAME    DENOM.                TRAINT                ERROR  VALUE
```

```
  1    B1  LNPriceP6   NUM.      1      0     NONE     1.2561     .1500   8.37
  2    B2 LNPriceP12   NUM.      1      0     NONE    -6.6402     .3054 -21.74
  3    B3 LNPriceP24   NUM.      1      0     NONE     3.2115     .2677  12.00
  4 THETA              MA        1      1     NONE      .8644     .0250  34.62

 EFFECTIVE NUMBER OF OBSERVATIONS . .           383
 R-SQUARE . . . . . . . . . . . . . .         0.694
 RESIDUAL STANDARD ERROR. . . . . . .  0.663259E+00

AUTOCORRELATIONS OF RESIDUALS

  1- 12    -.00 -.01 -.05 -.03  .05 -.01 -.02  .08  .01 -.04 -.03  .01
  ST.E.     .05  .05  .05  .05  .05  .05  .05  .05  .05  .05  .05  .05

 13- 24     .01 -.02  .01  .11  .03 -.06 -.01 -.01  .01 -.01 -.02  .06
  ST.E.     .05  .05  .05  .05  .05  .05  .05  .05  .05  .05  .05  .05
```

The estimate of θ is close to one. This model is equivalent to one that relates differences of log sales to differences of log prices, with a moving average error component that is close to one. Recall that first differences of logs are equivalent to percentage changes.

The “design” of the price data is interesting, as there are periods where prices are rather flat. Look at the time series graph of (log) prices. One notices a certain “industry price” which stores use as the base when reducing their prices. Every once in a while the industry price changes. One could argue that it is not the actual price, but the “un-anticipated” price that matters and affects sales. One could measure the “un-anticipated” price component by considering the difference between the current price, p_t, and the exponentially weighted average of past prices. That is, one could consider

$$p_t - (1-\alpha)[p_{t-1} + \alpha p_{t-2} + \alpha^2 p_{t-3} + ...] = p_t - (1-\alpha)\frac{B}{1-\alpha B}p_t = \frac{1-B}{1-\alpha B}p_t$$

as the relevant regressor variable. The parameter α determines how quickly price information is discounted. [Here B is the backshift operator. Check that the left hand side of the above expression can be written this way. For simplicity of exposition we have considered a single price series.]

Regressing y_t on the un-anticipated price component leads to the model

$$y_t = \beta_0 + \beta_1[(1-B)/(1-\alpha B)]p_t + ... + \varepsilon_t$$

or,

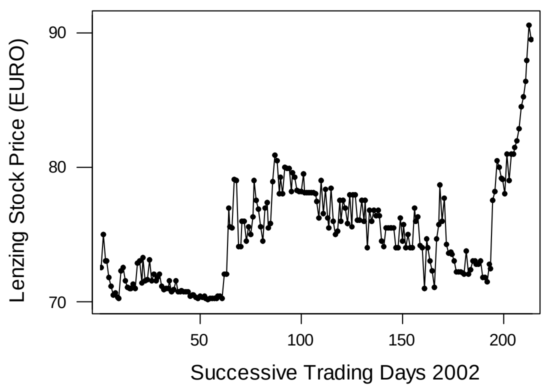
$$(1-\alpha B)y_t = \beta_0^* + \beta_1(1-B)p_t + ... + (1-\alpha B)\varepsilon_t$$

Note that this derivation assumes that α is the same for all three price series. The estimation results for this model are shown below. The estimate of α is close to one. In essence, this model goes back to the model with differences in all variables (response as well as regressor variables) and a moving average parameter that is close to one. The estimates of the regression coefficients (1.11, -6.57, 2.95) are similar to the coefficients in the earlier regression time series model (1.26, -6.64, 3.21).

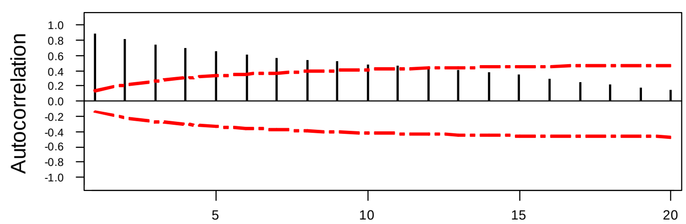
```
-----------------------------------------------------------------------
 VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
           VARIABLE   OR CENTERED

 LNSalesP12    RANDOM     ORIGINAL     NONE
 LNSalesP12    RANDOM     ORIGINAL     NONE
                                           1
  LNPriceP6    RANDOM     ORIGINAL     (1-B  )
                                           1
 LNPriceP12    RANDOM     ORIGINAL     (1-B  )
                                           1
 LNPriceP24    RANDOM     ORIGINAL     (1-B  )
 -----------------------------------------------------------------------

  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
    LABEL       NAME    DENOM.                 TRAINT               ERROR  VALUE

   1  CNST             CNST      1       0     NONE      .6756      .1573   4.30
   2 THETA  LNSalesP12   NUM.    1       1    EQ 01      .9214      .0162  56.70
   3   B1    LNPriceP6   NUM.    1       0     NONE     1.1090      .1778   6.24
   4   B2   LNPriceP12   NUM.    1       0     NONE    -6.5664      .3564 -18.42
   5   B3   LNPriceP24   NUM.    1       0     NONE     2.9520      .3159   9.35
 *** THETA  LNSalesP12    MA     1       1    EQ 01      .9214      .0162  56.70


 EFFECTIVE NUMBER OF OBSERVATIONS . .           383
 R-SQUARE . . . . . . . . . . . . . .         0.581
 RESIDUAL STANDARD ERROR. . . . . . .  0.776764E+00
```

10.9 The time series graph shows that the level of the series changes over time. The series is not stationary. Stock price data are usually nonstationary, with changing levels and locally changing trends. Note that we treat the time series observations as equally spaced, despite the fact that there is no trading on weekends and holidays. The autocorrelation function of the series is slow to die down. This is yet another indication of nonstationary.

Exercise 10.9: Lenzing Stock Prices



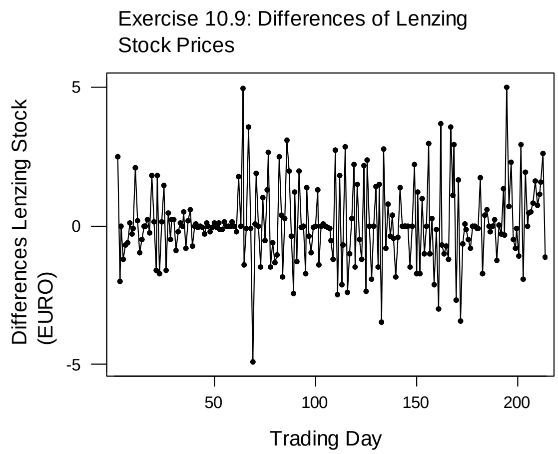
Autocorrelation Function for Lenzing Stock

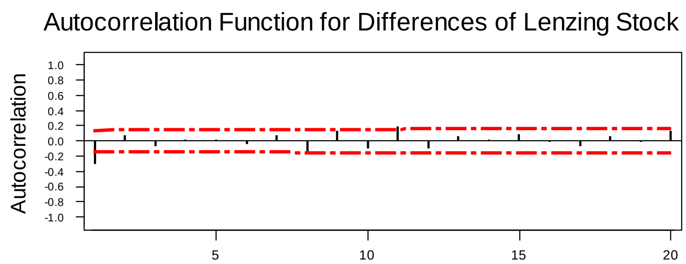


Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.89	13.02	171.96	8	0.54	2.73	872.83	15	0.35	1.50	1182.52
2	0.82	7.48	319.43	9	0.52	2.56	934.43	16	0.30	1.29	1203.56
3	0.75	5.56	444.15	10	0.49	2.32	988.64	17	0.25	1.08	1218.60
4	0.71	4.58	553.68	11	0.47	2.19	1039.62	18	0.21	0.91	1229.47
5	0.66	3.91	649.53	12	0.44	1.98	1083.35	19	0.18	0.75	1237.05
6	0.61	3.40	732.73	13	0.41	1.81	1121.25	20	0.15	0.62	1242.26
7	0.58	3.05	807.56	14	0.38	1.67	1154.60				

First differences of the series have a constant level and are stationary. Autocorrelations of first differences die down rapidly. In fact, only the lag one autocorrelation exceeds twice the standard error, $1/\sqrt{213} = 0.07$. Adjacent changes of stock prices are correlated. Note that also, the lag 11 autocorrelation exceeds twice the standard error. However, we doubt that changes 11 steps apart are really correlated, and we attribute this autocorrelation to chance.

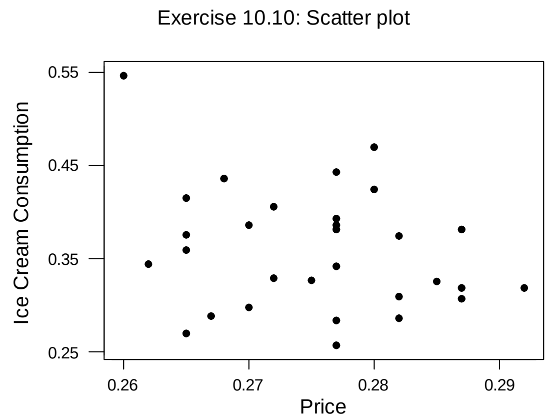
The time series graph of first differences shows periods where there is more (and less) variability (also called volatility). Time series models that incorporate components for changing variability (ARCH and GARCH models) are studied in the finance literature.

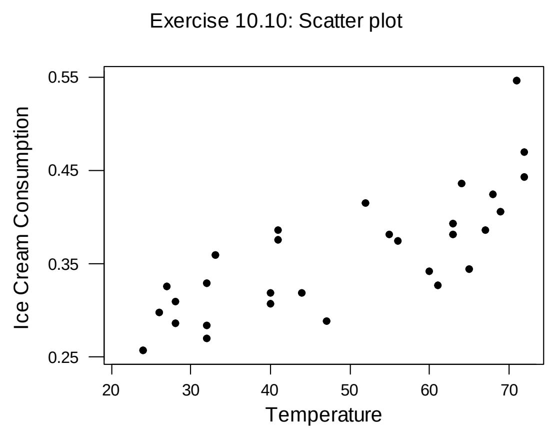


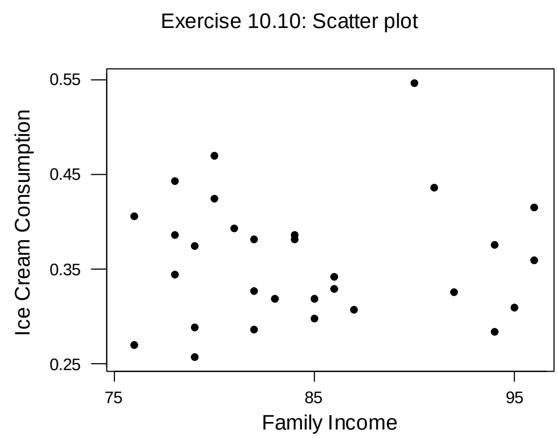


Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.31	-4.47	20.30	8	-0.14	-1.87	28.76	15	0.08	1.02	48.32
2	0.07	0.94	21.36	9	0.13	1.64	32.34	16	-0.02	-0.26	48.42
3	-0.08	-1.04	22.68	10	-0.11	-1.39	34.97	17	-0.08	-0.94	49.80
4	0.01	0.18	22.72	11	0.20	2.49	43.70	18	0.06	0.77	50.76
5	0.01	0.18	22.76	12	-0.10	-1.24	45.99	19	-0.01	-0.13	50.78
6	-0.04	-0.59	23.19	13	0.05	0.66	46.67	20	0.14	1.68	55.37
7	0.07	0.92	24.27	14	0.01	0.14	46.70				

10.10 Scatter plots of ice cream consumption on price, family income, and temperature, and results of fitting the regression model $\text{Cons}_t = \beta_0 + \beta_1 \text{Price}_t + \beta_2 \text{Inc}_t + \text{Temp}_t + \varepsilon_t$ are shown below. The Durbin-Watson statistic is much smaller than the desired value 2 and unacceptable. The small value of the Durbin-Watson statistic indicates positive lag one autocorrelation. The first six autocorrelations of the residuals are also shown. Especially the lag one autocorrelation (r_1 = 0.32) is relatively large when compared to its standard error $1/\sqrt{30} = 0.18$.


Exercise 10.10: Scatter plot
Ice Cream Consumption
0.55
0.45
0.35
0.25
0.26
0.27
0.28
0.29
Price


Exercise 10.10: Scatter plot
Ice Cream Consumption
0.55
0.45
0.35
0.25
20
30
40
50
60
70
Temperature


Exercise 10.10: Scatter plot
Ice Cream Consumption
0.55
0.45
0.35
0.25
75
85
95
Family Income

```
The regression equation is
Consumption = 0.197 - 1.04 Price + 0.00331 Income + 0.00346
Temperature

Predictor        Coef     SE Coef         T       P
Constant       0.1973      0.2702      0.73   0.472
Price         -1.0444      0.8344     -1.25   0.222
Income       0.003308    0.001171      2.82   0.009
Temperature 0.0034584   0.0004455      7.76   0.000

S = 0.03683     R-Sq = 71.9%     R-Sq(adj) = 68.7%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         3    0.090251    0.030084     22.17    0.000
Residual Error    26    0.035273    0.001357
Total             29    0.125523

Durbin-Watson statistic = 1.02

Autocorrelations of Residuals

   0.329772   0.036248   0.011063  -0.093395  -0.318641  -0.205802
```

The errors in this regression are not independent, and the error model needs to be revised. We consider two different error models: a first-order moving average and a first-order autoregressive error model. Note that in the regression with independent errors the coefficient for price is not significant. However, we keep this variable in the model as the significance may have been affected by the correlations in the errors. If it turns out that this coefficient is still insignificant, it can be removed at a later stage.

Estimation results for the two models are shown below. We use SCA to carry out the estimation. Alternatively, one can use EVIEWS. The residuals of the revised models are uncorrelated. The regression coefficients for income and temperature are significant (t-ratios exceed two). Income and temperature have positive regression coefficients; ice cream sales increase with increasing income and rising temperature. The coefficient of price is negative and not very significant (t-ratios of -1.77 and -1.18, respectively).

MA(1): $\text{Cons}_t = \beta_0 + \beta_1 \text{Price}_t + \beta_2 \text{Inc}_t + \beta_3 \text{Temp}_t + (1-\theta B)a_t$

PARAMETER	LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	B0		CNST	1	0	NONE	.3287	.2661	1.24
2	B1	PRICE	NUM.	1	0	NONE	-1.3886	.7829	-1.77
3	B2	INCOME	NUM.	1	0	NONE	.0029	.0014	2.15
4	B3	TEMP	NUM.	1	0	NONE	.0034	.0005	6.64
5	THETA	ICE	MA	1	1	NONE	-.5031	.1760	-2.86

```
EFFECTIVE NUMBER OF OBSERVATIONS . .           30
 R-SQUARE . . . . . . . . . . . . . .        0.771
 RESIDUAL STANDARD ERROR. . . . . . .  0.309303E-01

 AUTOCORRELATIONS OF RESIDUALS

   1- 12     .02  .06 -.01  .02 -.30  .01 -.14 -.13 -.01 -.17 -.13  .07
   ST.E.     .18  .18  .18  .18  .18  .20  .20  .20  .21  .21  .21  .21

  13- 24     .32  .10  .02  .07  .13 -.15 -.04  .05 -.03 -.18  .01 -.23
   ST.E.     .21  .23  .23  .23  .23  .23  .24  .24  .24  .24  .24  .24
```

AR(1): $\text{Cons}_t = \beta_0 + \beta_1 \text{Price}_t + \beta_2 \text{Inc}_t + \beta_3 \text{Temp}_t + \frac{1}{(1-\phi B)} a_t$

PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	B0		CNST	1	0	NONE	.1495	.2697	.55
2	B1	PRICE	NUM.	1	0	NONE	-.8889	.7532	-1.18
3	B2	INCOME	NUM.	1	0	NONE	.0033	.0014	2.33
4	B3	TEMP	NUM.	1	0	NONE	.0035	.0005	6.57
5	PHI	ICE	D-AR	1	1	NONE	.4016	.1866	2.15

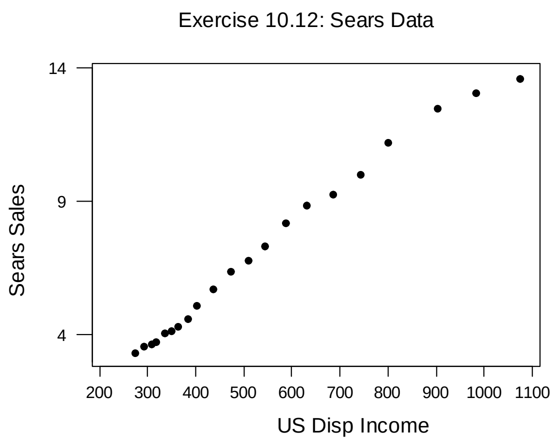
```
 EFFECTIVE NUMBER OF OBSERVATIONS . .           29
 R-SQUARE . . . . . . . . . . . . . .        0.790
 RESIDUAL STANDARD ERROR. . . . . . .  0.296282E-01

AUTOCORRELATIONS OF RESIDUALS

   1- 12     .09 -.11 -.02  .04 -.15  .08 -.10 -.09  .01 -.29 -.24  .09
   ST.E.     .19  .19  .19  .19  .19  .19  .19  .20  .20  .20  .21  .22

  13- 24     .38  .07 -.01  .00  .14 -.06  .02  .06  .03 -.09 -.12 -.20
   ST.E.     .22  .24  .24  .24  .24  .25  .25  .25  .25  .25  .25  .25
```

10.11 The scatter plot of lake levels against sunspots and the results of fitting the regression $\text{LakeLevel}_t = \beta_0 + \beta_1 \text{Sunspots}_t + \varepsilon_t$ are shown below. The Durbin-Watson statistic (1.71) and the autocorrelations of the residuals (with standard error $1/\sqrt{20} = 0.22$) indicate that there is no problem with serial correlation. The errors can be assumed independent.

```
The regression equation is
LakeLevel = - 8.04 + 0.413 Sunspots

Predictor        Coef     SE Coef          T        P
Constant       -8.042       2.556      -3.15    0.006
Sunspots      0.41281     0.05275       7.83    0.000

S = 6.466       R-Sq = 77.3%     R-Sq(adj) = 76.0%
```

```
Analysis of Variance

Source            DF        SS        MS       F      P
Regression         1    2560.4    2560.4   61.24  0.000
Residual Error    18     752.5      41.8
Total             19    3313.0

Unusual Observations
Obs  Sunspots   LakeLeve       Fit     SE Fit   Residual   St Resid
  5        54      29.00     14.25       1.62      14.75      2.36R
 16       104      35.00     34.89       3.67       0.11      0.02 X

R denotes an observation with a large standardized residual
X denotes an observation whose X value gives it large influence.

Durbin-Watson statistic = 1.71

Autocorrelations of residuals
   0.100203   0.027064   0.284582   0.100791
```

10.12 (a) The scatter plots of sales on disposable income and the results of fitting the regression model $\text{Sales}_t = \beta_0 + \beta_1 \text{Income}_t + \varepsilon_t$ are shown below. The Durbin-Watson statistics is much smaller than the desired value 2, and is unacceptable. The small value of the Durbin-Watson statistic indicates positive lag one autocorrelation. The first four autocorrelations of the residuals are also shown. The lag one autocorrelation (r_1 = 0.48) exceeds twice its standard error $1/\sqrt{21} = 0.22$.




```
The regression equation is
Sales = - 0.524 + 0.0140 Income

Predictor        Coef     SE Coef        T      P
Constant      -0.5243      0.1884    -2.78  0.012
Income      0.0140496   0.0003185    44.11  0.000
```

```
S = 0.3435      R-Sq = 99.0%     R-Sq(adj) = 99.0%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      229.60      229.60   1945.85    0.000
Residual Error    19        2.24        0.12
Total             20      231.85

Durbin-Watson statistic = 0.63

Autocorrelations of residuals

   0.478152   0.075695   0.060663   0.200269
```

(b) The errors in the regression are not independent, and the error model needs to be revised. We consider a noisy random walk (the ARIMA(0,1,1)) model

$$\text{Sales}_t = \beta_0 + \beta_1 \text{Income}_t + \frac{1-\theta B}{(1-B)} a_t \ , \quad \text{or}$$

$$(1-B)\text{Sales}_t = \beta_1 (1-B)\text{Income}_t + (1-\theta B) a_t$$

Because of the differencing operation it is no longer possible to estimate the intercept β_0 in the earlier regression model. The SCA estimation results show that this model fits much better. The residuals are uncorrelated; especially the lag one autocorrelation is much smaller. Note that with a small data set such as this (n = 21), various other noise models could be considered to approximate the autocorrelation of the errors in the regression model in (a).

```
----------------------------------------------------------------------
 VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
           VARIABLE   OR CENTERED
                                            1
  SALES     RANDOM     ORIGINAL     (1-B  )
                                            1
   INC      RANDOM     ORIGINAL     (1-B  )
 ----------------------------------------------------------------------

  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
    LABEL       NAME    DENOM.                TRAINT               ERROR  VALUE

   1    B1       INC     NUM.      1      0     NONE      .0107     .0014   7.77
   2  THETA     SALES     MA       1      1     NONE     -.7308     .1504  -4.86

 EFFECTIVE NUMBER OF OBSERVATIONS . .           20
 R-SQUARE . . . . . . . . . . . . . . .      0.997
 RESIDUAL STANDARD ERROR. . . . . . .  0.190304E+00

AUTOCORRELATIONS OF RESIDUALS

   1- 6      .08 -.24 -.27  .31  .11 -.15
   ST.E.     .22  .23  .24  .25  .27  .27
```

(c) For $\theta = 0$ (which, however, is not indicated from the data), the model in (b) simplifies to a regression of $ChangeSales_t = (Sales_t - Sales_{t-1})$ on changes in disposable income $ChangeIncome_t = (Income_t - Income_{t-1})$. The results of this regression are given below. The Durbin-Watson statistic is still much smaller than the desired value 2, and the first four autocorrelations of the residuals are barely within two standard errors ($1/\sqrt{20} = 0.22$). The results indicate that a moving average component (and hence the ARIMA(0,1,1) model in part (b)) are needed.

```
The regression equation is
ChangeSales = 0.149 + 0.00916 ChangeIncome

20 cases used 1 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant      0.14892     0.09770       1.52    0.145
ChangeInc    0.009155    0.002034       4.50    0.000

S = 0.2397      R-Sq = 53.0%     R-Sq(adj) = 50.3%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      1.1646      1.1646     20.27    0.000
Residual Error    18      1.0344      0.0575
Total             19      2.1990

Durbin-Watson statistic = 1.12
Autocorrelation of residuals
   0.332880  -0.398696  -0.191203   0.333627
```

(d) The results of the regression of $RelChangeSales_t = (Sales_t - Sales_{t-1})/Sales_{t-1} \cong \ln(Sales_t) - \ln(Sales_{t-1})$ on relative changes in disposable income, $RelChangeIncome_t = (Income_t - Income_{t-1})/Income_{t-1} \cong \ln(Income_t) - \ln(Income_{t-1})$ are shown below. The results are similar to those in part (c) of the exercise.

```
The regression equation is
RelChaSales = 0.0219 + 0.732 RelChaInc

20 cases used 1 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant      0.02187     0.02449       0.89    0.384
RelChaIn       0.7322      0.3290       2.23    0.039

S = 0.03219     R-Sq = 21.6%     R-Sq(adj) = 17.2%
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1    0.005131    0.005131      4.95    0.039
Residual Error    18    0.018648    0.001036
Total             19    0.023778
```

```
Durbin-Watson statistic = 1.30

Autocorrelations of residuals
   0.284050  -0.322172  -0.086133   0.319946
```

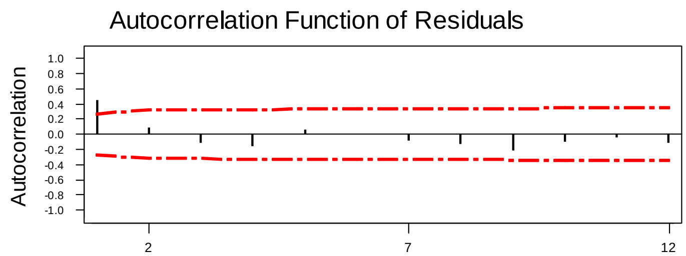
(e) The model in part (b) gives a good description of the data.

10.13 (a) Results of the regression

$$\text{FTEShares}_t = \beta_0 + \beta_1 \text{CarProd}_{t-6} + \beta_2 \text{FTECom}_{t-7} + \varepsilon_t$$

are given below. The Durbin-Watson statistics is much smaller than the desired value 2 and is unacceptable. The small value of the Durbin-Watson statistic indicates positive lag one autocorrelation. The autocorrelation function of the residuals indicates significant autocorrelations, especially at lag 1 ($r_1 = 0.45$, compared to its standard error $1/\sqrt{22} = 0.13$). The extremely significant estimates for lagged car production and lagged commodity index are surprising, because results in the finance literature indicate that stock prices are best predicted by the current value of the stock, but not by other economic variables.

```
The regression equation is
FTEShare = 595 + 0.000514 CarLag6 - 5.54 ComLag7

55 cases used 7 cases contain missing values

Predictor         Coef     SE Coef          T        P
Constant        594.51       60.65       9.80    0.000
CarLag6     0.00051422  0.00003406      15.10    0.000
ComLag7        -5.5439      0.6727      -8.24    0.000

S = 25.06       R-Sq = 88.2%     R-Sq(adj) = 87.8%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      244274      122137    194.46    0.000
Residual Error    52       32661         628
Total             54      276935

Durbin-Watson statistic = 0.87
```



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.45	3.36	11.94	8	-0.13	-0.80	16.84
2	0.09	0.57	12.42	9	-0.22	-1.31	20.19
3	-0.11	-0.69	13.16	10	-0.10	-0.59	20.92
4	-0.17	-1.04	14.91	11	-0.04	-0.24	21.05
5	0.05	0.32	15.08	12	-0.12	-0.71	22.17
6	0.01	0.04	15.08				
7	-0.09	-0.56	15.64				

(b) We consider the noisy random walk as a model for the errors, and fit the regression model

$$\text{FTEShares}_t = \beta_1 \text{CarProd}_{t-6} + \beta_2 \text{FTECom}_{t-7} + \frac{1-\theta B}{1-B} a_t .$$

Because of the differencing operation, it is no longer possible to estimate the intercept β_0 of the earlier regression model.

```
-----------------------------------------------------------------------
VARIABLE    TYPE OF    ORIGINAL     DIFFERENCING
           VARIABLE   OR CENTERED
                                          1
FTESHARE    RANDOM     ORIGINAL     (1-B   )
                                          1
  CAR       RANDOM     ORIGINAL     (1-B   )
                                          1
FTECOMM     RANDOM     ORIGINAL     (1-B   )
-----------------------------------------------------------------------

 PARAMETER    VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD      T
   LABEL        NAME    DENOM.                TRAINT               ERROR   VALUE

  1     B1   CarProd     NUM.     1       6     NONE      .0001 .8107E-04   1.81
  2     B2   FTECom      NUM.     1       7     NONE     -.6884    1.1833   -.58
  3  THETA   FTEShares   MA       1       1     NONE     -.1468     .1417  -1.04


EFFECTIVE NUMBER OF OBSERVATIONS . .            54
R-SQUARE . . . . . . . . . . . . . .         0.951
RESIDUAL STANDARD ERROR. . . . . . .  0.180416E+02

AUTOCORRELATIONS OF RESIDUALS

  1- 12     -.03 -.04  .00 -.12  .07  .02 -.15 -.00 -.32  .11 -.11 -.18
  ST.E.      .14  .14  .14  .14  .14  .14  .14  .14  .14  .15  .16  .16

 13- 24      .12 -.09  .03  .27 -.15  .09  .08  .10 -.01 -.17  .06 -.03
  ST.E.      .16  .16  .16  .16  .17  .17  .18  .18  .18  .18  .18  .18
```

(c) The estimate of θ is not much different from zero. We set it zero and estimate the parameters in the regression model with random walk errors

$$\text{FTEShares}_t = \beta_1 \text{CarProd}_{t-6} + \beta_2 \text{FTECom}_{t-7} + \frac{1}{1-B} a_t \ .$$

This model is a regression of differences of the response on differences of the regressor variables,

$$\Delta\text{FTEShares}_t = \beta_1 \Delta\text{CarProd}_{t-6} + \beta_2 \Delta\text{FTECom}_{t-7} + a_t \,.$$

The results given below show that there is no autocorrelation in the residuals. The model passes all diagnostic checks. The intercept and the regressors are not statistically significant (p-values of 0.085 and 0.51), implying that the model for the FTE share index is given by the random walk

$$\Delta\text{FTEShares}_t = \text{FTEShares}_t - \text{FTEShares}_{t-1} = a_t \,.$$

This result is expected. The finance literature shows that in efficient markets stock prices follow random walks and changes in stock prices are unrelated to economic variables. The "significant" regression in part (a) was spurious, implied by the incorrect model for the error terms; see the discussion of spurious regression in Section 10.2.

```
The regression equation is
DiffShare = 3.71 +0.000144 DiffCarLag6 - 0.79 DiffCommLag7

54 cases used 8 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant        3.712       2.547       1.46    0.151
DiffCarPr  0.00014414  0.00008218       1.75    0.085
DiffComm       -0.786       1.175      -0.67    0.507

S = 18.35       R-Sq = 6.1%      R-Sq(adj) = 2.4%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2      1107.6       553.8      1.64    0.203
Residual Error    51     17179.0       336.8
Total             53     18286.5

Durbin-Watson statistic = 1.72

Autocorrelations of residuals

   0.100215  -0.050381  -0.024561  -0.117140   0.070461   0.023664
  -0.147233  -0.070849  -0.317866   0.050094  -0.118589  -0.173089
```

CHAPTER 11

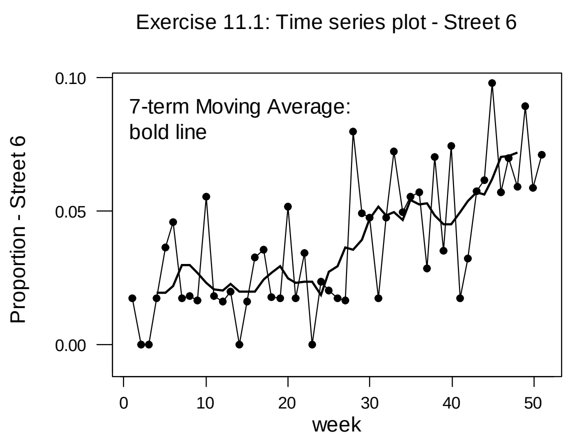
A note on computing with MINITAB (Version 14):

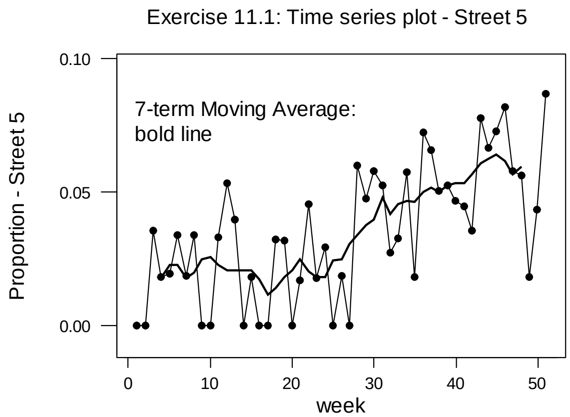
The **Minitab** software is used for fitting the logistic regression models in Chapter 11. Alternatively, one can use the SAS PROC GENMOD procedure; see the explanation in Chapter 12 of this solutions manual.

Minitab works like a spreadsheet program. We enter the data into the various columns of the spreadsheet and use the tabs: Stat > Regression > Binary logistic regression. We need to specify the response; either a column of zeros and ones if we work with individual cases, or the number of successes and the number of trials for each constellation if we work with aggregated data. We need to write out the model in model format. We can declare variables as factors – then Minitab will automatically create the needed indicator variables and test for factor effects. We can store the results (fitted values, residuals, …) in unused columns of the worksheet. All diagnostic graphs discussed in Chapter 11 of the book are available in Minitab.

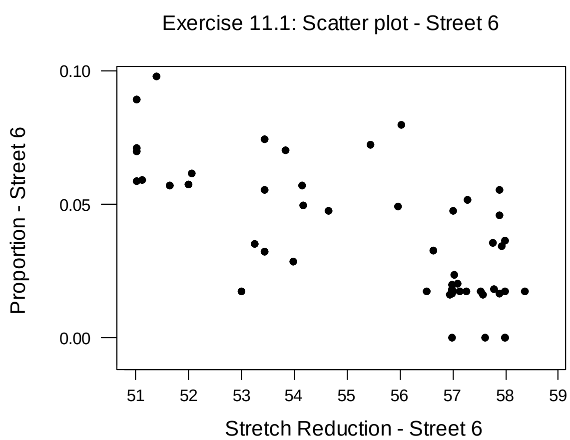
Options for various links (logit, probit, and complementary log-log links), starting values, maximum number of iterations, and number of classes in the Hosmer-Lemeshow test are available. Many other options are available. See the Minitab on-line help for detailed discussion and examples.

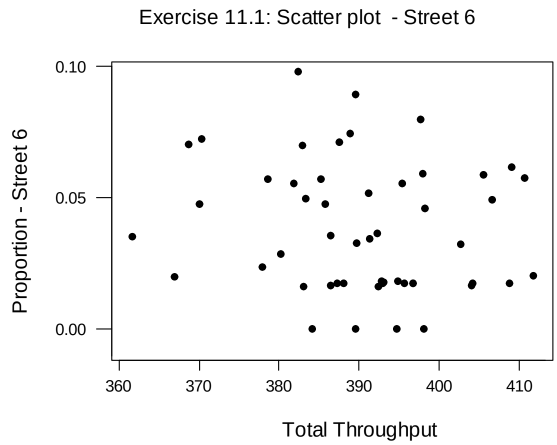
11.1 Time series graphs of weekly proportions of long fibers are given below. 7-term moving averages, $MA_t = (y_{t-2} + y_{t-1} + y_t + y_{t+1} + y_{t+2})/7$, are added to these graphs. Moving averages amplify the trend component in a time series graph of noisy observations. The proportions of long fibers increase during the second half of the year.

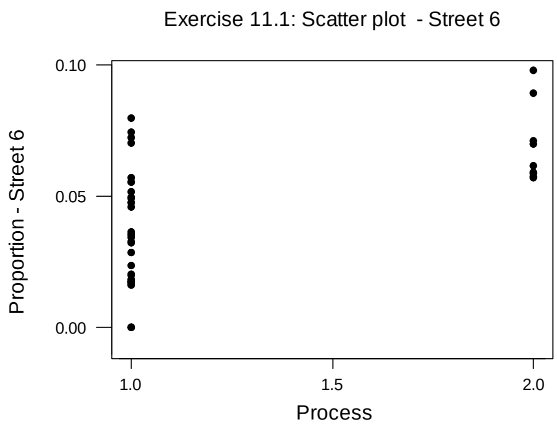




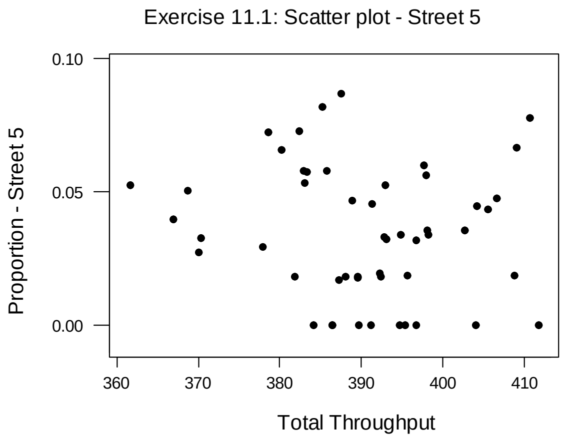
For each street (machine) separately, we construct scatter plots of the proportions of long fibers against stretch reduction, total throughput, and the type of process. The proportions of long fibers decrease with increased stretch reduction. The proportion of long fibers is larger under process 2.




Exercise 11.1: Scatter plot - Street 6
Proportion - Street 6
0.10
0.05
0.00
360
370
380
390
400
410
Total Throughput


Exercise 11.1: Scatter plot - Street 6
Proportion - Street 6
0.10
0.05
0.00
1.0
1.5
2.0
Process


Exercise 11.1: Scatter plot - Street 5
Proportion - Street 5
0.10
0.05
0.00
51
52
53
54
55
56
57
58
59
Stretch Reduction - Street 5



Logistic regression models for machine (street) 6:

Results for the following three logistic regression models are given below:

- model with stretch reduction, throughput, and process
- model with stretch reduction and throughput
- model with stretch reduction only

The total throughput and the type of process are insignificant. Stretch reduction remains as the only significant variable. An increase in the stretch reduction of one unit (percent) changes the odds for long fibers by a (multiplicative) factor of 0.85. That is, an increase in the stretch reduction of one unit (percent) reduces the odds for the occurrence of long fibers by 15 percent. Or, to say this differently: A small stretch reduction increases the odds for quality problems.

The proportion of long fibers π can be obtained from

$$\pi(\mathrm{x}) = \frac{\exp(\beta_0 + \beta_1 \mathrm{x})}{1 + \exp(\beta_0 + \beta_1 \mathrm{x})} = \frac{\exp(5.928 - 0.1662\mathrm{x})}{1 + \exp(5.928 - 0.1662\mathrm{x})}$$

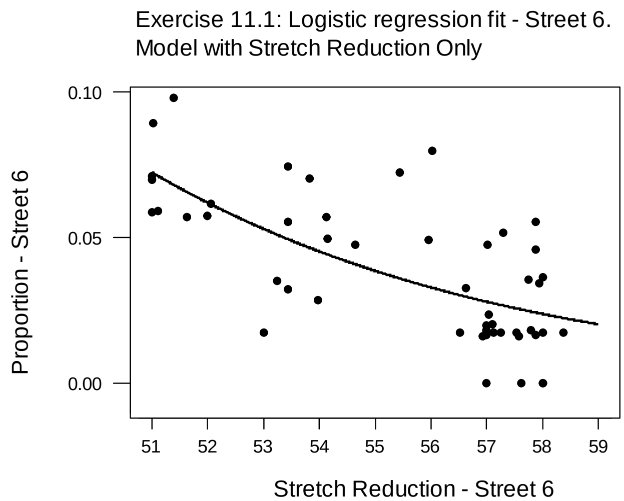
For stretch reduction x = 52, $\pi(\mathrm{x} = 52) = \frac{\exp(5.928 - 0.1662(52))}{1 + \exp(5.928 - 0.1662(52))} = 0.062$

For stretch reduction x = 53, $\pi(\mathrm{x} = 53) = \frac{\exp(5.928 - 0.1662(53))}{1 + \exp(5.928 - 0.1662(53))} = 0.053$

….

For stretch reduction x = 57, $\pi(x=57) = \frac{\exp(5.928 - 0.1662(57))}{1+\exp(5.928-0.1662(57))} = 0.028$

We have superimposed the fitted values (proportions of long fibers) in the scatter plot of the proportion of long fibers against stretch reduction (street 6). The main features of the scatter plot are well represented by the fitted model.



In this problem there are few exact replicates of the explanatory variable, stretch reduction. Minitab uses the approach by Hosmer and Lemeshow to group the cases on the basis of the estimated probabilities $\hat{\pi}_i = \hat{\pi}(x_i)$. It ranks the estimated probabilities from the smallest to the largest, and uses this ranking to break the cases into *g* = 10 groups of equal size. For each group *k* , *k = 1, 2, …,g*, it calculates the number of successes o_k and the number of failures $n_k - o_k$ that are associated with the n_k cases in the group. The observed frequencies are compared with the expected frequencies $n_k\bar{\pi}_k$ and $n_k(1-\bar{\pi}_k)$, where $\bar{\pi}_k = \frac{\sum_{i\in \text{Group}k} \hat{\pi}_i}{n_k}$ is the average estimated success probability in the k^{th} group. The Pearson chi-square statistic is calculated from the resulting 2 x *g* table, and

$$HL = \sum_{k=1}^{g} \frac{[o_k - n_k\bar{\pi}_k]^2}{n_k\bar{\pi}_k(1-\bar{\pi}_k)}$$

is referred to as the Hosmer-Lemeshow statistic. Hosmer and Lemeshow show that the distribution of *HL* is well approximated by a chi-square distribution with *g - 2* degrees of freedom. Large values of the Hosmer-Lemeshow statistic indicate lack of fit. In our problem the Hosmer-Lemeshow statistic is *HL* = 6.938. It is quite small when compared to the 95th percentile of chi-square distribution with 10 – 2 = 8 degrees of freedom (15.51). The associated large probability value, 0.435, confirms that the model gives a very adequate representation of the data.

The Pearson residual for each of the 52 weeks is calculated from the equation $r_i = r(y_i, \hat{\pi}_i) = \frac{y_i - n_i\hat{\pi}_i}{\sqrt{n_i\hat{\pi}_i(1-\hat{\pi}_i)}}$, where y_i and n_i are the number of occurrences and the total number of trials in week *i*, and where $\hat{\pi}_i = \hat{\pi}(x_i) = \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)} = \frac{\exp(5.928 - 0.1662 x_i)}{1+\exp(5.928 - 0.1662 x_i)}$ is the implied success probability. The autocorrelations for the first six lags are given by

```
0.00, 0.06, -0.22, 0.05, 0.16, -0.02.
```

Comparing these to their approximate standard error, $1/\sqrt{52} = 0.14$, indicates no serial correlation among the residuals.

<u>A note on residuals and fitted values:</u> Minitab stores the residuals and the diagnostic measures for each <u>constellation</u>, and the constellations change with different model specifications. When estimating the logistic regression on stretch reduction alone, there are data for 51 weeks, but there are only 43 different stretch constellations. For three weeks the stretch reduction on street 6 is 51.000, for four weeks it is 57.000, and for four weeks it is 58.000. Minitab aggregates the information and supplies vectors of fitted values and residuals for the 43 constellations. This is fine as far as the usual diagnostic checks are concerned, but it causes difficulties if one wants to calculate the autocorrelations of the residuals where time order is of importance. One cannot compute the autocorrelations of weekly residuals from the vector of the aggregated residuals.
One must first compute the residuals for each week. This can be done by using the weekly frequencies (number of successes and number of trials) and the event probabilities at the constellations (note that these are stored by Minitab). Alternatively, one can "trick" the program by adding small numbers to the replicates of stretch to make them slightly different (say 51.000, 51.001, and 51.003 for the three weeks with identical stretch reduction 51.000; etc). Then Minitab will treat them as separate constellations and will give you the vector of the 51 weekly residuals automatically.

<u>Model with stretch reduction, throughput, and process:</u>

```
Link Function:  Logit

Response  Information
```

```
Variable  Value          Count
Positive6 Success          139
          Failure         3225
samples6  Total           3364

Logistic Regression Table
                                                      Odds      95% CI
Predictor       Coef    SE Coef        Z      P      Ratio   Lower   Upper
Constant       8.854      4.044     2.19 0.029
stretch6    -0.16900    0.06532    -2.59 0.010     0.84     0.74    0.96
throughput -0.007084   0.008151    -0.87 0.385     0.99     0.98    1.01
process6     -0.0062     0.3606    -0.02 0.986     0.99     0.49    2.02

Log-Likelihood = -566.074
Test that all slopes are zero: G = 25.849, DF = 3, P-Value = 0.000

Goodness-of-Fit Tests

Method               Chi-Square    DF       P
Pearson                  29.837    47  0.976
Deviance                 32.323    47  0.949
Hosmer-Lemeshow           2.053     8  0.979

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)

                                    Group
Value      1     2     3     4     5     6     7     8     9    10   Total
Success
  Obs      8    10     7     8    13    20    17    19    26    11     139
  Exp     7.7   9.3   9.1  10.0  10.5  17.7  17.2  20.0  25.7  11.8
Failure
  Obs    335   378   337   348   331   365   321   323   342   145    3225
  Exp    335.3 378.7 334.9 346.0 333.5 367.3 320.8 322.0 342.3 144.2

  Total  343   388   344   356   344   385   338   342   368   156    3364
```

Model with stretch reduction and throughput:

```
Link Function:  Logit
Response  Information

Variable  Value          Count
Positive6 Success          139
          Failure         3225
samples6  Total           3364

Logistic Regression Table
                                                      Odds      95% CI
Predictor       Coef    SE Coef        Z      P      Ratio   Lower   Upper
Constant       8.818      3.461     2.55 0.011
stretch6    -0.16805    0.03391    -4.96 0.000     0.85     0.79    0.90
throughput -0.007146   0.007285    -0.98 0.327     0.99     0.98    1.01
```

```
Log-Likelihood = -566.075
Test that all slopes are zero: G = 25.849, DF = 2, P-Value = 0.000

Goodness-of-Fit Tests

Method               Chi-Square    DF      P
Pearson                  29.840    48  0.982
Deviance                 32.324    48  0.960
Hosmer-Lemeshow           2.063     8  0.979

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)

                                    Group
Value      1     2     3     4     5     6     7     8     9    10   Total
Success
  Obs      8    10     7     8    13    20    17    19    26    11     139
  Exp      7.7   9.3   9.1  10.0  10.5  17.6  17.2  20.0  25.7  11.8
Failure
  Obs    335   378   337   348   331   365   321   323   342   145    3225
  Exp    335.3 378.7 334.9 346.0 333.5 367.4 320.8 322.0 342.3 144.2

  Total  343   388   344   356   344   385   338   342   368   156    3364
```

<u>Model with stretch reduction only:</u>

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Positive6 Success          139
          Failure         3225
samples6  Total           3364

Logistic Regression Table
                                                        Odds      95% CI
Predictor       Coef    SE Coef        Z      P   Ratio   Lower   Upper
Constant       5.928      1.818     3.26 0.001
stretch6    -0.16619    0.03359    -4.95 0.000    0.85    0.79    0.90

Log-Likelihood = -566.554
Test that all slopes are zero: G = 24.891, DF = 1, P-Value = 0.000

Goodness-of-Fit Tests

Method               Chi-Square    DF      P
Pearson                  27.801    41  0.943
Deviance                 26.951    41  0.955
Hosmer-Lemeshow           6.938     7  0.435

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)
```

Value	1	2	3	4	5	6	7	8	9	Total
					Group					
Success										
Obs	9	8	11	6	20	21	17	25	22	139
Exp	9.2	8.8	10.4	11.1	13.1	18.5	19.4	26.0	22.4	
Failure										
Obs	379	345	373	389	367	383	340	363	286	3225
Exp	378.8	344.2	373.6	383.9	373.9	385.5	337.6	362.0	285.6	
Total	388	353	384	395	387	404	357	388	308	3364

Logistic regression models for machine (street) 5:

Results for the following two logistic regressions models are given below:

- model with stretch reduction and throughput
- model with stretch reduction only

Process does not enter here, as machine 5 operates under one production process.

Total throughput is insignificant. Stretch reduction remains as the only significant variable. An increase in the stretch reduction of one unit (percent) changes the odds for long fibers by a (multiplicative) factor of 0.85. That is, an increase in the stretch reduction of one unit (percent) reduces the odds for long fibers by 15 percent. Note that the odds-ratios for stretch reduction are the same on both streets.

The proportion of long fibers π can be obtained from

$$\pi(\mathrm{x}) = \frac{\exp(\beta_0 + \beta_1 \mathrm{x})}{1 + \exp(\beta_0 + \beta_1 \mathrm{x})} = \frac{\exp(6.006 - 0.1681\mathrm{x})}{1 + \exp(6.006 - 0.1681\mathrm{x})}$$

For stretch reduction x = 52, $\pi(\mathrm{x} = 52) = \frac{\exp(6.006 - 0.1681(52))}{1 + \exp(6.006 - 0.1681(52))} = 0.061$

For stretch reduction x = 53, $\pi(\mathrm{x} = 53) = \frac{\exp(6.006 - 0.1681(53))}{1 + \exp(6.006 - 0.1681(53))} = 0.052$

….

For stretch reduction x = 57, $\pi(\mathrm{x} = 57) = \frac{\exp(6.006 - 0.1681(57))}{1 + \exp(6.006 - 0.1681(57))} = 0.027$

We have superimposed the fitted proportions of long fibers in the scatter plot of the proportion of long fibers against stretch reduction (street 5). The main features of the scatter plot are well represented by the fitted model.

The Hosmer-Lemeshow statistic is *HL* = 5.146. It is quite small when compared with the 95th percentile of chi-square distribution with 10 – 2 = 8 degrees of freedom (15.51). The associated large probability value, 0.742, confirms that the model leads to a very adequate representation of the data.

Model with stretch reduction and throughput:

```
Link Function:  Logit

Response  Information

Variable  Value           Count
Positive5 Success           119
          Failure          3014
samples5  Total            3133


Logistic Regression Table
                                                      Odds      95% CI
Predictor        Coef     SE Coef         Z      P   Ratio   Lower   Upper
Constant        9.888       3.934      2.51 0.012
stretch5     -0.17408     0.03979     -4.38 0.000     0.84    0.78    0.91
throughput  -0.009107    0.007761     -1.17 0.241     0.99    0.98    1.01

Log-Likelihood = -496.083
Test that all slopes are zero: G = 19.664, DF = 2, P-Value = 0.000

Goodness-of-Fit Tests

Method              Chi-Square    DF       P
Pearson                 36.191    48   0.895
Deviance                47.990    48   0.473
Hosmer-Lemeshow          5.904     7   0.551
```

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)

	Group									
Value	1	2	3	4	5	6	7	8	9	Total
Success										
Obs	7	10	4	11	8	18	21	21	19	119
Exp	7.3	8.2	8.5	9.3	10.4	14.5	18.3	19.5	23.0	
Failure										
Obs	326	336	328	335	341	361	348	316	323	3014
Exp	325.7	337.8	323.5	336.7	338.6	364.5	350.7	317.5	319.0	
Total	333	346	332	346	349	379	369	337	342	3133

Model with stretch reduction only:

Link Function: Logit

Response Information

Variable	Value	Count
Positive5	Success	119
	Failure	3014
samples5	Total	3133

Logistic Regression Table

Predictor	Coef	SE Coef	Z	P	Odds Ratio	95% CI Lower	Upper
Constant	6.006	2.138	2.81	0.005			
stretch5	-0.16807	0.03917	-4.29	0.000	0.85	0.78	0.91

Log-Likelihood = -496.765
Test that all slopes are zero: G = 18.300, DF = 1, P-Value = 0.000

Goodness-of-Fit Tests

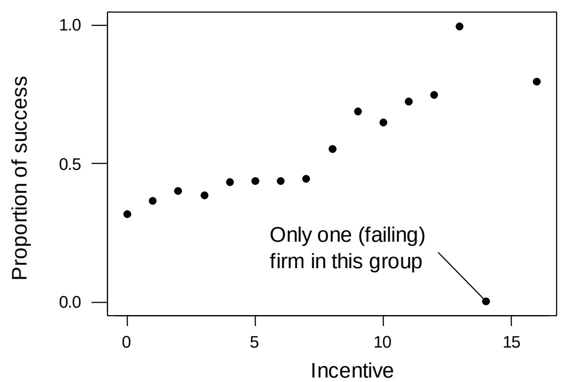
Method	Chi-Square	DF	P
Pearson	33.514	43	0.850
Deviance	44.313	43	0.416
Hosmer-Lemeshow	5.146	8	0.742

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)

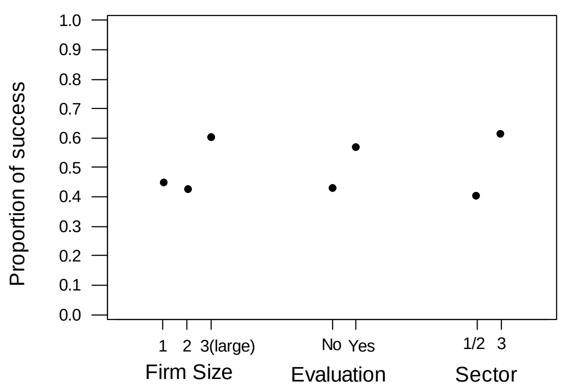
	Group										
Value	1	2	3	4	5	6	7	8	9	10	Total
Success											
Obs	8	8	10	6	12	11	21	20	22	1	119
Exp	7.9	8.5	8.5	9.2	11.0	11.4	16.6	20.2	21.9	3.9	
Failure											
Obs	329	335	314	336	355	304	330	334	323	54	3014
Exp	329.1	334.5	315.5	332.8	356.0	303.6	334.4	333.8	323.1	51.1	
Total	337	343	324	342	367	315	351	354	345	55	3133

11.2 Scatter plots of the success proportions against the incentive index, the size of the firm, the evaluation indicator, and the sector are given below. We learn that the chance for success increases with the number of offered incentives, and the size of the firm (large firms are usually more successful). Evaluation matters (evaluated firms tend to be more successful), and the sector appears to make a difference (larger success rate in the tertiary sector).

Exercise 11.2: Proportion of Success against Index of Incentive



Exercise 11.2: Proportion of Success against Firm Size, Evaluation, and Sector



We consider a logistic regression model with the following explanatory variables: incentive index (a linear component), size (a categorical variable with 3 possibilities; we include two parameters for the three groups), evaluation, and sector (since we consider just two sectors - the primary/secondary and the tertiary sectors - we need only one parameter).

Model with incentives, size, evaluation and sector:

```
Link Function:  Logit

Response Information

Variable  Value       Count
profit    1             209  (Event)
          0             220
          Total         429

Factor Information

Factor   Levels Values
size          3 1 2 3

Logistic Regression Table
                                                    Odds       95% CI
Predictor       Coef    SE Coef          Z      P    Ratio    Lower    Upper
Constant     -2.8964     0.5630      -5.15 0.000
incentive    0.13173    0.02945       4.47 0.000      1.14     1.08     1.21
size
 2           -0.0352     0.2519      -0.14 0.889      0.97     0.59     1.58
 3            0.2794     0.2512       1.11 0.266      1.32     0.81     2.16
evaluation    0.4811     0.2096       2.30 0.022      1.62     1.07     2.44
sector2       0.7747     0.2138       3.62 0.000      2.17     1.43     3.30


Log-Likelihood = -271.242
Test that all slopes are zero: G = 51.955, DF = 5, P-Value = 0.000

Goodness-of-Fit Tests

Method               Chi-Square    DF      P
Pearson                 137.900   135  0.415
Deviance                171.893   135  0.018
Hosmer-Lemeshow           6.499     8  0.592

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)

                                    Group
Value      1     2     3     4     5     6     7     8     9    10   Total
1
  Obs     10     9    19    18    29    21    28    27    30    18     209
  Exp      9.7  14.3  16.9  18.2  24.4  21.7  25.4  28.8  32.0  17.5
0
  Obs     32    38    29    27    23    21    16    17    14     3     220
  Exp     32.3  32.7  31.1  26.8  27.6  20.3  18.6  15.2  12.0   3.5

  Total   42    47    48    45    52    42    44    44    44    21     429
```

Next, we omit size of the firm (the two size indicators are insignificant), and fit the simpler logistic regression model with the incentive index, evaluation, and sector as explanatory variables.

We can construct a log-likelihood-ratio test to test the statistical significance of the factor "size." We illustrate in detail how one can test whether the size effect is significant. Comparing the log-likelihood = -271.242 of the full model with the log-likelihood of the restricted model (model without size; log-likelihood = -272.029) leads to the log-likelihood ratio test statistic $2(-271.242-(-272.029))=1.57$. Relating this statistic to a chi-square distribution with 2 degrees of freedom leads to the probability value $P(\chi^2(2) \geq 1.57) = 0.4561$. Since the probability value is considerably larger than 0.05, we conclude that the factor "size" is not significant. We can work with the simplified model.

All remaining variables are statistically significant. A one unit increase in the incentive index (while keeping the other variables in the model constant) increases the odds for success by 15 percent. Evaluating the firm (and keeping the other variables in the model fixed) increases the odds for success by 64 percent. The odds for success of firms with the same incentive structure and evaluation in the tertiary sector are 127 percent larger than the odds in the primary/secondary sector.

The small Hosmer-Lemeshow statistic (5.905) and its large associated probability value (0.551) indicate that we have found an adequate model.

<u>Model with size omitted from the model:</u>

```
Link Function:  Logit

Response Information

Variable  Value       Count
profit    1             209  (Event)
          0             220
          Total         429

Logistic Regression Table
                                                        Odds        95% CI
Predictor       Coef    SE Coef        Z      P    Ratio    Lower    Upper
Constant     -2.9768     0.5451    -5.46 0.000
incentive    0.13837    0.02893     4.78 0.000     1.15     1.09     1.22
evaluation    0.4926     0.2088     2.36 0.018     1.64     1.09     2.46
sector2       0.8206     0.2102     3.90 0.000     2.27     1.50     3.43

Log-Likelihood = -272.029
Test that all slopes are zero: G = 50.381, DF = 3, P-Value = 0.000

Goodness-of-Fit Tests

Method                Chi-Square    DF      P
Pearson                   44.332    54  0.823
Deviance                  49.861    54  0.635
Hosmer-Lemeshow            5.905     7  0.551

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)
```

Value	Group 1	2	3	4	5	6	7	8	9	Total
1										
Obs	12	10	17	24	29	29	32	27	29	209
Exp	12.3	13.8	18.0	21.6	26.8	25.7	29.6	28.8	32.4	
0										
Obs	39	34	33	27	26	18	16	15	12	220
Exp	38.7	30.2	32.0	29.4	28.2	21.3	18.4	13.2	8.6	
Total	51	44	50	51	55	47	48	42	41	429

11.3 The information can be arranged as a factorial, with the number of affected workers among the total number of workers in each group as the response variable. The 72 groups of the factorial arrangement are formed by all possible level combinations of the five explanatory variables: 3 (Dust) x 2 (Race) x 2 (Sex) x 2 (Smoking) x 3 (Employment). Seven of the 72 categories are empty and are ignored in our analysis. We use the binary logistic regression function in MINITAB, specifying the number of successes and the number of trials, and entering the explanatory variables as (categorical) factors. MINITAB creates the appropriate indicators for the factors automatically.

<u>Model with all five factors:</u>

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Yes       Success          165
          Failure         5254
Number    Total           5419

   65 cases were used
    7 cases contained missing values

Logistic Regression Table
                                                    Odds       95% CI
Predictor        Coef    SE Coef         Z      P   Ratio    Lower    Upper
Constant     -1.9452     0.2334    -8.33 0.000
Dust
 2           -2.5799     0.2921    -8.83 0.000     0.08     0.04     0.13
 3           -2.7306     0.2153   -12.68 0.000     0.07     0.04     0.10
Race
 2            0.1163     0.2072     0.56 0.574     1.12     0.75     1.69
Sex
 2            0.1239     0.2288     0.54 0.588     1.13     0.72     1.77
Smoking
 2           -0.6413     0.1944    -3.30 0.001     0.53     0.36     0.77
Employ
 2            0.5641     0.2617     2.16 0.031     1.76     1.05     2.94
 3            0.7531     0.2161     3.48 0.000     2.12     1.39     3.24

Log-Likelihood = -598.968
```

```
Test that all slopes are zero: G = 279.256, DF = 7, P-Value = 0.000

Goodness-of-Fit Tests

Method              Chi-Square    DF      P
Pearson                 37.934    57  0.976
Deviance                43.271    57  0.910
```

The test statistic for testing the overall significance of the regression (in equation (11.25)) is given by G = 279.256. Its sampling distribution (under the null hypotheses that none of the regressors have an influence on the response) is chi-square with 7 degrees of freedom. The test statistic G = 279.256 is huge compared to the percentiles from that distribution, and its associated probability value is tiny (p value < 0.0001). Hence the regressor variables (all or a subset) have a significant impact on the occurrence of byssinosis.

Race and Sex (both at two levels) have no significant effects. One can see this from the odds-ratios (they are roughly one), their t-ratios (Z-scores) and the associated probability values. The probability values for Race and Sex exceed the usual cutoff 0.05. The insignificance of the effects is also expressed by the confidence intervals of the odds-ratios; the confidence intervals cover one (indicating even odds).

The dustiness of the workplace, the smoking history, and the length of employment matter; the probability values of the estimated coefficients are smaller than 0.05, and the confidence intervals of the resulting odds-ratios do not cover the value one.

The deviance (in equation (11.26)) and the Pearson statistic (in equation (11.31)) compare the fit of the parameterized model (here with 8 = 7 + 1 (for constant) parameters) with the fit of the saturated model where each constellation of the explanatory variables is allowed its own distinct success probability. Here there are 65 = 72 - 7 constellations as seven cells are empty. The deviance is D = 37.9 and the Pearson statistic is $\chi^2 = 43.3$. Large values of these statistics indicate model inadequacy; the appropriate reference distribution is chi-square with 65 – 8 = 57 degrees of freedom. The deviance and the Pearson statistic are smaller than the critical percentile (the 95^{th} percentile is 75.62), implying that the probability values are considerably larger than 0.05. Hence there is no reason to question the adequacy of the model.

Here the deviance and the Pearson chi-square statistics are useful measures of (lack of) fit, as we have replicate observations at each configuration of the explanatory variable(s). In this example there is no reason to consider the Hosmer-Lemeshow statistic which becomes useful if we don’t have replicate observations (as is often the case with continuous covariates).

The next steps in the analysis remove the insignificant regressors, sex and race. Because of possible multicollinearity it is always safer to this one step at a time. We first omit race as this variable has the smaller insignificant t-ratio (or, equivalently, the larger probability value). The output of the simplified model is given below:

Model without race:

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Yes       Success          165
          Failure         5254
Number    Total           5419

   65 cases were used
    7 cases contained missing values

Logistic Regression Table
                                                         Odds       95% CI
Predictor       Coef    SE Coef         Z     P    Ratio    Lower    Upper
Constant     -1.8483     0.1549   -11.93 0.000
Dust
 2           -2.6118     0.2864    -9.12 0.000     0.07     0.04     0.13
 3           -2.7623     0.2079   -13.29 0.000     0.06     0.04     0.09
Sex
 2            0.1247     0.2286     0.55 0.586     1.13     0.72     1.77
Smoking
 2           -0.6411     0.1944    -3.30 0.001     0.53     0.36     0.77
Employ
 2            0.5238     0.2512     2.08 0.037     1.69     1.03     2.76
 3            0.6904     0.1844     3.74 0.000     1.99     1.39     2.86

Log-Likelihood = -599.126
Test that all slopes are zero: G = 278.940, DF = 6, P-Value = 0.000

Goodness-of-Fit Tests

Method              Chi-Square    DF      P
Pearson                 28.316    27  0.395
Deviance                29.716    27  0.327
```

The factor sex is insignificant (t-ratio 0.55, and probability value 0.59), and is omitted in the next model.

Model without race and sex:

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Yes       Success          165
          Failure         5254
Number    Total           5419
```

```
   65 cases were used
    7 cases contained missing values

Logistic Regression Table
                                                        Odds       95% CI
Predictor        Coef    SE Coef         Z      P    Ratio    Lower    Upper
Constant     -1.8336     0.1525   -12.03 0.000
Dust
 2           -2.5493     0.2614    -9.75 0.000     0.08     0.05     0.13
 3           -2.7175     0.1898   -14.31 0.000     0.07     0.05     0.10
Smoking
 2           -0.6210     0.1908    -3.26 0.001     0.54     0.37     0.78
Employ
 2            0.5060     0.2490     2.03 0.042     1.66     1.02     2.70
 3            0.6728     0.1813     3.71 0.000     1.96     1.37     2.80

Log-Likelihood = -599.274
Test that all slopes are zero: G = 278.645, DF = 5, P-Value = 0.000

Goodness-of-Fit Tests

Method                 Chi-Square    DF       P
Pearson                    13.570    12  0.329
Deviance                   12.094    12  0.438
```

No other variables can be omitted. Smoking is an important contributor to byssinosis. For a non-smoker the odds of contracting byssinosis are 0.54 the odds of a smoker. Everything else equal, not smoking reduces the odds of contracting byssinosis by 46 percent.

The length of employment in the cotton industry matters. The odds that a worker with 10 to 20 years employment contracts byssinosis are 1.66 times the odds of a worker with less than ten years in the industry. The odds for a worker with more than 20 years are twice (1.96) the odds of a worker with less than ten years in the industry.

Dustiness of the workplace clearly matters. The odds of contracting byssinosis at workplaces with medium and low levels of dustiness are considerably smaller than the odds for workplaces with a high level of dustiness (they are 0.08 and 0.07 times the odds of workplaces with high level of dustiness).

Next, we explore whether it is necessary to include interactions. The model with the three factors - smoking, length of employment, and dustiness of the workplace - and all two-factor interactions is given below.

<u>Model with two-factor interactions:</u>

```
Link Function:  Logit
Response  Information

Variable  Value          Count
Yes       Success          165
          Failure         5254
Number    Total           5419
```

```
Factor Information

Factor   Levels Values
Dust          3 1 2 3
Smoking       2 1 2
Employ L      3 1 2 3

   65 cases were used
    7 cases contained missing values

Logistic Regression Table
                                                        Odds       95% CI
Predictor             Coef    SE Coef        Z      P   Ratio    Lower    Upper
Constant           -1.9545     0.1922   -10.17 0.000
Dust
 2                 -2.7064     0.4775    -5.67 0.000     0.07     0.03     0.17
 3                 -2.4646     0.3274    -7.53 0.000     0.09     0.04     0.16
Smoking
 2                 -0.7242     0.3516    -2.06 0.039     0.48     0.24     0.97
Employ
 2                  0.8287     0.3324     2.49 0.013     2.29     1.19     4.39
 3                  0.9904     0.2551     3.88 0.000     2.69     1.63     4.44
Dust*Smoking
 2*2                1.1956     0.5501     2.17 0.030     3.31     1.12     9.72
 3*2                0.4546     0.4375     1.04 0.299     1.58     0.67     3.71
Dust*Employ
 2*2               -0.1908     0.7751    -0.25 0.806     0.83     0.18     3.78
 2*3               -0.5094     0.5881    -0.87 0.386     0.60     0.19     1.90
 3*2               -1.0915     0.6432    -1.70 0.090     0.34     0.10     1.18
 3*3               -0.4572     0.4103    -1.11 0.265     0.63     0.28     1.41
Smoking*Employ
 2*2               -0.0556     0.6162    -0.09 0.928     0.95     0.28     3.16
 2*3               -0.4911     0.4183    -1.17 0.240     0.61     0.27     1.39

Tests for terms with more than 1 degree of freedom

Term              Chi-Square    DF      P
Dust                  73.005     2  0.000
Employ                16.025     2  0.000
Dust*Smoking           4.863     2  0.088
Dust*Employ            3.712     4  0.446
Smoking*Employ         1.473     2  0.479

Log-Likelihood = -593.735
Test that all slopes are zero: G = 289.723, DF = 13, P-Value = 0.000

Goodness-of-Fit Tests

Method                Chi-Square    DF      P
Pearson                    1.005     4  0.909
Deviance                   1.016     4  0.907
```

The interactions between dust and employment length and between smoking history and employment length matter little, and are omitted from the model at the next step. The chi-square tests for the Dust*EmployLength interaction is 3.712 with probability value 0.446, and the Smoking*EmployLength interaction is 1.473 with probability value 0.479. These chi-square tests compare the full model with the model that restricts the interactions under consideration to zero.

Fitting the simpler model with the three factors smoking, length of employment, and dustiness of the workplace and the remaining 2-factor interaction between dust and smoking is shown below.

<u>Model with the dustiness by smoking interaction:</u>

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Yes       Success          165
          Failure         5254
Number    Total           5419

Factor Information

Factor   Levels Values
Dust          3 1 2 3
Smoking       2 1 2
Employ L      3 1 2 3

   65 cases were used
    7 cases contained missing values


Logistic Regression Table
                                                         Odds        95% CI
Predictor           Coef    SE Coef         Z     P    Ratio    Lower    Upper
Constant         -1.7573     0.1555   -11.30 0.000
Dust
 2               -2.9576     0.3565    -8.30 0.000     0.05     0.03     0.10
 3               -2.8325     0.2230   -12.70 0.000     0.06     0.04     0.09
Smoking
 2               -0.9573     0.2751    -3.48 0.001     0.38     0.22     0.66
Employ
 2                0.4990     0.2499     2.00 0.046     1.65     1.01     2.69
 3                0.6638     0.1819     3.65 0.000     1.94     1.36     2.77
Dust*Smoking
 2*2              1.1807     0.5490     2.15 0.031     3.26     1.11     9.55
 3*2              0.4864     0.4338     1.12 0.262     1.63     0.69     3.81

Tests for terms with more than 1 degree of freedom

Term          Chi-Square    DF      P
Dust             198.232     2  0.000
Employ            13.717     2  0.001
Dust*Smoking       4.840     2  0.089

Log-Likelihood = -596.848
Test that all slopes are zero: G = 283.496, DF = 7, P-Value = 0.000

Goodness-of-Fit Tests

Method                Chi-Square    DF      P
Pearson                    7.289    10  0.698
Deviance                   7.243    10  0.702
```

We illustrate in detail how one can test whether the interaction is significant. Comparing the log-likelihood = -596.848 of the full model with the log-likelihood of the restricted model (model without the interaction; log-likelihood = -599.274) leads to the log-likelihood ratio test statistic $2(-596.848-(-599.274)) = 4.84$. Relating this statistic to a chi-square distribution with 2 degrees of freedom leads to the probability value $P(\chi^2(2) \geq 4.84) = 0.089$. Note that the test-statistic (4.84) and the probability value (0.089) are given in the previous computer output. Since the probability value is larger than 0.05, we conclude that the interaction is not significant. Of course, at the ten percent significance level one would conclude that there is a smoking by dustiness interaction effect on the odds of contracting byssinosis. While there is some evidence of an interaction, the evidence is certainly not very strong.

How would one interpret the coefficients and the odds-ratios in the interaction component? One can write out the logistic regression model with the interaction terms and look at the odds for fixed levels of dustiness of the workplace.

(i) Comparing the odds for a non-smoker at a high-level dusty workplace (dust level 1), exp(constant - 0.9573), to those of a smoker at a high-level dusty workplace, exp(constant), leads to the odds-ratio exp(-0.9573) = 0.38. At a dusty workplace, nonsmoking reduces the odds of contracting byssinosis by 62 percent.

(ii) The odds-ratio for a non-smoker at a medium-level dusty workplace (dust level 2) is 0.38exp(1.1807) = (0.38)(3.26) = 1.25. At a medium-level dusty workplace the odds of contracting byssinosis for smokers and non-smokers are about the same. At medium-level dusty workplaces the smoking history has little influence on the odds of contracting the disease.

(iii) The odds-ratio for a non-smoker at a low-level dusty workplace (dust level 3) is 0.38exp(0.4864) = (0.38)(1.63) = 0.62. However, note the confidence interval for the interaction effect for (non)smoking and low dustiness (level 3) is quite wide (extending from 0.69 to 3.81) making the interpretation for low-level dustiness quite uncertain. The odds of contracting byssinosis for smokers and non-smokers may in fact be the same.

In summary, nonsmoking reduces the odds of contracting byssinosis, and the reduction is largest in very dusty workplaces.

11.4
Occurrence of proteinurea only:

Model with Smoking and Class: The test statistic for testing the overall significance of the logistic regression (in equation (11.25)) is G = 83.82. The sampling distribution (under the null hypotheses that none of the regressors have an influence on the response) is chi-square with 6 degrees of freedom. The test statistic is large compared to the percentiles from that distribution and the probability value is small (p value < 0.001). Hence the regressor variables (all or some) have a significant impact on the presence of proteinurea.

The deviance (in equation (11.26)) and the Pearson statistic (in equation (11.31)) compare the fit of the parameterized model (here with 7 = 6 + 1 (for constant) parameters) with the fit of the saturated model where each constellation of the explanatory variables is allowed its own distinct success probability. Here there are 15 = (5)(3) constellations. The deviance is D = 15.35 and the Pearson statistic is $\chi^2 = 16.08$. Large values of these statistics indicate model inadequacy; the appropriate reference distribution is chi-square with 15 – 7 = 8 degrees of freedom. The deviance and the Pearson statistic are roughly the same size as the critical percentile (the 95th percentile is 15.51), implying probability values that are about 0.05. This leaves some doubt whether the model is adequate.

Individually, the coefficients for the four classes (class 2 through 5) are insignificant. These four coefficients express the incremental effect of class 2 through class 5, with class 1 acting as the standard. One can see the insignificance from the odds-ratios (they are roughly one, hence not changing the odds of class 1), their t-ratios (Z-scores), and the associated probability values. The probability values exceed the usual cutoff 0.05, with the one for the second class coming closest to 0.05 (it is 0.087). All four confidence intervals of their odds-ratios cover the value one (even odds).

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Proteinu  Success         2715
          Failure        10669
Total     Total          13384

Factor Information

Factor   Levels Values
Smoking       3 1 2 3
Class         5 1 2 3 4 5

Logistic Regression Table
                                                  Odds       95% CI
Predictor       Coef    SE Coef         Z     P   Ratio    Lower    Upper
Constant     -1.2964     0.1078   -12.03 0.000
Smoking
 2          -0.38319    0.04770    -8.03 0.000    0.68     0.62     0.75
 3          -0.26838    0.09115    -2.94 0.003    0.76     0.64     0.91
Class
 2            0.2102     0.1227     1.71 0.087    1.23     0.97     1.57
 3            0.0802     0.1112     0.72 0.471    1.08     0.87     1.35
 4           -0.0088     0.1222    -0.07 0.943    0.99     0.78     1.26
 5            0.0071     0.1386     0.05 0.959    1.01     0.77     1.32
```

```
Tests for terms with more than 1 degree of freedom

Term      Chi-Square    DF       P
Smoking       66.685     2  0.000
Class          8.385     4  0.078

Log-Likelihood = -6708.093
Test that all slopes are zero: G = 83.819, DF = 6, P-Value = 0.000

Goodness-of-Fit Tests

Method                 Chi-Square    DF       P
Pearson                    16.077     8  0.041
Deviance                   15.351     8  0.053
```

Model with Smoking Only: The next step in the analysis is to remove the factor "class" from the model (that is, omitting all four class indicators). We can test whether the factor class (with its five categories) is significant.

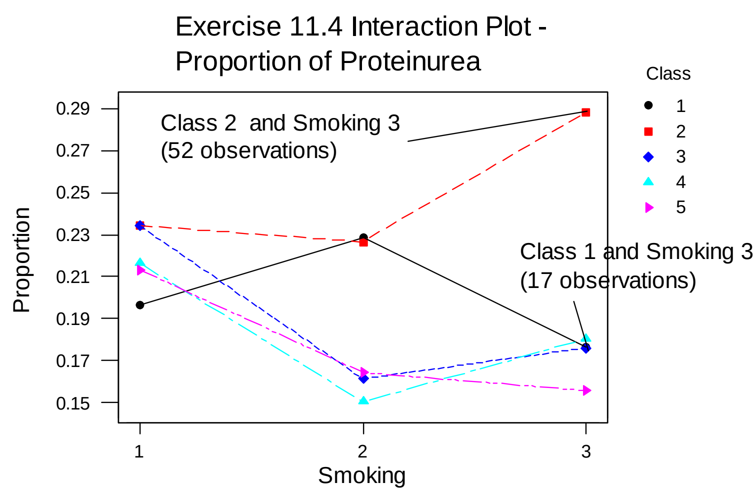
Comparing the log-likelihood -6,708.093 of the full model with the log-likelihood of the restricted model (model without class; log-likelihood = -6,712.254) leads to the log-likelihood ratio test statistic $2(-6{,}708.093-(-6{,}712.254))=8.38$. Relating the test statistic to a chi-square distribution with 4 degrees of freedom leads to the probability value $P(\chi^2(4)\geq 8.38)=0.078$. Since this probability value is larger than 0.05, we conclude that the factor "class" is insignificant. "Class" can be omitted from the model. Note that the test statistic and its probability value are part of the earlier output for the model with both smoking and class.

The odds-ratios for smoking (0.67 and 0.75) imply that smoking is beneficial in reducing the onset of proteinurea. It seems beneficial for mothers to smoke!! Other studies also found that toxemia is less frequent in smokers than in non-smokers. The medical explanation for this is unclear. Brown et al quote evidence that nicotine dilates the muscle capillaries. Furthermore, research suggests that the cyanide in tobacco is detoxicated in the body to thiocyanate which has a known effect on hypertension and may be the active agent in reducing toxaemia.

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Proteinu  Success         2715
          Failure        10669
Total     Total          13384

Factor Information

Factor   Levels Values
Smoking       3 1 2 3
```

```
Logistic Regression Table
                                                        Odds       95% CI
Predictor        Coef     SE Coef        Z      P      Ratio     Lower     Upper
Constant     -1.21512     0.02730   -44.51  0.000
Smoking
 2           -0.39654     0.04716    -8.41  0.000       0.67      0.61      0.74
 3           -0.29167     0.09052    -3.22  0.001       0.75      0.63      0.89

Tests for terms with more than 1 degree of freedom

Term       Chi-Square    DF       P
Smoking        73.867     2   0.000

Log-Likelihood = -6712.254
Test that all slopes are zero: G = 75.498, DF = 2, P-Value = 0.000

* NOTE * No goodness of fit tests performed.
       * The model uses all degrees of freedom.
```

<u>Comment:</u> The model with smoking and class considered above is barely adequate, with goodness-of-fit statistics right at the critical 95th percentile. This fact may be the result of an interaction effect. The following interaction plot shows that this lack of fit may originate from the data for class 1 and 2 at smoking level 3. Unfortunately these cells are the ones with the smallest numbers of trials, and the somewhat unusual proportions at these cells may be an artifact of the small sample size.



Occurrence of hypertension only:

<u>Model with Smoking and Class:</u> The test statistic for testing the overall significance of the regression (in equation (11.25)) is G = 29.27 (with probability value = 0.000). Hence the regressor variables (all or some) have a significant impact on the presence of hypertension. The deviance D = 8.1 and the Pearson statistic $\chi^2 = 6.9$, and their respective probability values 0.42 and 0.55, give us no reason to question the adequacy of the model.

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Hyperten  Success          589
          Failure        12795
Total     Total          13384

Factor Information
Factor   Levels Values
Smoking       3 1 2 3
Class         5 1 2 3 4 5

Logistic Regression Table
                                                     Odds       95% CI
Predictor        Coef    SE Coef         Z     P    Ratio    Lower    Upper
Constant      -3.0430     0.2024    -15.04 0.000
Smoking
 2            0.23179    0.08989      2.58 0.010     1.26     1.06     1.50
 3             0.3339     0.1575      2.12 0.034     1.40     1.03     1.90
Class
 2            -0.3829     0.2442     -1.57 0.117     0.68     0.42     1.10
 3            -0.2277     0.2095     -1.09 0.277     0.80     0.53     1.20
 4             0.0255     0.2254      0.11 0.910     1.03     0.66     1.60
 5             0.2582     0.2431      1.06 0.288     1.29     0.80     2.08

Log-Likelihood = -2400.886
Test that all slopes are zero: G = 29.270, DF = 6, P-Value = 0.000

Goodness-of-Fit Tests

Method             Chi-Square    DF      P
Pearson                 6.904     8  0.547
Deviance                8.122     8  0.422
```

<u>Model with Smoking only:</u> We assess whether it is possible to omit the factor "class" from the model. Comparing the log-likelihood -2,400.886 of the full model with the log-likelihood of the restricted model (model without class; log-likelihood = -2,409.267) leads to the log-likelihood ratio test statistic $2(-2{,}400.886-(-2{,}409.267))=16.76$. Relating it to a chi-square distribution with 4 degrees of freedom leads to the probability value $\text{P}(\chi^2(4)\geq 16.76)=0.0022$. Since this probability value is small, we conclude that the factor "class" is significant. It cannot be omitted from the model.

Smoking increases the odds for hypertension (odds-ratios of 1.26 and 1.40).

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Hyperten  Success          589
          Failure        12795
Total     Total          13384
```

```
Logistic Regression Table
                                                        Odds       95% CI
Predictor       Coef    SE Coef        Z      P    Ratio    Lower    Upper
Constant    -3.21356    0.05948   -54.03 0.000
Smoking
 2           0.27020    0.08861     3.05 0.002     1.31     1.10     1.56
 3            0.3966     0.1559     2.54 0.011     1.49     1.10     2.02

Log-Likelihood = -2409.276
Test that all slopes are zero: G = 12.492, DF = 2, P-Value = 0.002

* NOTE * No goodness of fit tests performed.
       * The model uses all degrees of freedom.
```

Occurrence of both hypertension and proteinurea:

Model with smoking and class:

```
Link Function:  Logit

Response  Information


Variable  Value          Count

Both hyp  Success          665
          Failure        12719
Total     Total          13384

Factor Information

Factor   Levels Values
Smoking       3 1 2 3
Class         5 1 2 3 4 5

Logistic Regression Table
                                                        Odds       95% CI
Predictor       Coef    SE Coef        Z      P    Ratio    Lower    Upper
Constant     -2.6142     0.1779   -14.70 0.000
Smoking
 2          -0.40768    0.08910    -4.58 0.000     0.67     0.56     0.79
 3           -0.5793     0.1903    -3.04 0.002     0.56     0.39     0.81
Class
 2           -0.4695     0.2191    -2.14 0.032     0.63     0.41     0.96
 3           -0.1641     0.1849    -0.89 0.375     0.85     0.59     1.22
 4           -0.1036     0.2049    -0.51 0.613     0.90     0.60     1.35
 5           -0.0101     0.2321    -0.04 0.965     0.99     0.63     1.56

Tests for terms with more than 1 degree of freedom

Term      Chi-Square    DF      P
Smoking       26.488     2  0.000
Class          7.884     4  0.096

Log-Likelihood = -2627.725
Test that all slopes are zero: G = 33.644, DF = 6, P-Value = 0.000
```

```
Goodness-of-Fit Tests

Method              Chi-Square    DF       P
Pearson                  6.673     8   0.572
Deviance                 7.240     8   0.511
```

Model with Smoking Only:

```
Link Function:  Logit

Response  Information

Variable  Value          Count
Both hyp  Success          665
          Failure        12719
Total     Total          13384

Factor Information

Factor   Levels Values
Smoking       3 1 2 3



Logistic Regression Table
                                                    Odds       95% CI
Predictor       Coef    SE Coef         Z      P    Ratio    Lower    Upper
Constant    -2.79260    0.04917    -56.80  0.000
Smoking
 2          -0.38545    0.08809     -4.38  0.000     0.68     0.57     0.81
 3           -0.5453     0.1893     -2.88  0.004     0.58     0.40     0.84


Log-Likelihood = -2631.881
Test that all slopes are zero: G = 25.332, DF = 2, P-Value = 0.000

* NOTE * No goodness of fit tests performed.
       * The model uses all degrees of freedom.
```

The factor “class” can be omitted from the model. Smoking decreases the odds of developing both hypertension and proteinurea.

Occurrence of either hypertension or proteinurea (or both):

Model with Smoking and Class:

```
Link Function:  Logit

Response  Information

Variable  Value          Count
EitherOr  Success         3969
          Failure         9415
Total     Total          13384
```

Factor Information

Factor	Levels	Values
Smoking	3	1 2 3
Class	5	1 2 3 4 5

Logistic Regression Table

Predictor	Coef	SE Coef	Z	P	Odds Ratio	95% CI Lower	95% CI Upper
Constant	-0.71346	0.09336	-7.64	0.000			
Smoking							
2	-0.33729	0.04142	-8.14	0.000	0.71	0.66	0.77
3	-0.25676	0.07919	-3.24	0.001	0.77	0.66	0.90
Class							
2	-0.0080	0.1078	-0.07	0.941	0.99	0.80	1.23
3	-0.02587	0.09641	-0.27	0.788	0.97	0.81	1.18
4	-0.0239	0.1056	-0.23	0.821	0.98	0.79	1.20
5	0.0748	0.1187	0.63	0.529	1.08	0.85	1.36

Tests for terms with more than 1 degree of freedom

Term	Chi-Square	DF	P
Smoking	69.021	2	0.000
Class	1.811	4	0.770

Log-Likelihood = -8100.026
Test that all slopes are zero: G = 72.512, DF = 6, P-Value = 0.000

Goodness-of-Fit Tests

Method	Chi-Square	DF	P
Pearson	13.141	8	0.107
Deviance	12.867	8	0.117

Model with Smoking Only:

Link Function: Logit

Response Information

Variable	Value	Count
EitherOr	Success	3969
	Failure	9415
Total	Total	13384

Factor Information

Factor	Levels	Values
Smoking	3	1 2 3

Logistic Regression Table

Predictor	Coef	SE Coef	Z	P	Odds Ratio	95% CI Lower	95% CI Upper
Constant	-0.73006	0.02448	-29.82	0.000			
Smoking							
2	-0.33465	0.04093	-8.18	0.000	0.72	0.66	0.78
3	-0.25381	0.07864	-3.23	0.001	0.78	0.67	0.91

```
Tests for terms with more than 1 degree of freedom

Term      Chi-Square    DF      P
Smoking       69.868     2  0.000

Log-Likelihood = -8100.923
Test that all slopes are zero: G = 70.719, DF = 2, P-Value = 0.000

* NOTE * No goodness of fit tests performed.
       * The model uses all degrees of freedom.
```

The factor “class” has no influence on the odds of developing either hypertension or proteinurea. Smoking decreases the odds of developing either one of these conditions.

CHAPTER 12

A note on computing with SAS (Version 9):

The **SAS GENMOD** procedure is used for fitting the Poisson regression models of Chapter 12. This procedure is very general. It can also be used for the logistic regression models in Chapter 11, as well as most generalized linear models.

SAS works slightly different than the previously considered spreadsheet programs Minitab, SPSS, or EXCEL. In SAS one needs to write out a line code. The line code gets entered into a program editor, and is executed by clicking the SAS "run" and "submit" tabs. Here we list an example of the line code, with a detailed discussion of important options. Many more options are available, and they can be reviewed by looking at the on-line help pages within SAS.

We list the input for Exercise 12.1:

```
data exer12n1;
```
specifies the file name for data set
```
input type year period ms nudamage;
```
specifies the input variables
```
lnms=log(ms);
```
specifies a transformation; here the natural log transformation
```
datalines;
1     1     1     127    0
1     1     2     63     0
1     2     1     1095   3
1     2     2     1095   4
1     3     1     1512   6
1     3     2     3353   18
1     4     2     2244   11
2     1     1     44882  39
2     1     2     17176  29
2     2     1     28609  58
2     2     2     20370  53
2     3     1     7064   12
2     3     2     13099  44
2     4     2     7117   18
3     1     1     1179   1
3     1     2     552    1
3     2     1     781    0
3     2     2     676    1
3     3     1     783    6
3     3     2     1948   2
3     4     2     274    1
```

```
4     1     1     251    0
4     1     2     105    0
4     2     1     288    0
4     2     2     192    0
4     3     1     349    2
4     3     2     1208   11
4     4     2     2051   4
5     1     1     45     0
5     2     1     789    7
5     2     2     437    7
5     3     1     1157   5
5     3     2     2161   12
5     4     2     542    1
;
proc genmod data=exer12n1;
```

PROC GENMOD is called

```
class type / param=ref ref=first;
class year / param=ref ref=first;
class period / param=ref ref=first;
```

specifies that type, year, and period are class (factor) variables; SAS creates the appropriate indicator variables automatically. The first numeric value is taken as the base for comparisons.

```
model nudamage=type year period lnms / d=poisson obstats
covb corrb lrci type3;
```

Here the model gets specified. The response is nudamage. The first three variables on the right hand side of the equal sign are factors. The last variable (lnms) is a covariate (not a factor). Options are listed after the slash.

d=Poisson: Poisson link.

Covb, Corrb: Covariance and correlation matrices of the parameter estimates are displayed.

Obstats: results in detailed output (fitted values, residuals, …)

Lrci requests that two-sided confidence intervals for all model parameters are computed based on the profile likelihood function. This is sometimes called the partially maximized likelihood function. Two-sided Wald confidence intervals are calculated, if lrci is not specified.

Likelihood ratio-based confidence intervals, also known as profile likelihood confidence intervals, of parameter estimates in generalized linear models can be explained as follows. Suppose that the parameter vector is $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_p)'$ and one wants a confidence interval for β_i. The profile likelihood function for β_i is defined as $l^*(\beta_i) = \max_{\tilde{\boldsymbol{\beta}}} l(\boldsymbol{\beta})$, where $\tilde{\boldsymbol{\beta}}$ is the vector $\boldsymbol{\beta}$ with the ith element fixed at β_i and $l = l(\boldsymbol{\beta})$ is the log likelihood function. Let $l = l(\hat{\boldsymbol{\beta}})$ be the log likelihood evaluated at the maximum likelihood estimate $\hat{\boldsymbol{\beta}}$. Under the assumption that β_i is the true parameter value, $2(l - l^*(\beta_i))$ has a limiting chi-square distribution with one degree of freedom. A $100(1-\alpha)$ percent confidence interval for β_i is

$$\{\beta_i : l^*(\beta_i) \geq l - 0.5\chi^2(1-\alpha;1)\}$$

where $\chi^2(1-\alpha;1)$ is the $100(1-\alpha)$ percentile of the chi-square distribution with one degree of freedom. The endpoints of the confidence interval can be found by solving numerically for values of β_i that satisfy the equality in the preceding relation.

Type 3: requests that statistics for Type 3 contrasts be computed for each class variable (factor) specified in the MODEL statement. This means that likelihood-ratio tests are calculated for the contrasts of the class variables.Type 3 means that these are partial tests, comparing the full model with the restricted model that lacks the indicated class variable (factor).

OFFSET = lnms: specifies a variable in the input data set (here lnms) to be used as an offset variable. This variable cannot be a CLASS variable. In our example it seems reasonable to suppose that the number of damage incidents is directly proportional to MS, the months of service, and one can expect that the coefficient in the Poisson regression model that corresponds to ln(MS) is one. OFFSET = lnms restricts this parameter to one.

Scale = deviance: Overdispersion is a phenomenon that sometimes occurs in data that are modeled with the Poisson (and also binomial - see Chapter 11) distributions. If the estimate of dispersion after fitting, as measured by the deviance or Pearson's chi-square divided by the degrees of freedom, is not near 1, then the data may be overdispersed if the dispersion estimate is greater than 1, or underdispersed if the dispersion estimate is less than 1. A simple way to model this situation is to allow the variance function of the Poisson distribution to have a multplicative overdispersion factor, $\text{Var}(\mu) = \phi\mu$ (or $\text{Var}(\mu) = \phi\mu(1-\mu)$ for the binomial link).

The models are fit in the usual way. The parameter estimates are not affected by the value of ϕ. The covariance matrix, however, is multiplied by ϕ, and the scaled deviance and log likelihoods used in likelihood ratio tests are divided by ϕ.

The SCALE= option in the MODEL statement enables you to specify a value of ϕ for the Poisson (and also binomial) distributions. If you specify the SCALE=DEVIANCE option in the MODEL statement, the procedure uses the deviance divided by the degrees of freedom as an estimate of ϕ, and all statistics are adjusted appropriately. You can use Pearson's chi-square instead of the deviance by specifying the SCALE=PEARSON option.

```
run;
```

Executes the program

Many other options are available. See the SAS on-line help for further discussion and examples.

12.1

(a) We use SAS GENMOD to estimate the Poisson regression model with link

$$\ln \mu = \beta_0 + \beta_1 \ln(\text{MS}) + \beta_2 \text{X2} + ... + \beta_5 \text{X5} + \beta_6 \text{Z2} + ... + \beta_8 \text{Z4} + \beta_9 \text{W2}$$

Here X1 through X5 are the indicator variables for the type of ship (a class variable with five possibilities), Z1 through Z4 are the indicator variables for the year of construction (a class variable with four possibilities), and W1 and W2 are the indicator variables for the period of operation (a class variable with two possibilities). SAS GENMOD creates the associated indicator variables for the specified class variables automatically. The first outcome is declared as the reference.

The (type 3) test statistics at the end of the program output test the significance of the class variables. For example, the test statistic for “type” is obtained by comparing the log-likelihood of the full model (768.4585) with the log-likelihood of the restricted model that is missing that factor (the model with year, period, and ln(MS)). The log-likelihood of the restricted model is 762.1757. Hence the log-likelihood statistic is 2(768.4582 - 762.1757) = 12.57. Comparing this value to a chi-square with 4 degrees of freedom (since there are 4 restrictions), leads to the probability value

$P(\chi^2(4) \geq 12.57) = 0.0136$. These are the values given at the end of the output. The tests for the other factors can be obtained similarly. They indicate that one can not simplify the model. All three factors are needed to explain the number of damage claims.

Ships of type 3 report the smallest number of damage incidents. Ships constructed in years 2 (1965-1969) and 3 (1970-1974) experience the highest number of reported damage incidents. The second period of operation (1975-79) is associated with a higher number of reported damage incidents.

<u>Fitting results for the full model:</u>

```
                    The GENMOD Procedure

                     Model Information

           Data Set              WORK.EXER12N1
           Distribution                Poisson
           Link Function                   Log
           Dependent Variable         nudamage
           Observations Used                34
                 Class Level Information

       Class      Value         Design Variables

       type       1           0      0      0      0
                  2           1      0      0      0
                  3           0      1      0      0
                  4           0      0      1      0
                  5           0      0      0      1

       year       1           0      0      0
                  2           1      0      0
                  3           0      1      0
                  4           0      0      1

       period     1           0
                  2           1

                  Parameter Information

   Parameter       Effect       type    year    period

   Prm1            Intercept
   Prm2            lnms
   Prm3            type         2
   Prm4            type         3
   Prm5            type         4
   Prm6            type         5
   Prm7            year                 2
   Prm8            year                 3
   Prm9            year                 4
   Prm10           period                       2
```

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	24	37.8043	1.5752
Scaled Deviance	24	37.8043	1.5752
Pearson Chi-Square	24	39.4494	1.6437
Scaled Pearson X2	24	39.4494	1.6437
Log Likelihood		768.4585	

Algorithm converged.

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6	Prm7	Prm8	Prm9	Prm10
Prm1	1.0000	-0.9688	0.6048	-0.3172	-0.3046	-0.3304	-0.3405	-0.4538	-0.4298	-0.1729
Prm2	-0.9688	1.0000	-0.7587	0.2328	0.2200	0.2234	0.2291	0.3364	0.3495	0.1216
Prm3	0.6048	-0.7587	1.0000	0.0990	0.1226	0.1958	-0.1165	-0.0967	-0.1341	-0.0768
Prm4	-0.3172	0.2328	0.0990	1.0000	0.2798	0.3483	0.0899	0.1225	0.1660	0.0258
Prm5	-0.3046	0.2200	0.1226	0.2798	1.0000	0.3706	0.0788	0.1001	0.0024	0.0225
Prm6	-0.3304	0.2234	0.1958	0.3483	0.3706	1.0000	0.0466	0.0428	0.1200	0.0522
Prm7	-0.3405	0.2291	-0.1165	0.0899	0.0788	0.0466	1.0000	0.6612	0.5146	-0.0770
Prm8	-0.4538	0.3364	-0.0967	0.1225	0.1001	0.0428	0.6612	1.0000	0.5938	-0.1854
Prm9	-0.4298	0.3495	-0.1341	0.1660	0.0024	0.1200	0.5146	0.5938	1.0000	-0.2444
Prm10	-0.1729	0.1216	-0.0768	0.0258	0.0225	0.0522	-0.0770	-0.1854	-0.2444	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-5.5940	0.8724	-7.3038	-3.8841	41.12	<.0001
lnms		1	0.9027	0.1018	0.7032	1.1022	78.63	<.0001
type	2	1	-0.3499	0.2702	-0.8795	0.1797	1.68	0.1954
type	3	1	-0.7631	0.3382	-1.4259	-0.1003	5.09	0.0240
type	4	1	-0.1355	0.2971	-0.7178	0.4469	0.21	0.6484
type	5	1	0.2739	0.2418	-0.1999	0.7478	1.28	0.2572
year	2	1	0.6625	0.1536	0.3614	0.9637	18.60	<.0001
year	3	1	0.7597	0.1777	0.4115	1.1079	18.29	<.0001
year	4	1	0.3697	0.2458	-0.1121	0.8516	2.26	0.1326
period	2	1	0.3703	0.1181	0.1387	0.6018	9.82	0.0017
Scale		0	1.0000	0.0000	1.0000	1.0000		

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
lnms	1	101.28	<.0001
type	4	12.57	0.0136
year	3	27.20	<.0001
period	1	9.97	0.0016

<u>Fitting results for the restricted model without type of ship:</u>

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N1
Distribution	Poisson
Link Function	Log
Dependent Variable	nudamage
Observations Used	34

Class Level Information

Class	Value	Design Variables		
year	1	0	0	0
	2	1	0	0
	3	0	1	0
	4	0	0	1
period	1	0		
	2	1		

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	28	50.3699	1.7989
Scaled Deviance	28	50.3699	1.7989
Pearson Chi-Square	28	46.7116	1.6683
Scaled Pearson X2	28	46.7116	1.6683
Log Likelihood		762.1757	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-5.2229	0.4826	-6.1688	-4.2771	117.12	<.0001
lnms		1	0.8311	0.0460	0.7409	0.9213	326.13	<.0001
year	2	1	0.6735	0.1503	0.3790	0.9681	20.08	<.0001
year	3	1	0.7967	0.1702	0.4631	1.1303	21.91	<.0001
year	4	1	0.3978	0.2337	-0.0603	0.8560	2.90	0.0887
period	2	1	0.3546	0.1168	0.1256	0.5837	9.21	0.0024
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

(b) It seems reasonable to suppose that the number of damage incidents is directly proportional to MS, the months of service, and one can expect that the coefficient β_1 is one. The literature refers to the term $\ln(\text{MS})$ as an "offset." Let us test for the offset, and test whether $\beta_1 = 1$. The estimate is $\hat{\beta}_1 = 0.9027$, and the 95 percent Wald confidence interval is given by $0.9027 \pm (1.96)(0.1018)$, 0.90 ± 0.20, or $0.70 \le \beta_1 \le 1.10$. The interval includes one, which makes the off-set interpretation plausible.

(c) We assume an "offset" for aggregate months of service (that is, we impose the restriction $\beta_1 = 1$) and estimate the model with link

$$\ln \mu = \beta_0 + \ln(\text{MS}) + \beta_2 \text{X2} + ... + \beta_5 \text{X5} + \beta_6 \text{Z2} + ... + \beta_8 \text{Z4} + \beta_9 \text{W2}$$

The results of the estimation are similar to the ones of the full model in (a).

Fitting results for the model with an offset:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N1
Distribution	Poisson
Link Function	Log
Dependent Variable	nudamage
Offset Variable	lnms
Observations Used	34

Class Level Information

Class	Value	Design Variables			
type	1	0	0	0	0
	2	1	0	0	0
	3	0	1	0	0
	4	0	0	1	0
	5	0	0	0	1
year	1	0	0	0	
	2	1	0	0	
	3	0	1	0	
	4	0	0	1	
period	1	0			
	2	1			

Parameter Information

Parameter	Effect	type	year	period
Prm1	Intercept			
Prm2	type	2		
Prm3	type	3		
Prm4	type	4		
Prm5	type	5		
Prm6	year		2	
Prm7	year		3	
Prm8	year		4	
Prm9	period			2

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	25	38.6951	1.5478
Scaled Deviance	25	38.6951	1.5478
Pearson Chi-Square	25	42.2753	1.6910
Scaled Pearson X2	25	42.2753	1.6910
Log Likelihood		768.0131	

Algorithm converged.

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6	Prm7	Prm8	Prm9
Prm1	1.0000	-0.8114	-0.3784	-0.3706	-0.4699	-0.4843	-0.5501	-0.4015	-0.2161
Prm2	-0.8114	1.0000	0.4332	0.4468	0.5707	0.0856	0.2714	0.2285	0.0254
Prm3	-0.3784	0.4332	1.0000	0.2375	0.3136	0.0358	0.0455	0.0971	-0.0031
Prm4	-0.3706	0.4468	0.2375	1.0000	0.3338	0.0277	0.0286	-0.0966	-0.0047
Prm5	-0.4699	0.5707	0.3136	0.3338	1.0000	-0.0041	-0.0371	0.0528	0.0269

Prm6	-0.4843	0.0856	0.0358	0.0277	-0.0041	1.0000	0.6335	0.4755	-0.1201
Prm7	-0.5501	0.2714	0.0455	0.0286	-0.0371	0.6335	1.0000	0.5482	-0.2636
Prm8	-0.4015	0.2285	0.0971	-0.0966	0.0528	0.4755	0.5482	1.0000	-0.3154
Prm9	-0.2161	0.0254	-0.0031	-0.0047	0.0269	-0.1201	-0.2636	-0.3154	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-6.4059	0.2174	-6.8321	-5.9797	867.89	<.0001
type	2	1	-0.5433	0.1776	-0.8914	-0.1953	9.36	0.0022
type	3	1	-0.6874	0.3290	-1.3323	-0.0425	4.36	0.0367
type	4	1	-0.0760	0.2906	-0.6455	0.4936	0.07	0.7938
type	5	1	0.3256	0.2359	-0.1367	0.7879	1.91	0.1675
year	2	1	0.6971	0.1496	0.4038	0.9904	21.70	<.0001
year	3	1	0.8184	0.1698	0.4857	1.1512	23.24	<.0001
year	4	1	0.4534	0.2332	-0.0036	0.9104	3.78	0.0518
period	2	1	0.3845	0.1183	0.1527	0.6163	10.57	0.0012
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
type	4	23.67	<.0001
year	3	31.41	<.0001
period	1	10.66	0.0011

(d) Let us look at the deviance goodness-of-fit statistics. Comparing the deviance D = 37.8043 to a chi-square with 24 degrees of freedom, leads to the probability value $P(\chi^2(24) \geq 37.80) = 1 - 0.9637 = 0.0363$. The deviance exceeds the 95th percentile and the probability value is slightly smaller than 0.05. This is a sign of overdispersion. We adjust the analysis for overdispersion by allowing the variance function of the Poisson distribution to have a multplicative overdispersion factor, $\text{Var}(\mu) = \phi\mu$. The model is fit in the usual way, and the parameter estimates are not affected by the value of ϕ. The covariance matrix, however, is multiplied by ϕ, and the scaled deviance and log likelihoods used in likelihood ratio tests are divided by ϕ. The SCALE=DEVIANCE option in the MODEL statement enables us to specify a value of ϕ for the Poisson distribution. The procedure uses the deviance divided by the degrees of freedom as an estimate of ϕ, and all statistics are adjusted appropriately.

The results are basically unchanged. The test statistics indicate that all three factors are statistically significant. Ships of types 2 and 3 experience the smallest numbers of reported damage incidents. Ships constructed in years 2 (1965-1969) and 3 (1970-1974) experience the largest numbers of reported damage incidents. The second period of operation (1975-79) is associated with a higher number of reported damage incidents.

Fitting results for the model with scale adjustment:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N1
Distribution	Poisson
Link Function	Log
Dependent Variable	nudamage
Offset Variable	lnms
Observations Used	34

Class Level Information

Class	Value	Design Variables			
type	1	0	0	0	0
	2	1	0	0	0
	3	0	1	0	0
	4	0	0	1	0
	5	0	0	0	1
year	1	0	0	0	
	2	1	0	0	
	3	0	1	0	
	4	0	0	1	
period	1	0			
	2	1			

Parameter Information

Parameter	Effect	type	year	period
Prm1	Intercept			
Prm2	type	2		
Prm3	type	3		
Prm4	type	4		
Prm5	type	5		
Prm6	year		2	
Prm7	year		3	
Prm8	year		4	
Prm9	period			2

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	25	38.6951	1.5478
Scaled Deviance	25	25.0000	1.0000
Pearson Chi-Square	25	42.2753	1.6910
Scaled Pearson X2	25	27.3131	1.0925
Log Likelihood		496.1960	

Algorithm converged.

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6	Prm7	Prm8	Prm9
Prm1	1.0000	-0.8114	-0.3784	-0.3706	-0.4699	-0.4843	-0.5501	-0.4015	-0.2161
Prm2	-0.8114	1.0000	0.4332	0.4468	0.5707	0.0856	0.2714	0.2285	0.0254
Prm3	-0.3784	0.4332	1.0000	0.2375	0.3136	0.0358	0.0455	0.0971	-0.0031
Prm4	-0.3706	0.4468	0.2375	1.0000	0.3338	0.0277	0.0286	-0.0966	-0.0047
Prm5	-0.4699	0.5707	0.3136	0.3338	1.0000	-0.0041	-0.0371	0.0528	0.0269

Prm6	-0.4843	0.0856	0.0358	0.0277	-0.0041	1.0000	0.6335	0.4755	-0.1201
Prm7	-0.5501	0.2714	0.0455	0.0286	-0.0371	0.6335	1.0000	0.5482	-0.2636
Prm8	-0.4015	0.2285	0.0971	-0.0966	0.0528	0.4755	0.5482	1.0000	-0.3154
Prm9	-0.2161	0.0254	-0.0031	-0.0047	0.0269	-0.1201	-0.2636	-0.3154	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-6.4059	0.2705	-6.9361	-5.8757	560.72	<.0001
type	2	1	-0.5433	0.2209	-0.9764	-0.1103	6.05	0.0139
type	3	1	-0.6874	0.4094	-1.4898	0.1149	2.82	0.0931
type	4	1	-0.0760	0.3615	-0.7845	0.6326	0.04	0.8336
type	5	1	0.3256	0.2935	-0.2496	0.9007	1.23	0.2672
year	2	1	0.6971	0.1862	0.3323	1.0620	14.02	0.0002
year	3	1	0.8184	0.2112	0.4044	1.2324	15.01	0.0001
year	4	1	0.4534	0.2901	-0.1151	1.0220	2.44	0.1180
period	2	1	0.3845	0.1471	0.0961	0.6729	6.83	0.0090
Scale		0	1.2441	0.0000	1.2441	1.2441		

NOTE: The scale parameter was estimated by the square root of DEVIANCE/DOF.

LR Statistics For Type 3 Analysis

Source	Num DF	Den DF	F Value	Pr > F	Chi-Square	Pr > ChiSq
type	4	25	3.82	0.0147	15.29	0.0041
year	3	25	6.76	0.0017	20.29	0.0001
period	1	25	6.89	0.0146	6.89	0.0087

(e) A model with every possible two-factor interaction contains
1 (const) + 4 + 3 + 1 (main effects) + 4*3 + 4*1 + 3*1 (2-factor interactions) = 28 parameters. This is a highly non-parsimonious model, considering that there are only 34 observations. The number of parameters in the fully saturated model (with the 3-factor interaction added) exceeds the number of observations.

Here we enter each two-factor interaction one at-a-time. The type 3 test results for the models with the type by period interaction (4 additional parameters) and the year by period interaction (3 additional parameters) are given below. The model with the type by year interaction (12 additional parameters) experienced convergence problems, probably due to the large number of additional parameters and the sparseness of the data. The results indicate that interaction components are not needed. Note that type 3 LR test statistics are partial tests, always testing whether the factor in question is significant when added last to the model. The period effect is insignificant when adding it to the model with type, year, and the type by period interaction. However, it becomes significant when the type by period interaction is omitted.

<u>Fitting results for the model with interaction:</u>

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq

type	4	12.13	0.0164
year	3	30.70	<.0001
period	1	1.57	0.2105
type*period	4	4.94	0.2936

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
type	4	23.71	<.0001
year	3	25.26	<.0001
period	1	7.29	0.0069
year*period	3	4.00	0.2613

(f) See parts (a) – (e)

12.2 PROC GENMOD is used to estimate the Poisson regression model with link

$$\ln \mu = \beta_0 + \alpha \ln(\mathrm{H}) + \beta_1 \mathrm{A}_2 + \beta_2 \mathrm{T}_2 + \beta_3 \mathrm{T}_3$$

where *H* is the number of policies and A_1, A_2 and $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ are the corresponding indicator variables for the two age groups and three car types.

The type 3 test statistics at the end of the program output are tests of the significance of the class variables. For example, the test statistic for "age" is obtained by comparing the log-likelihood of the full model (838.1594) with the log-likelihood of the restricted model (the model with type and ln(H); log-likelihood is 817.8596). The log-likelihood statistic is 40.60. Comparing this values to a chi-square with 1 degree of freedom (since there is only restrictions), leads to the probability value $\mathrm{P}(\chi^2(1) \geq 40.60) = 0.0000$.

The type 3 test statistics indicate that both age and type are highly significant. Both factors are needed to explain the number of claims. Looking at the individual parameter estimates, we see that the second age group experiences more claims than the first. The second and third car type experience fewer claims than the first, and the third car type experiences fewer claims than the second.

It seems reasonable to suppose that the number of claims is directly proportional to the number of policies, and that one can expect the coefficient β_1 to be one. Let us test whether $\beta_1 = 1$. The estimate is $\hat{\beta}_1 = 0.6189$, and the 95 percent Wald confidence interval is given by $0.6189 \pm (1.96)(0.3113)$, 0.62 ± 0.61, or $0.01 \leq \beta_1 \leq 1.23$. The interval is quite wide because there are only very few (six) observations. However, it includes one, which makes the off-set interpretation plausible.

Note that this run also asked for an additional table of statistics to be displayed. For each observation, the following items are displayed: the value of the response variable y_i, the values of the regressor variables, the predicted mean $\hat{\mu}_i = \exp(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}})$, the standard error in the linear predictor $\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}$, the value of the Hessian weight at the final iteration (diagonal elements of the matrix in equation (12.12)), lower and upper confidence limits of the predicted value of the mean (see equation (12.19), the raw residual, the Pearson residual (equation (12.23)), the standardized Pearson residual, the deviance residual (equation (12.22)), the standardized deviance residual, and the likelihood residual. Most of these statistics are explained in Chapter 12.

Fitting results for the full model:

```
                    The GENMOD Procedure

                     Model Information

             Data Set              WORK.EXER12N2
             Distribution                Poisson
             Link Function                   Log
             Dependent Variable         nuclaims
             Observations Used                 6

                  Class Level Information

                                  Design
                Class     Value     Variables

                age       1          0
                          2          1

                car       1          0      0
                          2          1      0
                          3          0      1

                   Parameter Information

           Parameter       Effect       age    car

           Prm1            Intercept
           Prm2            lnnupol
           Prm3            age          2
           Prm4            car                 2
           Prm5            car                 3

              Criteria For Assessing Goodness Of Fit

 Criterion                 DF           Value        Value/DF

 Deviance                   1          1.4084          1.4084
 Scaled Deviance            1          1.4084          1.4084
 Pearson Chi-Square         1          1.2742          1.2742
 Scaled Pearson X2          1          1.2742          1.2742
 Log Likelihood                      838.1594

Algorithm converged.
```

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5
Prm1	1.0000	-0.9979	-0.7731	0.7040	-0.4578
Prm2	-0.9979	1.0000	0.7416	-0.7275	0.4500
Prm3	-0.7731	0.7416	1.0000	-0.4975	0.2953
Prm4	0.7040	-0.7275	-0.4975	1.0000	-0.2073
Prm5	-0.4578	0.4500	0.2953	-0.2073	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-0.1920	1.9964	-4.1048	3.7208	0.01	0.9234
lnnupol		1	0.6189	0.3113	0.0089	1.2290	3.95	0.0468
age	2	1	1.1313	0.2005	0.7383	1.5244	31.83	<.0001
car	2	1	-0.5266	0.1856	-0.8904	-0.1628	8.05	0.0046
car	3	1	-1.9130	0.3045	-2.5099	-1.3161	39.46	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
lnnupol	1	4.34	0.0372
age	1	40.60	<.0001
car	2	74.23	<.0001

Observation Statistics (need to read across)

Observation	nuclaims	lnnupol	age	car	Pred	Xbeta	Std
		HessWgt	Lower		Upper	Resraw	Reschi
		Resdev	StResdev		StReschi	Reslik	
1	42	6.2146081	1	1	38.649518	3.6545343	0.1413353
		38.649518	29.298086		50.985762	3.350482	0.5389336
		0.5314155	1.1130493		1.128796	1.125226	
2	37	7.0900768	1	2	39.243343	3.6697818	0.1513849
		39.243343	29.168017		52.798927	-2.243343	-0.358107
		-0.361603	-1.139821		-1.128802	-1.129916	
3	1	4.6051702	1	3	2.1071828	0.7453519	0.5078463
		2.1071828	0.7787941		5.7014036	-1.107183	-0.762725
		-0.850683	-1.259006		-1.12883	-1.190029	
4	101	5.9914645	2	1	104.3505	4.6477554	0.0936696
		104.3505	86.848591		125.37942	-3.350497	-0.327991
		-0.32977	-1.134924		-1.128801	-1.129319	
5	73	6.2146081	2	2	70.756662	4.2592467	0.1155164
		70.756662	56.420843		88.73503	2.2433384	0.2666927
		0.2653017	1.1229119		1.1287992	1.1284714	
6	14	5.7037825	2	3	12.89285	2.5566729	0.2679085
		12.89285	7.6261405		21.796816	1.10715	0.3083415
		0.3040793	1.1131926		1.1287961	1.1276393	

Next, we assume an "offset" for the number of policies (that is, we impose the restriction $\beta_1 = 1$) and estimate the model with link

$$\ln \mu = \beta_0 + \ln(\mathrm{H}) + \beta_1 \mathrm{A}_2 + \beta_2 \mathrm{T}_2 + \beta_3 \mathrm{T}_3 .$$

The results are given below. The interpretation of the earlier model is largely unchanged. Both age and type are highly significant. The second age group experiences more claims than the first, the second and third car type experience fewer claims than the first, and the third car type experiences fewer claims than the second.

Goodness-of-fit statistics: Comparing the deviance D = 2.82 (in the model with the offset) to a chi-square with 2 degrees of freedom, leads to the probability value $P(\chi^2(2) \geq 2.82) = 1 - 0.7559 = 0.2441$. The deviance does not exceed the critical 95th percentile (5.99) and the probability value is larger than 0.05. Hence there is no sign of overdispersion and there is no need to adjust the analysis.

<u>Fitting results for the model with an offset:</u>

```
                          The GENMOD Procedure

                           Model Information

                 Data Set              WORK.EXER12N2
                 Distribution                Poisson
                 Link Function                   Log
                 Dependent Variable         nuclaims
                 Offset Variable             lnnupol
                 Observations Used                 6

                        Class Level Information

                                        Design
                    Class     Value     Variables

                    age       1          0
                              2          1

                    car       1          0      0
                              2          1      0
                              3          0      1

                         Parameter Information

               Parameter       Effect       age    car

               Prm1            Intercept
               Prm2            age          2
               Prm3            car                 2
               Prm4            car                 3

                Criteria For Assessing Goodness Of Fit

     Criterion                 DF           Value        Value/DF

     Deviance                   2          2.8207          1.4103
     Scaled Deviance            2          2.8207          1.4103
     Pearson Chi-Square         2          2.8416          1.4208
     Scaled Pearson X2          2          2.8416          1.4208
     Log Likelihood                      837.4533

Algorithm converged.


                      Estimated Correlation Matrix
```

	Prm1	Prm2	Prm3	Prm4
Prm1	1.0000	-0.7729	-0.5286	-0.1298
Prm2	-0.7729	1.0000	0.1487	-0.0841
Prm3	-0.5286	0.1487	1.0000	0.1877
Prm4	-0.1298	-0.0841	0.1877	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-2.6367	0.1318	-2.8950	-2.3784	400.20	<.0001
age	2	1	1.3199	0.1359	1.0536	1.5863	94.34	<.0001
car	2	1	-0.6928	0.1282	-0.9441	-0.4414	29.18	<.0001
car	3	1	-1.7643	0.2724	-2.2981	-1.2304	41.96	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
age	1	104.64	<.0001
car	2	72.82	<.0001

Finally, we estimate the model with an interaction term. This is a saturated model with the same number of parameters as observations. The output is given below. The type 3 analysis indicates that the interaction is not needed. Now you may wonder why it is possible to test for an interaction term in a saturated model. In the usual (normal) linear model this would not be possible as the saturated model leaves no degrees of freedom for the error term. With a Poisson link, however, the variance is the same as the mean and there is no extra parameter (variance or dispersion parameter) that needs to be estimated; the program indicates this fact when it says that the scale parameter was held fixed. Hence we can compare the log-likelihood of the full (saturated) model (838.8636) with the log-likelihood of the model without the interaction (837.4533) and compute the log- likelihood ratio test statistic 2(838.8636-837.4533)=2.82. Since its probability value $\mathrm{P}(\chi^2(2) \geq 2.82) = 1 - 0.7559 = 0.2441$ exceeds 0.05, the interaction is insignificant and we can use the simpler model without interaction. Note that the likelihood ratio test statistic for the interaction in the saturated model is identical to the deviance in the model without the interaction component.

Fitting results for the model with interaction:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N2
Distribution	Poisson
Link Function	Log
Dependent Variable	nuclaims
Offset Variable	lnnupol
Observations Used	6

Class Level Information

Class	Value	Design Variables	
age	1	0	
	2	1	
car	1	0	0
	2	1	0
	3	0	1

Parameter Information

Parameter	Effect	age	car
Prm1	Intercept		
Prm2	age	2	
Prm3	car		2
Prm4	car		3
Prm5	age*car	2	2
Prm6	age*car	2	3

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	0	0.0000	.
Scaled Deviance	0	0.0000	.
Pearson Chi-Square	0	0.0000	.
Scaled Pearson X2	0	0.0000	.
Log Likelihood		838.8636	

Algorithm converged.

The GENMOD Procedure

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6
Prm1	1.0000	-0.8404	-0.6844	-0.1525	0.5656	0.1468
Prm2	-0.8404	1.0000	0.5751	0.1282	-0.6730	-0.1747
Prm3	-0.6844	0.5751	1.0000	0.1044	-0.8264	-0.1005
Prm4	-0.1525	0.1282	0.1044	1.0000	-0.0862	-0.9625
Prm5	0.5656	-0.6730	-0.8264	-0.0862	1.0000	0.1175
Prm6	0.1468	-0.1747	-0.1005	-0.9625	0.1175	1.0000

Analysis Of Parameter Estimates

Parameter			DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept			1	-2.4769	0.1543	-2.7794	-2.1745	257.68	<.0001
age	2		1	1.1006	0.1836	0.7407	1.4605	35.93	<.0001
car	2		1	-1.0022	0.2255	-1.4441	-0.5603	19.76	<.0001
car	3		1	-2.1282	1.0118	-4.1114	-0.1451	4.42	0.0354
age*car	2	2	1	0.4544	0.2728	-0.0803	0.9892	2.77	0.0958
age*car	2	3	1	0.4399	1.0513	-1.6206	2.5003	0.18	0.6757
Scale			0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq

```
age                1      40.03        <.0001
car                2      23.43        <.0001
age*car            2       2.82        0.2441
```

12.3 We use SAS GENMOD to estimate the Poisson regression model with link

$$\ln \mu = \lambda_1 \mathrm{T}_1 + \lambda_2 \mathrm{T}_2$$

The deviance is D = 4.00 and the standardized deviance is 0.67. While the standardized deviance is somewhat smaller than one, the deviance is not small enough to suggest underdispersion ($\mathrm{P}(\chi^2(6) \le 4.00) = 0.32$).

The estimate of λ_2 is not significantly different from zero; the likelihood ratio test statistic is 0.81 with probability value 0.3685 (larger than 0.05). Alternatively, one can look at the confidence interval for λ_2; it covers zero.

The model without T_2 (that is, the Poisson regression with link $\ln \mu = \lambda_1 T_1$) is estimated next). The estimate of λ_1 is significant. A scatter plot of the observations against T_1, and the Poisson fit $\hat{\mu} = \exp(\hat{\lambda}_1 T_1)$ are shown below.

Fitting results for the model with T_1 and T_2:

```
                              The GENMOD Procedure

                               Model Information

                     Data Set                  WORK.EXER12N3
                     Distribution                    Poisson
                     Link Function                       Log
                     Dependent Variable               nufail
                     Observations Used                     9

                             Parameter Information

                          Parameter       Effect

                          Prm1            Intercept
                          Prm2            time1
                          Prm3            time2

                     Criteria For Assessing Goodness Of Fit

         Criterion                 DF           Value        Value/DF

         Deviance                   6          4.0033          0.6672
         Scaled Deviance            6          4.0033          0.6672
         Pearson Chi-Square         6          3.9505          0.6584
         Scaled Pearson X2          6          3.9505          0.6584
         Log Likelihood                      362.7354

Algorithm converged.

                           Estimated Correlation Matrix
```

	Prm1	Prm2	Prm3
Prm1	1.0000	-0.7791	-0.2690
Prm2	-0.7791	1.0000	-0.3272
Prm3	-0.2690	-0.3272	1.0000

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	2.1752	0.2555	1.6745	2.6759	72.50	<.0001
time1	1	0.0070	0.0024	0.0023	0.0118	8.34	0.0039
time2	1	0.0025	0.0028	-0.0030	0.0081	0.81	0.3685
Scale	0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

Fitting results for the model without T_2:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N3
Distribution	Poisson
Link Function	Log
Dependent Variable	nufail
Observations Used	9

Parameter Information

Parameter	Effect
Prm1	Intercept
Prm2	time1

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	7	4.8078	0.6868
Scaled Deviance	7	4.8078	0.6868
Pearson Chi-Square	7	4.6345	0.6621
Scaled Pearson X2	7	4.6345	0.6621
Log Likelihood		362.3331	

Algorithm converged.

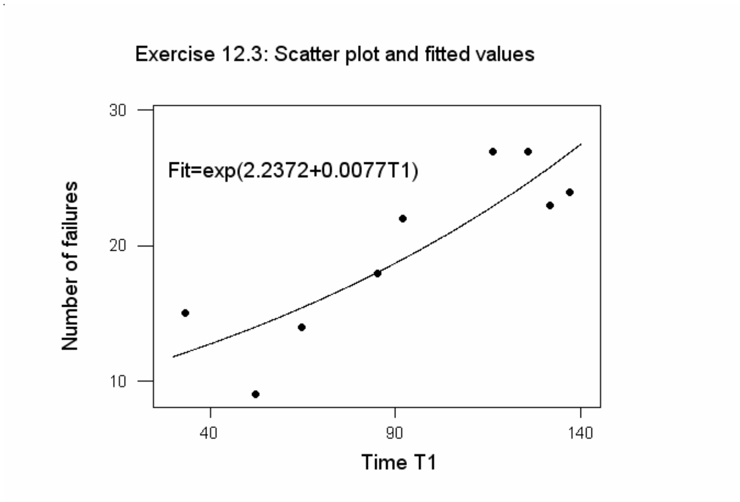
Estimated Correlation Matrix

	Prm1	Prm2
Prm1	1.0000	-0.9515
Prm2	-0.9515	1.0000

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	2.2372	0.2431	1.7608	2.7136	84.72	<.0001
time1	1	0.0077	0.0023	0.0033	0.0121	11.58	0.0007
Scale	0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.



12.4

(a) Cancer incidence should be directly proportional to the size of the population. Hence it is reasonable to consider $\ln(\text{POP})$ as an offset. Age is a categorical variable. We use indicator variables for the eight age groups (X1 through X8) and consider the Poisson regression with link

$$\ln \mu = \beta_0 + \ln(\text{POP}) + \beta_2 \text{X2} + ... + \beta_8 \text{X8} + \beta_9 \text{Town}$$

The results of the model fit are shown below. Both age and town are significant; you can see this from the (partial; type 3) likelihood-ratio test statistics and their probability values at the end of the output. The estimate of the town effect is $\hat{\beta}_9 = 0.85$, with standard error 0.06. There is a significant location effect; women in Texas have a $100[\exp(0.85) - 1] = 134$ percent higher incidence of skin cancer. The deviance and the Pearson Chi-Square statistics are approximately one and indicate no problem with over/under-dispersion.

<u>Fitting results for the full model with an offset:</u>

```
                    The GENMOD Procedure

                     Model Information

            Data Set                 WORK.EXER12N4
            Distribution                   Poisson
            Link Function                      Log
            Dependent Variable             nucases
            Offset Variable                  lnpop
            Observations Used                   15

                  Class Level Information

Class      Value                  Design Variables

age        1          0      0      0      0      0      0      0
           2          1      0      0      0      0      0      0
```

3	0	1	0	0	0	0	0
4	0	0	1	0	0	0	0
5	0	0	0	1	0	0	0
6	0	0	0	0	1	0	0
7	0	0	0	0	0	1	0
8	0	0	0	0	0	0	1

Parameter Information

Parameter	Effect	age
Prm1	Intercept	
Prm2	town	
Prm3	age	2
Prm4	age	3
Prm5	age	4
Prm6	age	5
Prm7	age	6
Prm8	age	7
Prm9	age	8

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	6	5.2089	0.8682
Scaled Deviance	6	5.2089	0.8682
Pearson Chi-Square	6	5.1482	0.8580
Scaled Pearson X2	6	5.1482	0.8580
Log Likelihood		6204.3156	

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6	Prm7	Prm8	Prm9
Prm1	1.0000	-0.0944	-0.9521	-0.9788	-0.9868	-0.9885	-0.9900	-0.9819	-0.9730
Prm2	-0.0944	1.0000	-0.0031	-0.0047	-0.0037	-0.0024	0.0007	0.0927	0.0039
Prm3	-0.9521	-0.0031	1.0000	0.9410	0.9486	0.9501	0.9513	0.9349	0.9347
Prm4	-0.9788	-0.0047	0.9410	1.0000	0.9753	0.9769	0.9781	0.9610	0.9610
Prm5	-0.9868	-0.0037	0.9486	0.9753	1.0000	0.9847	0.9860	0.9689	0.9687
Prm6	-0.9885	-0.0024	0.9501	0.9769	0.9847	1.0000	0.9875	0.9706	0.9703
Prm7	-0.9900	0.0007	0.9513	0.9781	0.9860	0.9875	1.0000	0.9721	0.9715
Prm8	-0.9819	0.0927	0.9349	0.9610	0.9689	0.9706	0.9721	1.0000	0.9554
Prm9	-0.9730	0.0039	0.9347	0.9610	0.9687	0.9703	0.9715	0.9554	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-11.6921	0.4492	-12.5725	-10.8116	677.43	<.0001
town		1	0.8527	0.0596	0.7358	0.9696	204.54	<.0001
age	2	1	2.6290	0.4675	1.7128	3.5452	31.63	<.0001
age	3	1	3.8456	0.4547	2.9545	4.7367	71.54	<.0001
age	4	1	4.5938	0.4510	3.7098	5.4778	103.74	<.0001
age	5	1	5.0864	0.4503	4.2038	5.9690	127.59	<.0001
age	6	1	5.6457	0.4497	4.7642	6.5272	157.58	<.0001
age	7	1	6.2032	0.4575	5.3065	7.0999	183.83	<.0001
age	8	1	6.1757	0.4577	5.2785	7.0728	182.02	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
town	1	226.52	<.0001
age	7	2199.01	<.0001

(b) The estimation results for the more general model

$$\ln \mu = \beta_0 + \beta_1 \ln(\text{POP}) + \beta_2 \text{X2} + ... + \beta_8 \text{X8} + \beta_9 \text{Town}$$

are given below. It seems reasonable to suppose that the number of cancers is directly proportional to the population, and that one can expect that the coefficient β_1 is one. Let us test whether $\beta_1 = 1$. The estimate is $\hat{\beta}_1 = 1.96$, and the 95 percent Wald confidence interval is given by $1.96 \pm (1.96)(0.63)$, 1.96 ± 1.23, or $0.73 \le \beta_1 \le 3.18$. The interval is quite wide (because there are few observations). The interval includes one, which makes the off-set interpretation plausible.

<u>Fitting results for the full model without an offset:</u>

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N4
Distribution	Poisson
Link Function	Log
Dependent Variable	nucases
Observations Used	15

Class Level Information

Class	Value	Design Variables						
age	1	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0
	3	0	1	0	0	0	0	0
	4	0	0	1	0	0	0	0
	5	0	0	0	1	0	0	0
	6	0	0	0	0	1	0	0
	7	0	0	0	0	0	1	0
	8	0	0	0	0	0	0	1

Parameter Information

Parameter	Effect	age
Prm1	Intercept	
Prm2	lnpop	
Prm3	town	
Prm4	age	2
Prm5	age	3
Prm6	age	4
Prm7	age	5
Prm8	age	6
Prm9	age	7
Prm10	age	8

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	5	2.8539	0.5708
Scaled Deviance	5	2.8539	0.5708
Pearson Chi-Square	5	2.8439	0.5688
Scaled Pearson X2	5	2.8439	0.5688
Log Likelihood		6205.4931	

The GENMOD Procedure

Algorithm converged.

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6	Prm7	Prm8	Prm9	Prm10
Prm1	1.0000	-0.9982	0.7154	-0.3692	-0.5729	-0.6353	-0.7844	-0.8810	-0.9360	-0.9851
Prm2	-0.9982	1.0000	-0.7206	0.3160	0.5241	0.5888	0.7465	0.8516	0.9138	0.9736
Prm3	0.7154	-0.7206	1.0000	-0.2317	-0.3831	-0.4284	-0.5401	-0.6131	-0.6327	-0.7003
Prm4	-0.3692	0.3160	-0.2317	1.0000	0.9260	0.9135	0.8357	0.7422	0.6489	0.5100
Prm5	-0.5729	0.5241	-0.3831	0.9260	1.0000	0.9800	0.9448	0.8830	0.8112	0.6971
Prm6	-0.6353	0.5888	-0.4284	0.9135	0.9800	1.0000	0.9691	0.9192	0.8560	0.7520
Prm7	-0.7844	0.7465	-0.5401	0.8357	0.9448	0.9691	1.0000	0.9802	0.9444	0.8742
Prm8	-0.8810	0.8516	-0.6131	0.7422	0.8830	0.9192	0.9802	1.0000	0.9852	0.9453
Prm9	-0.9360	0.9138	-0.6327	0.6489	0.8112	0.8560	0.9444	0.9852	1.0000	0.9783
Prm10	-0.9851	0.9736	-0.7003	0.5100	0.6971	0.7520	0.8742	0.9453	0.9783	1.0000

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-23.2489	7.5392	-38.0256	-8.4723	9.51	0.0020
lnpop		1	1.9613	0.6259	0.7345	3.1880	9.82	0.0017
town		1	0.7556	0.0862	0.5866	0.9245	76.81	<.0001
age	2	1	2.8684	0.4927	1.9027	3.8341	33.89	<.0001
age	3	1	4.2766	0.5339	3.2303	5.3230	64.17	<.0001
age	4	1	5.0990	0.5580	4.0053	6.1927	83.49	<.0001
age	5	1	5.8623	0.6768	4.5358	7.1888	75.02	<.0001
age	6	1	6.7681	0.8579	5.0866	8.4496	62.23	<.0001
age	7	1	7.7827	1.1265	5.5748	9.9906	47.73	<.0001
age	8	1	9.1783	2.0057	5.2473	13.1094	20.94	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
lnpop	1	9.77	0.0018
town	1	81.60	<.0001
age	7	988.50	<.0001

Additional model: We estimate a model that includes an interaction between town and age. We want to check whether the town effect depends on the age group. The results are given below. The likelihood-ratio test for the town by age interaction is insignificant. Note that such a test is possible in the saturated Poisson regression model, as the variance is the same as the mean; the scale parameter is kept fixed.

<u>Fitting results for the model with interaction:</u>

```
                         The GENMOD Procedure

                          Model Information

                   Data Set                WORK.EXER12N4
                   Distribution                  Poisson
                   Link Function                     Log
                   Dependent Variable            nucases
                   Offset Variable                 lnpop
                   Observations Used                  15

                       Class Level Information

      Class     Value                  Design Variables

      age       1          0      0      0      0      0      0      0
                2          1      0      0      0      0      0      0
                3          0      1      0      0      0      0      0
                4          0      0      1      0      0      0      0
                5          0      0      0      1      0      0      0
                6          0      0      0      0      1      0      0
                7          0      0      0      0      0      1      0
                8          0      0      0      0      0      0      1
      town      0          0
                1          1

                        Parameter Information

                 Parameter       Effect       age    town

                 Prm1            Intercept
                 Prm2            town                1
                 Prm3            age          2
                 Prm4            age          3
                 Prm5            age          4
                 Prm6            age          5
                 Prm7            age          6
                 Prm8            age          7
                 Prm9            age          8
                 Prm10           age*town     2      1
                 Prm11           age*town     3      1
                 Prm12           age*town     4      1
                 Prm13           age*town     5      1
                 Prm14           age*town     6      1
                 Prm15           age*town     7      1
                 Prm16           age*town     8      1

                Criteria For Assessing Goodness Of Fit

        Criterion                 DF           Value        Value/DF

        Deviance                   0          0.0000           .
        Scaled Deviance            0          0.0000           .
        Pearson Chi-Square         0          0.0000           .
        Scaled Pearson X2          0          0.0000           .
        Log Likelihood                     6206.9201

  Algorithm converged.
```

Analysis Of Parameter Estimates

Parameter			DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept			1	-12.0592	1.0000	-14.0191	-10.0992	145.42	<.0001
town	1		1	1.3373	1.1180	-0.8540	3.5286	1.43	0.2316
age	2		1	3.1113	1.0308	1.0910	5.1316	9.11	0.0025
age	3		1	3.9860	1.0165	1.9937	5.9784	15.38	<.0001
age	4		1	4.8917	1.0070	2.9180	6.8655	23.60	<.0001
age	5		1	5.4975	1.0049	3.5280	7.4671	29.93	<.0001
age	6		1	6.0167	1.0038	4.0492	7.9842	35.92	<.0001
age	7		1	6.5703	1.0038	4.6029	8.5376	42.85	<.0001
age	8		1	6.7207	1.0124	4.7364	8.7050	44.07	<.0001
age*town	2	1	1	-0.6446	1.1571	-2.9124	1.6232	0.31	0.5774
age*town	3	1	1	-0.1917	1.1365	-2.4193	2.0359	0.03	0.8661
age*town	4	1	1	-0.3922	1.1263	-2.5998	1.8154	0.12	0.7277
age*town	5	1	1	-0.5455	1.1241	-2.7487	1.6578	0.24	0.6275
age*town	6	1	1	-0.4901	1.1229	-2.6910	1.7107	0.19	0.6625
age*town	7	1	0	0.0000	0.0000	0.0000	0.0000	.	.
age*town	8	1	1	-0.7581	1.1360	-2.9845	1.4683	0.45	0.5045
Scale			0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
town	1	1.78	0.1817
age	7	845.79	<.0001
age*town	7	5.21	0.6342

Another model: Finally, we introduce age as a continuous variable, and not as a factor as was done in the previous models. The output is shown below. Both age and town are significant. A graph of the number of cancer deaths against age (with the two towns indicated by different plotting symbols) and the Poisson model fit is given in the following graph. Every ten years the cancer rate (deaths per population) increases by a factor of exp(0.6133) = 1.85; that is, by 85 percent.

<u>Fitting results for the model with age as continuous variable:</u>

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N4
Distribution	Poisson
Link Function	Log
Dependent Variable	nucases
Offset Variable	lnpop
Observations Used	15

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	12	184.8091	15.4008
Scaled Deviance	12	184.8091	15.4008
Pearson Chi-Square	12	141.4307	11.7859

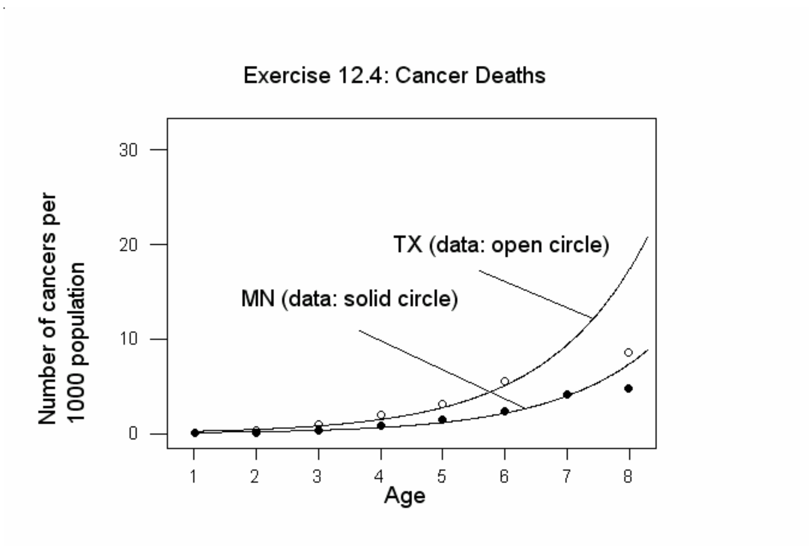
Scaled Pearson X2	12	141.4307	11.7859
Log Likelihood		6114.5155	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	-9.8191	0.0902	-9.9959	-9.6423	11846.5	<.0001
town	1	0.8584	0.0545	0.7515	0.9652	247.95	<.0001
age	1	0.6133	0.0142	0.5855	0.6411	1871.42	<.0001
Scale	0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.



12.5 We use SAS GENMOD to estimate the Poisson regression model with link

$$\ln \mu = \beta_0 + \beta_1 \ln(\text{Pop}) + \beta_2 \text{X2} + ... + \beta_9 \text{X9} + \beta_{10} \text{Z2} + \beta_{11} \text{Z3} + \beta_{12} \text{Z4}$$

The output shows that age and smoking are statistically significant factors. Lung cancer deaths increase monotonically with age. Lung cancer deaths also increase with smoking. The situation is worst for people who smoke cigarettes only (smoking = 4). The surprising fact that people who smoke cigarettes and pipe (or cigar) have lower incidences is probably explained by the number of cigarettes smoked (which is not recorded). People who smoke cigarettes only probably smoke more cigarettes than people who smoke both cigarettes and pipe (or cigar).

The deviance is D = 16.38 and the standardized deviance is 0.71. While the standardized deviance is somewhat smaller than one, the deviance is not small enough to suggest underdispersion ($P(\chi^2(23) \le 16.38) = 0.16$).

Let us test whether $\beta_1 = 1$. The estimate is $\hat{\beta}_1 = 1.0761$, and the 95 percent Wald confidence interval is given by $1.0761 \pm (1.96)(0.0340)$, 1.076 ± 0.067, or $1.01 \le \beta_1 \le 1.14$. The interval fails to cover one – however just barely (the lower limit is about one). While we would reject at the 0.05 significance level that $\beta_1 = 1$, the off-set interpretation is not entirely implausible.

Fitting results for the model without an offset:

```
                              The GENMOD Procedure

                               Model Information

                     Data Set                WORK.EXER12N5
                     Distribution                  Poisson
                     Link Function                     Log
                     Dependent Variable            nudeath
                     Observations Used                  36

                            Class Level Information

           Class       Value                       Design Variables

           age         1          0      0      0      0      0      0      0      0
                       2          1      0      0      0      0      0      0      0
                       3          0      1      0      0      0      0      0      0
                       4          0      0      1      0      0      0      0      0
                       5          0      0      0      1      0      0      0      0
                       6          0      0      0      0      1      0      0      0
                       7          0      0      0      0      0      1      0      0
                       8          0      0      0      0      0      0      1      0
                       9          0      0      0      0      0      0      0      1
           smoking     1          0      0      0
                       2          1      0      0
                       3          0      1      0
                       4          0      0      1

                             Parameter Information

                 Parameter       Effect       age    smoking

                 Prm1            Intercept
                 Prm2            lnpop
                 Prm3            age          2
                 Prm4            age          3
                 Prm5            age          4
                 Prm6            age          5
                 Prm7            age          6
                 Prm8            age          7
                 Prm9            age          8
                 Prm10           age          9
                 Prm11           smoking             2
                 Prm12           smoking             3
                 Prm13           smoking             4

                              The GENMOD Procedure

                     Criteria For Assessing Goodness Of Fit

              Criterion                 DF           Value        Value/DF

              Deviance                  23         16.3820          0.7123
```

Scaled Deviance	23	16.3820	0.7123
Pearson Chi-Square	23	16.3745	0.7119
Scaled Pearson X2	23	16.3745	0.7119
Log Likelihood		45620.8854	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-4.2192	0.2505	-4.7103	-3.7282	283.61	<.0001
lnpop		1	1.0761	0.0340	1.0095	1.1427	1002.25	<.0001
age	2	1	0.5855	0.0812	0.4263	0.7447	51.97	<.0001
age	3	1	1.0304	0.0800	0.8736	1.1872	165.93	<.0001
age	4	1	1.3814	0.0653	1.2535	1.5093	447.97	<.0001
age	5	1	1.6401	0.0629	1.5169	1.7634	680.41	<.0001
age	6	1	2.0158	0.0633	1.8917	2.1398	1014.09	<.0001
age	7	1	2.3330	0.0701	2.1957	2.4704	1108.03	<.0001
age	8	1	2.6721	0.0848	2.5060	2.8383	993.31	<.0001
age	9	1	2.9916	0.0970	2.8015	3.1817	951.64	<.0001
smoking	2	1	0.0148	0.0494	-0.0820	0.1117	0.09	0.7643
smoking	3	1	0.1159	0.0598	-0.0012	0.2330	3.76	0.0524
smoking	4	1	0.3485	0.0503	0.2498	0.4471	47.91	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
lnpop	1	1244.55	<.0001
age	8	3254.75	<.0001
smoking	3	143.40	<.0001

The regression results treating ln(POP) as an offset are given next. The interpretation of the results is mostly unchanged.

Fitting results for the model with an offset:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N5
Distribution	Poisson
Link Function	Log
Dependent Variable	nudeath
Offset Variable	lnpop
Observations Used	36

Class Level Information

Class	Value	Design Variables							
age	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0
	3	0	1	0	0	0	0	0	0

	4	0	0	1	0	0	0	0	0
	5	0	0	0	1	0	0	0	0
	6	0	0	0	0	1	0	0	0
	7	0	0	0	0	0	1	0	0
	8	0	0	0	0	0	0	1	0
	9	0	0	0	0	0	0	0	1
smoking	1	0	0	0					
	2	1	0	0					
	3	0	1	0					
	4	0	0	1					

Parameter Information

Parameter	Effect	age	smoking
Prm1	Intercept		
Prm2	age	2	
Prm3	age	3	
Prm4	age	4	
Prm5	age	5	
Prm6	age	6	
Prm7	age	7	
Prm8	age	8	
Prm9	age	9	
Prm10	smoking		2
Prm11	smoking		3
Prm12	smoking		4

The GENMOD Procedure

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	24	21.4867	0.8953
Scaled Deviance	24	21.4867	0.8953
Pearson Chi-Square	24	20.6194	0.8591
Scaled Pearson X2	24	20.6194	0.8591
Log Likelihood		45618.3330	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-3.6800	0.0682	-3.8138	-3.5463	2908.35	<.0001
age	2	1	0.5539	0.0800	0.3971	0.7107	47.95	<.0001
age	3	1	0.9804	0.0768	0.8298	1.1309	162.88	<.0001
age	4	1	1.3795	0.0653	1.2515	1.5074	446.80	<.0001
age	5	1	1.6542	0.0626	1.5316	1.7769	699.00	<.0001
age	6	1	1.9982	0.0628	1.8751	2.1212	1012.79	<.0001
age	7	1	2.2714	0.0644	2.1453	2.3975	1245.78	<.0001
age	8	1	2.5586	0.0678	2.4257	2.6914	1424.74	<.0001
age	9	1	2.8469	0.0724	2.7050	2.9889	1545.27	<.0001
smoking	2	1	0.0478	0.0470	-0.0443	0.1399	1.03	0.3090
smoking	3	1	0.2180	0.0387	0.1421	0.2938	31.73	<.0001
smoking	4	1	0.4170	0.0399	0.3387	0.4952	109.14	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
age	8	3889.22	<.0001
smoking	3	170.24	<.0001

Results of the Poisson regression with the factors age, smoking and the interaction between age and smoking is given below. The interaction between age and smoking turns out to be insignificant.

Fitting results for the model with interaction:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N5
Distribution	Poisson
Link Function	Log
Dependent Variable	nudeath
Offset Variable	lnpop
Observations Used	36

Class Level Information

Class	Value	Design Variables							
age	1	0	0	0	0	0	0	0	0
	2	1	0	0	0	0	0	0	0
	3	0	1	0	0	0	0	0	0
	4	0	0	1	0	0	0	0	0
	5	0	0	0	1	0	0	0	0
	6	0	0	0	0	1	0	0	0
	7	0	0	0	0	0	1	0	0
	8	0	0	0	0	0	0	1	0
	9	0	0	0	0	0	0	0	1
smoking	1	0	0	0					
	2	1	0	0					
	3	0	1	0					
	4	0	0	1					

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	0	0.0000	.
Scaled Deviance	0	0.0000	.
Pearson Chi-Square	0	0.0000	.
Scaled Pearson X2	0	0.0000	.
Log Likelihood		45629.0764	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-3.5958	0.2357	-4.0578	-3.1338	232.73	<.0001
age	2	1	0.8035	0.3178	0.1806	1.4264	6.39	0.0115
age	3	1	1.0228	0.3289	0.3781	1.6674	9.67	0.0019
age	4	1	1.1542	0.2715	0.6220	1.6865	18.07	<.0001
age	5	1	1.3854	0.2532	0.8891	1.8816	29.94	<.0001
age	6	1	1.9325	0.2479	1.4467	2.4183	60.79	<.0001
age	7	1	2.2789	0.2473	1.7942	2.7635	84.94	<.0001
age	8	1	2.4944	0.2528	1.9990	2.9898	97.39	<.0001
age	9	1	2.7702	0.2528	2.2747	3.2656	120.11	<.0001
smoking	2	1	-0.6878	0.7454	-2.1487	0.7731	0.85	0.3561
smoking	3	1	0.1810	0.2495	-0.3080	0.6701	0.53	0.4681

smoking	4		1	0.2816	0.2522	-0.2128	0.7760	1.25	0.2642
age*smoking	2	2	1	0.2220	0.9225	-1.5861	2.0301	0.06	0.8098
age*smoking	2	3	1	-0.2752	0.3371	-0.9359	0.3855	0.67	0.4143
age*smoking	2	4	1	-0.2615	0.3409	-0.9296	0.4067	0.59	0.4431
age*smoking	3	2	1	-0.2255	0.9703	-2.1273	1.6762	0.05	0.8162
age*smoking	3	3	1	-0.0715	0.3465	-0.7507	0.6076	0.04	0.8364
age*smoking	3	4	1	-0.0010	0.3487	-0.6844	0.6825	0.00	0.9978
age*smoking	4	2	1	0.8480	0.7746	-0.6702	2.3663	1.20	0.2736
age*smoking	4	3	1	0.1651	0.2867	-0.3967	0.7270	0.33	0.5646
age*smoking	4	4	1	0.3096	0.2894	-0.2576	0.8768	1.14	0.2846
age*smoking	5	2	1	0.8851	0.7569	-0.5985	2.3687	1.37	0.2423
age*smoking	5	3	1	0.2300	0.2680	-0.2952	0.7552	0.74	0.3907
age*smoking	5	4	1	0.3452	0.2710	-0.1860	0.8764	1.62	0.2028
age*smoking	6	2	1	0.6489	0.7531	-0.8272	2.1251	0.74	0.3889
age*smoking	6	3	1	0.0221	0.2632	-0.4937	0.5379	0.01	0.9330
age*smoking	6	4	1	0.1250	0.2664	-0.3971	0.6470	0.22	0.6390
age*smoking	7	2	1	0.6471	0.7522	-0.8272	2.1215	0.74	0.3896
age*smoking	7	3	1	-0.0630	0.2636	-0.5797	0.4536	0.06	0.8110
age*smoking	7	4	1	0.0178	0.2674	-0.5063	0.5420	0.00	0.9469
age*smoking	8	2	1	0.7795	0.7537	-0.6977	2.2566	1.07	0.3010
age*smoking	8	3	1	-0.0292	0.2712	-0.5608	0.5024	0.01	0.9142
age*smoking	8	4	1	0.1081	0.2768	-0.4344	0.6507	0.15	0.6961
age*smoking	9	2	1	0.7608	0.7536	-0.7161	2.2378	1.02	0.3127
age*smoking	9	3	1	0.0428	0.2755	-0.4971	0.5827	0.02	0.8765
age*smoking	9	4	1	-0.0402	0.2964	-0.6211	0.5406	0.02	0.8920
Scale			0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
age	8	382.21	<.0001
smoking	3	3.80	0.2843
age*smoking	24	21.49	0.6099

Finally, we consider the Poisson regression that includes smoking as a factor (with the three indicators) and age as a continuous variable. The results are given below. We can test whether a class factor for age is needed or whether it is sufficient to include age as a continuous variable. The log-likelihood of the model that considers age as a factor (the full model) is 45,618.3330; the log-likelihood of the model that considers age as a continuous variable (the restricted model) is 45,591.2091. We compare the log-likelihood ratio statistic, 2(45,618.3330-45,591.2091) = 54.25, to a chi-square with 7 degrees of freedom (the nine intercepts in the unrestricted model, one for each age group, are tested against the linear formulation which includes two parameters, the intercept and the slope). The test statistic is large (probability value $P(\chi^2(7) \geq 54.25) < 0.001$ is small), indicating that it is not adequate to consider a linear component of size. Size must be treated as a class factor.

<u>Fitting results for the model with age as continuous variable:</u>

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N5

Distribution	Poisson
Link Function	Log
Dependent Variable	nudeath
Offset Variable	lnpop
Observations Used	36

Class Level Information

Class	Value	Design Variables		
smoking	1	0	0	0
	2	1	0	0
	3	0	1	0
	4	0	0	1

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	31	75.7345	2.4430
Scaled Deviance	31	75.7345	2.4430
Pearson Chi-Square	31	71.9749	2.3218
Scaled Pearson X2	31	71.9749	2.3218
Log Likelihood		45591.2091	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-3.7389	0.0500	-3.8369	-3.6409	5589.70	<.0001
age		1	0.3330	0.0056	0.3220	0.3440	3547.25	<.0001
smoking	2	1	0.0329	0.0469	-0.0590	0.1248	0.49	0.4826
smoking	3	1	0.2364	0.0386	0.1607	0.3120	37.50	<.0001
smoking	4	1	0.4379	0.0398	0.3599	0.5160	121.07	<.0001
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
age	1	3834.97	<.0001
smoking	3	196.00	<.0001

12.6 The output from estimating the Poisson regression with link

$$\ln \mu = \beta_0 + \beta_1 \text{DIST} + \beta_2 \text{INC} + \beta_3 \text{SIZE2} + \beta_4 \text{SIZE3} + \beta_5 \text{SIZE4} + \beta_6 \text{SIZE5}$$

is shown below. Here we treat SIZE as a class variable, specifying 4 indicators for the factor with five outcomes (1 through 5 people; size 1 is the baseline). Income does not affect the number of visits to the lake (probability value = 0.27) and is omitted in the next run. The deviance and the Pearson chi-square statistics are roughly the size of the critical 95th percentile (280.36).

Fitting results for the full model:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N6
Distribution	Poisson
Link Function	Log
Dependent Variable	nuvisits
Observations Used	250

Class Level Information

Class	Value	Design Variables			
size	1	0	0	0	0
	2	1	0	0	0
	3	0	1	0	0
	4	0	0	1	0
	5	0	0	0	1

Parameter Information

Parameter	Effect	size
Prm1	Intercept	
Prm2	dist	
Prm3	inc	
Prm4	size	2
Prm5	size	3
Prm6	size	4
Prm7	size	5

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	243	313.6999	1.2909
Scaled Deviance	243	313.6999	1.2909
Pearson Chi-Square	243	286.2022	1.1778
Scaled Pearson X2	243	286.2022	1.1778
Log Likelihood		11.3651	

Algorithm converged.

Estimated Correlation Matrix

	Prm1	Prm2	Prm3	Prm4	Prm5	Prm6	Prm7
Prm1	1.0000	-0.4081	-0.6605	-0.4409	-0.5760	-0.5506	-0.5740
Prm2	-0.4081	1.0000	-0.0275	0.0247	0.1431	0.0573	0.0438
Prm3	-0.6605	-0.0275	1.0000	-0.0391	0.0536	0.0374	0.0749
Prm4	-0.4409	0.0247	-0.0391	1.0000	0.5739	0.5990	0.6019
Prm5	-0.5760	0.1431	0.0536	0.5739	1.0000	0.6386	0.6437
Prm6	-0.5506	0.0573	0.0374	0.5990	0.6386	1.0000	0.6678
Prm7	-0.5740	0.0438	0.0749	0.6019	0.6437	0.6678	1.0000

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	1.6578	0.1907	1.2840	2.0315	75.57	<.0001
dist	1	-0.0215	0.0016	-0.0245	-0.0184	190.19	<.0001

inc		1	0.0203	0.0184	-0.0158	0.0563	1.22	0.2700
size	2	1	-0.0249	0.1595	-0.3375	0.2877	0.02	0.8758
size	3	1	0.1032	0.1521	-0.1949	0.4014	0.46	0.4973
size	4	1	0.3344	0.1454	0.0495	0.6194	5.29	0.0214
size	5	1	0.4731	0.1442	0.1904	0.7558	10.76	0.0010
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

LR Statistics For Type 3 Analysis

Source	DF	Chi-Square	Pr > ChiSq
dist	1	213.97	<.0001
inc	1	1.22	0.2699
size	4	21.19	0.0003

The output of the simplified Poisson regression with link

$$\ln \mu = \beta_0 + \beta_1 \text{DIST} + \beta_3 \text{SIZE2} + \beta_4 \text{SIZE3} + \beta_5 \text{SIZE4} + \beta_6 \text{SIZE5}$$

is shown below.

Fitting results for the restricted model without income:

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N6
Distribution	Poisson
Link Function	Log
Dependent Variable	nuvisits
Observations Used	250

Class Level Information

Class	Value	Design Variables			
size	1	0	0	0	0
	2	1	0	0	0
	3	0	1	0	0
	4	0	0	1	0
	5	0	0	0	1

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	244	314.9173	1.2906
Scaled Deviance	244	314.9173	1.2906
Pearson Chi-Square	244	284.7341	1.1669
Scaled Pearson X2	244	284.7341	1.1669
Log Likelihood		10.7564	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	1.7957	0.1431	1.5152	2.0762	157.44	<.0001
dist	1	-0.0214	0.0016	-0.0245	-0.0184	189.88	<.0001

size	2	1	-0.0184	0.1594	-0.3308	0.2939	0.01	0.9079
size	3	1	0.0941	0.1519	-0.2035	0.3917	0.38	0.5355
size	4	1	0.3283	0.1453	0.0436	0.6130	5.11	0.0238
size	5	1	0.4610	0.1439	0.1790	0.7429	10.27	0.0014
Scale		0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.

Additional model: Treating SIZE as a continuous variable and not as a factor leads to the Poisson link

$$\ln \mu = \beta_0 + \beta_1 \text{DIST} + \beta_2 \text{INC} + \beta_3 \text{SIZE}\,.$$

The estimation results show that income can be omitted (output not shown). Omitting income leads to the results shown below. Both distance and family size are statistically significant. A change in distance by 10 miles reduces the mean number of visits by a factor of exp(-0.0212(10)) = 0.81, or 19 percent. A change in the family size by one unit increases the mean number of visits by a factor exp(0.1358) = 1.145, or 14.5 percent.

We can test whether a class factor for size is needed or whether it is sufficient to treat size as a continuous variable. The log-likelihood of the model that considers size as a factor (the full model) is 10.7564; the log-likelihood of the model that considers size as a continuous variable (the restricted model) is 9.8849. We compare the log-likelihood ratio statistic, 2(10.7564-9.8849) =1.74, to a chi-square with 3 degrees of freedom (the five intercepts in the unrestricted model, one for each of the five size groups, are tested against the linear formulation which includes two parameters, the intercept and the slope). The test statistic is small (probability value $P(\chi^2(3) \geq 1.74) = 1 - 0.37 = 0.63$ is large), indicating that it is sufficient to consider a linear component of size. A scatter plot of the number of visits against distance and fitted values from the Poisson regression against distance is also shown.

<u>Fitting results for the model with size as continuous variable:</u>

The GENMOD Procedure

Model Information

Data Set	WORK.EXER12N6
Distribution	Poisson
Link Function	Log
Dependent Variable	nuvisits
Observations Used	250

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	247	316.6602	1.2820
Scaled Deviance	247	316.6602	1.2820
Pearson Chi-Square	247	287.1920	1.1627
Scaled Pearson X2	247	287.1920	1.1627
Log Likelihood		9.8849	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	1.5453	0.1367	1.2774	1.8133	127.75	<.0001
dist	1	-0.0212	0.0015	-0.0242	-0.0182	191.10	<.0001
size	1	0.1358	0.0317	0.0736	0.1980	18.32	<.0001
Scale	0	1.0000	0.0000	1.0000	1.0000		

NOTE: The scale parameter was held fixed.