since both m and $M-m$ are even. Assume that $g(\tau) \neq 0$. Then one can easily see, from (A.11), that $g(j)=0$ for all $j \neq \tau$. In other words, the optimum $g(k)$ has the form $g(k)=\alpha\delta(k-\tau)$. Thus, we have completed the proof.

REFERENCES

[1] M. T. Silvia and E. A. Robinson, *Geophysical Signal Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1980.

[2] S. M. Kay, *Modern Spectral Estimation.* Englewood Cliffs, NJ: Prentice-Hall, 1988.

[3] C.-Y. Chi and M.-C. Wu, "A unified class of inverse filter criteria using two cumulants for blind deconvolution and equalization," in *Proc. ICASSP-95*, Detroit, MI, USA, May 1995, pp. 1960–1963.

[4] ——, "Inverse filter criteria for blind deconvolution and equalization using two cumulants," *Signal Processing*, vol. 43, no. 1, pp. 55–63, Apr. 1995.

[5] R. A. Wiggins, "Minimum entropy deconvolution," *Geoexploration*, vol. 16, pp. 21–35, 1978.

[6] O. Shalvi and E. Weinstein, "New criteria for blind deconvolution of nonminimum phase systems (channels)," *IEEE Trans. Inform. Theory*, vol. 36, no. 2, pp. 312–321, Mar. 1990.

[7] J. K. Tugnait, "Estimation of linear parametric model using inverse filter criteria and higher order statistics," *IEEE Trans. Signal Processing*, vol. 41, no. 11, pp. 3196–3199, Nov. 1993.

[8] C.-Y. Chi and J.-Y. Kung, "A new cumulant based inverse filtering algorithm for identification and deconvolution of nonminimum-phase systems," in *Proc. IEEE 6th SP Workshop Stat. Signal Array Processing*, Victoria, Canada, Oct. 1992, pp. 144–147.

[9] ——, "A new identification algorithm for allpass systems by higher-order statistics," *Signal Processing*, vol. 41, no. 2, pp. 239–256, Jan. 1995.

[10] C. L. Nikias and A. P. Petropulu, *Higher-Order Spectra Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

Fractal Estimation Using Models on Multiscale Trees

Paul W. Fieguth and Alan S. Willsky

Abstract—**In this correspondence, we estimate the Hurst parameter H of fractional Brownian motion (or, by extension, the fractal exponent φ of stochastic processes having $1/f^{\varphi}$-like spectra) by applying a recently introduced multiresolution framework. This framework admits an efficient likelihood function evaluation, allowing us to compute the maximum likelihood estimate of this fractal parameter with relative ease. In addition to yielding results that compare well with other proposed methods, and in contrast with other approaches, our method is directly applicable with, at most, very simple modification in a variety of other contexts including fractal estimation given irregularly sampled data or nonstationary measurement noise and the estimation of fractal parameters for 2-D random fields.**

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The authors are with the Laboratory for Information and Decision Systems, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

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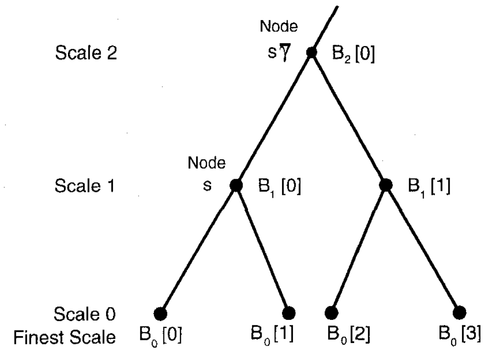

Fig. 1. Dyadic tree structure used for the estimator of this correspondence.

I. INTRODUCTION

Many natural and human phenomena have been found to possess $1/f$-like spectral properties, which has led to considerable study of $1/f$ processes. One class of such processes that is frequently used because of its analytical convenience and tractability is the class of fractional Brownian motion (fBm) processes, which were introduced by Mandelbrot and Van Ness [8]. For practical computation purposes, we consider only sampled versions of continuous time fBm processes $B(t)$, i.e.

$$B[k] = B(k\Delta t) \quad k \in Z \tag{1}$$

for which the associated nonstationary covariance is

$$E\{B[k], B[m]\} = \frac{\sigma^2}{2}(\Delta t)^{2H}(|k|^{2H} + |m|^{2H} - |k-m|^{2H}) \tag{2}$$

where σ and H are scalar parameters that completely characterize the process, and H is the quantity we wish to estimate. Previous estimators have been developed addressing this problem, notably those of Wornell and Oppenheim [11], Kaplan and Kuo [4], Tewfik and Deriche [10], and Flandrin [3]. The exact maximum likelihood (ML) calculation for H is computationally difficult (see [10]); to address this difficulty, fractal estimators typically fall into one of the two following classes to achieve computational efficiency:

1) optimal algorithms, admitting efficient solutions, based on $1/f$-like models other than fBm;
2) approximate or suboptimal algorithms developed directly from the fBm model.

Our approach and that of [11] fall into the former category, whereas the methods in [3], [4], and [9] fall into the latter. In particular, the approach in [11] is based on a $1/f$-like process constructed using wavelets in which the wavelet coefficients are independent, with variances that vary geometrically with scale with exponent H. The method in [4] determines the exact statistics of the Haar wavelet coefficients of the discrete fractional Gaussian noise (DFGN) process $F[k] = B[k+1] - B[k]$ and then develops an estimator by assuming, with some approximation, that the coefficients are uncorrelated.

The goal of our research, on the other hand, is the development of a fast estimator for H that functions under a broader variety of measurement circumstances, for example, the presence of gaps in the measured sequence, measurement noise having a time-varying variance, and higher dimensional processes (e.g., 2-D random fields). The basis for accomplishing this is the utilization of a recently

TABLE I

SCALE TO SCALE RATIOS OF THE STANDARD DEVIATION OF THE HAAR WAVELET DETAIL COEFFICIENTS OF fBm FOR FOUR VALUES OF H : g_m REPRESENTS THE STANDARD DEVIATION OF THE WAVELET DETAIL COEFFICIENT AT SCALE m, WHERE THE FINEST SCALE IS $m = 0$. THE DEVIATION OF THE VARIANCE PROGRESSION FROM AN EXPONENTIAL LAW IS MOST PRONOUNCED AT FINE SCALES AND FOR LOW VALUES OF H

Variance Ratio:	$H = 0.25$	$H = 0.50$	$H = 0.75$	$H = 0.9$
$\log_2(g_9/g_8)$	0.250			
$\log_2(g_8/g_7)$	0.249	⋮		
$\log_2(g_7/g_6)$	0.247	0.500	⋮	
$\log_2(g_6/g_5)$	0.242	0.499	0.750	⋮
$\log_2(g_5/g_4)$	0.228	0.496	0.749	0.900
$\log_2(g_4/g_3)$	0.188	0.484	0.745	0.898
$\log_2(g_3/g_2)$	0.091	0.437	0.727	0.892
$\log_2(g_2/g_1)$	-0.084	0.292	0.650	0.861

TABLE II

COMPARISON OF TWO MULTISCALE ESTIMATORS: THE ESTIMATES IN THE TOP ROW ARE BASED ON CHOICES OF COEFFICIENT VARIANCES g_m^2, WHICH ARE AN EXPONENTIAL FUNCTION OF SCALE m (15). THE ESTIMATES IN THE BOTTOM ROW ARE BASED ON THE EXACT fBm VARIANCES OF (13). THE RESULTS ARE BASED ON 64 fBm SAMPLE PATHS, EACH OF LENGTH 2048 SAMPLES, WITH NO ADDITIVE NOISE

Variance Rule:		$H = 0.25$	$H = 0.50$	$H = 0.75$	$H = 0.90$
Variances assumed exponential with scale	$\hat{H}$:	0.05	0.40	0.70	0.91
Variances based on exact result	$\hat{H}$:	0.24	0.51	0.75	0.92

introduced multiscale framework [1], [5]. The next section gives a brief description of this framework, followed by the development of the estimator, and finally a description of estimation results.

II. MULTISCALE FRAMEWORK

In the framework developed in [1] and [5], stochastic models are constructed recursively in scale on multilevel trees. Specifically, let s index the nodes of a tree $\mathcal{T}$ (refer to Fig. 1), which, for the purposes of this correspondence, may be considered to be a dyadic tree, although the framework permits much greater flexibility. Let $s_o \in \mathcal{T}$ designate the root node of $\mathcal{T}$; in addition, let $s\bar{\gamma}$ denote the parent node of $s \neq s_o$. Each node $s \in \mathcal{T}$ has associated with it a state vector $x(s)$ and, possibly, an observation vector $y(s)$. Stochastic models are written recursively on $\mathcal{T}$.

$$x(s) = A(s)x(s\bar{\gamma}) + G(s)w(s) \quad \forall s \in \mathcal{T},\, s \neq s_o \tag{3}$$

where $w(s)$ is a white Gaussian noise process with identity covariance. Similarly, noisy observations of the process are permitted on an arbitrary subset of the tree nodes:

$$y(s) = C(s)x(s) + v(s) \quad \forall s \in \mathcal{O} \subseteq \mathcal{T} \tag{4}$$

where $v(s)$ is white and Gaussian with covariance $R(s)$. In general, tree variables at all scales may be physically meaningful and measurable, or they may be abstract—a by-product of achieving the desired statistics on the finest level of the tree. In this correspondence, coarse scale nodes are abstract, with a $1/f$-like process residing on the finest scale.

For those multiscale stochastic models that can be written in the form (3) and (4), the following two problems possess extremely efficient algorithmic realizations [1], [6]:

1) given observations $y()$, determine the optimum least-square estimate for $x()$.
2) determine the likelihood $l[A(), G(), C(), R(), y()]$ of a set of observations $y()$.

The latter algorithm permits the estimation of any parameter embedded in the multiscale model, which is the very problem we have set out to solve: By formulating an appropriate multiscale model $A(s, H)$, $G(s, H)$, $C(s)$, and $R(s)$, an estimator for H may be written abstractly as

$$\hat{H} = \arg_H \max\, l\{A(s, H), G(s, H), C(s), R(s), B[k]\}. \tag{5}$$

As was the case with Wornell and Oppenheim [11], we do not construct an exact model of fBm, but rather choose an appropriate approximation—in our case within this multiscale framework. The selection of such a multiscale model is achieved in the next section.

III. FRACTAL ESTIMATOR

We will now develop a multiscale *design model* for fBm, i.e., a model strictly to be used for designing an estimator for H and *not* for the simulation of fractal processes. The basis for our design model is the resolution-to-resolution scaling law of fBm. Similar design models were proposed by Kaplan and Kuo [4], who applied the Haar wavelet to the incremental process $F[k]$, and by Wornell and Oppenheim [11], who applied higher order Daubechies wavelets to $B[k]$. We will use the multiscale framework of the previous section to develop a Haar wavelet multiscale stochastic model that applies directly to $B[k]$. This choice of wavelet is motivated by the particularly simple realization of the Haar wavelet in our multiscale framework by using a dyadic tree structure (see Fig. 1):

$$\text{Coarse Scales:} \quad \begin{cases} \text{If } s \text{ is left descendant of its parent} \\ \quad x(s) = \begin{bmatrix} 1 & +1 \\ 0 & 0 \end{bmatrix} x(s\bar{\gamma}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(s, H)w(s) \\ \text{If } s \text{ is the right descendant of its parent} \\ \quad x(s) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} x(s\bar{\gamma}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(s, H)w(s) \end{cases} \tag{6}$$

$$\text{Finest Scale:} \quad \begin{cases} \text{If } s \text{ is the left descendant of its parent} \\ \quad x(s) = [1 \quad +1]\, x(s\bar{\gamma}) + 0 \cdot w(s) \\ \text{If } s \text{ is the right descendant of its parent} \\ \quad x(s) = [1 \quad -1]\, x(s\bar{\gamma}) + 0 \cdot w(s) \\ y(s) = x(s) + v(s). \end{cases} \tag{7}$$

That is, at coarse scales, $x(s)$ consists of two scalars: a coarse approximation to the $1/f$ process and a wavelet detail coefficient, where this detail coefficient equals the difference in the coarse $1/f$-like representation between node s and its two children. At the finest scale $x(s)$ is a single scalar, representing a sample of a $1/f$-like process, and measurements of the actual fBm sequence appear as observations $y(s)$ at the finest scale.

Our design model of (6) and (7) does not yield a finest scale process having exact fBm statistics. Specifically, the design model approximates the wavelet detail coefficients as being uncorrelated. Consequently, just as with the technique in [11], our model does not exactly match the statistics of the process to be estimated.

The elements that remain to be determined in the above multiscale model are the $g(s, H)$: the variance of the detail wavelet coefficient at each node s. Expressions for the statistics of the wavelet decomposition of fBm have been determined by others [3], [9]; however, the self statistics for the special case of the Haar wavelet are easily computed as follows:

TABLE III
ESTIMATION RESULTS FOR THREE ESTIMATORS, BASED ON 64 fBm SAMPLE PATHS, EACH OF LENGTH 2048 SAMPLES, WITH NO ADDITIVE NOISE. THE EXPERIMENTAL RESULTS FOR THE FIRST TWO ESTIMATORS ARE FROM [4]

Estimator		$H=0.25$	$H=0.50$	$H=0.75$	$H=0.90$
W.O.	$\hat{H}$	0.082	0.398	0.683	0.846
	$\sigma_{\hat{H}}$	0.022	0.021	0.026	0.021
	$(H-\hat{H})_{\text{RMS}}$	0.169	0.109	0.072	0.058
K.K.	$\hat{H}$	0.252	0.499	0.748	0.899
	$\sigma_{\hat{H}}$	0.017	0.017	0.017	0.017
	$(H-\hat{H})_{\text{RMS}}$	0.017	0.017	0.017	0.017
Multiscale	$\hat{H}$	0.249	0.503	0.768	0.919
Haar	$\sigma_{\hat{H}}$	0.011	0.019	0.050	0.109
	$(H-\hat{H})_{\text{RMS}}$	0.011	0.019	0.054	0.110

- Let $B_o[k] = B[k]$, which is the fBm process of interest.
- Define $B_m[k]$ as the process obtained by coarsening $B[k]$ m times

$$B_m[k] = (B_{m-1}[2k] + B_{m-1}[2k+1])/2 \tag{8}$$

which is a relation that follows from the multiscale model of (6).
- Recall that F denotes the increments process of B. Then, from (8), it follows that

$$B_{m-1}[k] = \sum_{i=0}^{2^{m-1}-1} \frac{B[2^{m-1}k+i]}{2^{m-1}} \tag{9}$$

$$B_{m-1}[k+1] - B_{m-1}[k] = \sum_{i=0}^{2(2^{m-1}-1)} F[2^{m-1}k+i]\,\frac{2^{m-1} - |2^{m-1}-i-1|}{2^{m-1}} \tag{10}$$

$$\equiv \sum_{i=0}^{2(2^{m-1}-1)} F[2^{m-1}k+i]c_i. \tag{11}$$

- From the stationarity of the increments process $F[k]$, and from (8), the desired variance expression may be deduced from (11):

$$E[(B_m[k] - B_{m-1}[2k])^2] = \tfrac{1}{4} E[(B_{m-1}[2k] - B_{m-1}[2k+1])^2] \tag{12}$$

$$= \tfrac{1}{4} \sum_{i=-2(2^{m-1}-1)}^{2(2^{m-1}-1)} \lambda_F[i] \sum_{j=-\min(0,i)}^{2(2^{m-1}-1)+\min(0,-i)} c_j c_{i+j} \tag{13}$$

$$= g_m^2(H)$$

where λ_F is the covariance function of $F[k]$:

$$\lambda_F[i] = \frac{\sigma^2}{2}[|i+1|^{2H} + |i-1|^{2H} - 2|i|^{2H}](\Delta t)^{2H}. \tag{14}$$

It should be noted that the result in (13) is equal to the variance of (50) and (51) derived in [3].

By way of comparison, in [11] an exponential variation with scale for g_m was proposed to estimate the fractal exponent φ of $1/f^{\varphi}$ processes. Since a fBm process having parameter H has an associated $1/f^{2H+1}$ spectrum, [11] leads to

$$g_m^2(H) = \beta 2^{2mH}$$

i.e.,

$$\log_2 \frac{g_m(H)}{g_{m-1}(H)} = H. \tag{15}$$

TABLE IV
PERFORMANCE OF THE fBm ESTIMATOR FOR TWO EXAMPLES: IRREGULAR SAMPLING (UNDER THE ASSUMPTION THAT ALL OF THE INTERSAMPLE SPACINGS ARE INTEGER MULTIPLES OF SOME PERIOD T, i.e., THE MEASURED SEQUENCE MAY BE REPRESENTED AS A UNIFORMLY SAMPLED SEQUENCE WITH GAPS) AND NONSTATIONARY MEASUREMENT NOISE. THE TOP ROW LISTS ESTIMATION RESULTS GIVEN THE fBm SAMPLE PATHS OF TABLE III, WITH 10% OF THE MEASUREMENTS DISCARDED AT RANDOM. THE BOTTOM ROW SHOWS ESTIMATION RESULTS GIVEN THE SAMPLE PATHS OF TABLE III WITH ADDED GAUSSIAN NOISE HAVING A DIAGONAL COVARIANCE WHERE THE DIAGONAL ENTRIES ARE GIVEN BY $R[k] = \frac{1}{4}, \exp\{-2[(k-1024)/500]^2\}$. IN BOTH CASES, THE RESULTS ARE BASED ON 64 fBm SAMPLE PATHS, EACH OF LENGTH 2048 SAMPLES

Circumstance		$H=0.25$	$H=0.50$	$H=0.75$	$H=0.90$
Irregular	$\hat{H}$	0.246	0.507	0.781	0.937
Sampling	$\sigma_{\hat{H}}$	0.033	0.044	0.076	0.124
	$(H-\hat{H})_{\text{RMS}}$	0.033	0.045	0.082	0.128
Nonstationary	$\hat{H}$	0.268	0.511	0.769	0.918
Measurement	$\sigma_{\hat{H}}$	0.011	0.019	0.051	0.109
Noise Variance	$(H-\hat{H})_{\text{RMS}}$	0.021	0.022	0.054	0.110

Table I shows the actual fBm scale to scale variance ratios as predicted by (13). The deviation from the approximate scaling law of (15) is most pronounced at low H; it is this deviation that leads to a bias for those estimators based on (15), as shown in Table II.

The actual estimator for H, based on the multiscale model of (6) and (7) and the variances of (13), takes precisely the form as outlined in (5), in which the likelihood maximization is performed using standard nonlinear techniques (e.g., the section search method of MATLAB).

IV. EXPERIMENTAL RESULTS

Sixty four fBm sample paths, each having a length of 2048 samples, were generated using the Cholesky decomposition method of [7]; this is precisely the same approach as in Kaplan and Kuo [4], whose experimental results form the basis of comparison with ours.

The performance of three fBm estimators is compared in Table III. The bias in the estimator of [11] for low H, as was argued earlier based on Table I, is evident. In addition, recall that the multiscale model of (6) and (7) assumed the wavelet detail coefficients to be uncorrelated; this assumption becomes progressively poorer as H increases [3], leading to an increase in the error variance for our estimator at large H. Nevertheless, our method still performs reasonably well over quite a wide range of values of H. Moreover, using the techniques developed in [5], we can construct higher order multiscale models that account for most of the residual correlation in the wavelet coefficients. However, since fBm itself is an idealization, the benefit in practice of such higher order models over that based on the low-order model (6) and (7) depends on the application.

Our approach also applies equally well in a variety of other settings. In particular, the multiscale measurement model (4) does not assume that $R(s)$ is constant over the finest scale (permitting nonstationary measurement noise), and it does not assume that a measurement exists at each finest scale node (permitting nonuniformly sampled processes in which each intersample spacing is an integer multiple of some period T). Both of these special cases are accomplished with essentially no change in the algorithm; an example of each is illustrated in Table IV. In addition, by using a quadtree rather than a dyadic tree, we can also apply these techniques in 2-D. An example of such an application to the estimation of nonisotropic fractal parameters for a 2-D random field based on irregular, nonstationary data is given in [2].

REFERENCES

[1] K. Chou, A. Willsky, and A. Benveniste, "Multiscale recursive estimation, data fusion, and regularization," *IEEE Trans. Automat. Contr.*, vol. 39, no. 3, pp. 464–478, 1994.

[2] P. Fieguth, "Application of multiscale estimation to large scale multidimensional imaging and remote sensing problems," Ph.D. thesis, Dept. of Elect. Eng. and Comput. Sci., Mass. Inst. of Technol., 1995.

[3] P. Flandrin, "Wavelet analysis and synthesis of fractional Brownian motion," *IEEE Trans. Inform. Theory,* vol. 38, no. 2, pp. 910–917, 1992.

[4] L. Kaplan and C. C. Kuo, "Fractal estimation from noisy data via discrete fractional Gaussian noise (DFGN) and the Haar basis," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3554–3562, 1993.

[5] M. Luettgen, W. Karl, A. Willsky, and R. Tenney, "Multiscale representations of Markov random fields," *IEEE Trans. Signal Processing,* vol. 41, no. 12, pp. 3377–3396, 1993.

[6] M. Luettgen and A. Willsky, "Likelihood calculation for a class of multiscale stochastic models, with application to texture discrimination," *IEEE Trans. Image Processing*, vol. 4, no. 2, Feb. 1994.

[7] T. Lundahl, W. Ohley, S. Kay, and R. Siffert, "Fractional Brownian motion: A maximum likelihood estimator and its application to image texture," *IEEE Trans. Med. Imaging*, vol. MI-5, pp. 152–161, 1986.

[8] B. Mandelbrot and J. van Ness, "Fractional Brownian motions, fractional noises, and applications," *SIAM Rev.*, vol. 10, pp. 422–437, 1968.

[9] A. Tewfik and M. Kim, "Correlation structure of the discrete wavelet coefficients of fractional Brownian motion," *IEEE Trans. Inform. Theory,* vol. 38, no. 2, pp. 904–909, 1992.

[10] A. Tewfik and M. Deriche, "Maximum likelihood estimation of the parameters of discrete fractionally differenced Gaussian noise processes," *IEEE Trans. Signal Processing,* vol. 41, no. 10, pp. 2977–2989, 1993.

[11] G. Wornell and A. Oppenheim, "Estimation of fractal signals from noisy measurements using wavelets," *IEEE Trans. Signal Processing,* vol. 40, pp. 611–623, 1992.

Corrections to "Signal Processing Applications of Oblique Projection Operators"

Richard T. Behrens and Louis L. Scharf

In Section VI of the above paper,[1] we presented an application of our results to intersymbol interference. Although the equations in that section are correct, we drew an erroneous conclusion in the text. The incorrect statement is that "...we can at least make sure that the first element of $\hat{\boldsymbol{y}}_i$, namely $\hat{y}(i)$, has all ISI removed by choosing as $\mathbf{S}$ the first m columns of $\mathbf{H}$ since these are the only columns that contribute to the first element of $\boldsymbol{y}_i$."

On the contrary, the remaining columns of $\mathbf{H}$ can, and generally do, contribute to the first element of $\boldsymbol{y}_i$, and it is not generally possible to construct a zero-forcing equalizer in this manner.

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R. T. Behrens is with Cirrus Logic Inc., Broomfield, CO 80021 USA.

L. L. Scharf is with the Department of Electrical and Computer Engineering, University of Colorado, Boulder, CO 80309-0425 USA.

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[1] ____, *IEEE Transactions on Signal Processing,* vol. 42, no. 6, pp. 1413–1424, June 1994.

A Delta MYWE Algorithm for Parameter Estimation of Noisy AR Processes

Qiang Li, H. (Howard) Fan, and Erlandur Karlsson

Abstract—**In this correspondence, we develop a delta-operator-based modified Yule–Walker equation algorithm (MYWE) for parameter estimation of a noisy autoregressive (AR) process. The methodology in developing this new algorithm is similar to the previous works on pure AR processes. Computer simulation results are given to show the improvement of performance in estimating AR parameters in white noise over the q-operator MYWE algorithm.**

I. INTRODUCTION

In [1] and [2], δ operator off-line and on-line Levinson-type algorithms have been developed. These algorithms have shown their advantage over q-operator-based algorithms for their superior numerical ability for ill-conditioned data. However, It is well known that Levinson-type algorithms can efficiently estimate noise-free AR processes but perform poorly when estimating a noisy AR process [5]. Therefore, these δ operator algorithms still cannot overcome the difficulty of estimating noisy AR processes accurately, which is also the case with q-operator Levinson algorithms.

One of the methods to estimate noisy AR processes is to use an autoregressive moving average (ARMA) model for a noisy AR process and the so-called modified Yule–Walker equation (MYWE) algorithm [5] to estimate the AR parameters under this situation. However, when it is applied to an ill-conditioned noisy AR process and implemented in finite precision, the poor performance of this q-operator-based algorithm becomes obvious. In other words, when poles of this AR process are close to the so-called lightly damped low-frequency (LDLF) region that corresponds to fast sampling, the estimation error of the AR parameters using the traditional q-operator MYWE algorithm will become large and unacceptable in finite precision implementation.

In this correspondence, a δ_b-operator MYWE algorithm is developed by transforming the q-operator MYWE algorithm into the δ_b domain. A backward difference is defined as $\delta_b = \frac{1-q^{-1}}{\Delta}$ where Δ is a positive constant and is often chosen to be the same as the sampling interval, and q^{-1} is a delay operator. Some δ_b-operator-related issues such as the relationship and comparison of δ_b and δ operators were addressed in [2] and are not repeated in this correspondence. Computer simulation results are given to show the improvement of the accuracy of parameter estimation performed by the δ_b-operator MYWE algorithm over q-operator MYWE algorithm for finite precision implementation.

II. DEVELOPMENT OF A δ_b-OPERATOR MYWE ALGORITHM

A noisy discrete-time pth order AR process can be modeled as

$$x(t) = \frac{e(t)}{A(q^{-1})} + v(t)$$

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Q. Li is with Hughes Network Systems, Germantown, MD 20876 USA.

H. Fan is with the Department of Electrical and Computer Engineering, University of Cincinnati, Cincinnati, OH 45221 USA (e-mail: h.fan@uc.edu).

E. Karlsson is with Systems and Control Group, Uppsala University, Uppsala, Sweden.

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