PROBABILISTIC CONTINUOUS EDGE DETECTION USING LOCAL SYMMETRY

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ABSTRACT

We describe a new model for the detection of edges in a given image. The model takes the invariance of local features of the image w.r.t translational symmetry operations into account. This is done by expressing the symmetries as a local Lie group and their associated Lie algebras in the regularizer of our model. Central to our work is the formulation of an *energy density* for the regularizer which itself is invariant under the action of a Lie algebra. Formulated as a Gaussian Markov Random Field, the parameters of the model are estimated by the EM principle.

I. INTRODUCTION

The idea of combining multiple image processing tasks into a single model has gained popularity, triggered by the seminal paper of Mumford and Shah and related work [1], [2], [3], which addressed the problem of image denoising. Given a noisy image Y, the denoised image X and edge-set S are jointly estimated by maximizing the posterior

$$\begin{aligned} P\left(X,S|Y\right) &= P\left(X|Y\right)\cdot P\left(S|X\right) \\ -\ln P\left(X|Y\right) &= \mu \int\limits_{\Omega} \left(X-Y\right)^2 \qquad (1) \\ -\ln P\left(S|X\right) &= \frac{1}{2}\int\limits_{\Omega\backslash S} \left|\nabla X\right|^2 dx + \nu\mathcal{H}\left(S\right) \end{aligned}$$

where μ and ν parameters. This approach was developed to further combine optical flow estimation and image denoising [4], image deblurring and segmentation [5], [6], [7], level-set segmentation [8], [9] and image registration [3].

At the center of the approach is the Hausdorff measure $\mathcal{H}\left(S\right)$ constraining the length of the edge-set S. Since the discretization of the edge-set is a tedious problem, [10] introduced a phase-field approach in which the edge-set S is implicitly described as the null-space of a function ϕ called a *phasefield*

$$S = \left\{x|\phi\left(x\right)=0\right\} \qquad (2)$$

[10] also showed that there exists an approximation to the conditional $P\left(S|X\right)$ in the form of a limit procedure with a limiting parameter ϵ

$$\lim_{\epsilon\to 0} P\left(\phi|X,\epsilon\right) \to P\left(S|X\right), \qquad (3)$$

with further extensions [11], [12], [7] to multi-phase formulations to jointly produce multiple segmentions of a given image. The approximating conditional $P\left(\phi|X,\epsilon\right)$ contains no prior information about the geometry of S, however such information is important, particularly for the reconstruction of object boundaries.

The purpose of this paper is theoretical, to propose a new prior for ϕ, embedding assumptions on the geometry of S. We will present an overview of [10] and highlight the problems of the conditional $P\left(\phi|X,\epsilon\right)$; we will then introduce the concept of conservation which we use in the development of our prior.

II. CONTINUOUS SEGMENTATION

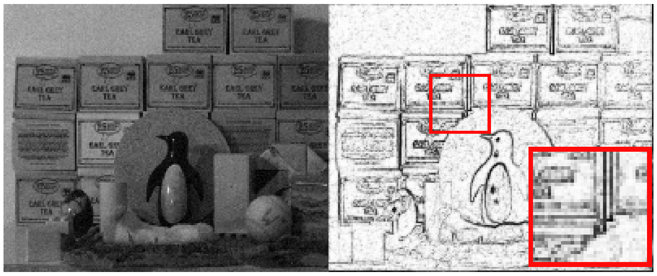
Our focus is on $P\left(\phi|X,\epsilon\right)$, the posterior for ϕ. The following posterior was proposed in [10]:

$$\begin{aligned} P\left(\phi|X,\epsilon\right) &\sim P\left(X|\phi\right)\cdot P\left(\phi|\epsilon\right) \qquad (4) \\ -\ln P\left(X|\phi\right) &= \int\limits_{\Omega}\left(\frac{1}{2}\phi\left(x\right)^2\left\|\nabla X\right\|^2\right)dx \qquad (5) \\ -\ln P\left(\phi|\epsilon\right) &= \int\limits_{\Omega}\left(\frac{1}{2\epsilon}\left(\phi\left(x\right)-1\right)^2+\frac{\epsilon}{2}\left\|\nabla\phi\right\|^2\right)dx \end{aligned}$$

The likelihood $P\left(X|\phi\right)$ forces ϕ to 0 at discontinuities in X ($\left\|\nabla X\right\| \gg 0$). The prior $P\left(\phi|\epsilon\right)$ states the assumptions $\phi = 1$ almost everywhere and that ϕ should be continuous. Furthermore in [10] it is proven that in the limit $\epsilon \to 0$ the maximum a posteriori (MAP) of the posterior is the exact edge-set S of the image X

$$\tilde{S} = \left\{x|\tilde{\phi}\left(x\right)=0\right\} \qquad \tilde{\phi} = \underset{\phi}{\operatorname{argmax}}\left\{\lim_{\epsilon\to 0} P\left(\phi|X,\epsilon\right)\right\}$$

While this model leads to good results on images with no noise, it performs poorly on noisy images, as can be seen in Fig. 1b, which plots the edge function ϕ learned from a noisy image. The edge corruption is caused by the fact that in the limit $\epsilon \to 0$ the prior $P\left(\phi|\epsilon\right)$ does not impose any regularity conditions on ϕ. For precisely this reason, our focus is to construct a new prior for $\nabla\phi$, $P\left(\nabla\phi\right)$. $P\left(\nabla\phi\right)$ will impose regularity on the tangential component of S.



(a) (b)

Fig. 1: 1a: Noisy image Y, 1b: MAP of ϕ from (4)

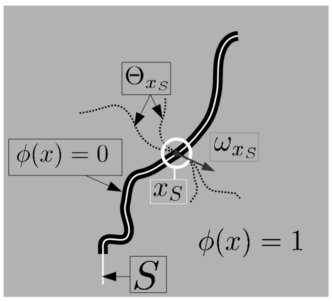


Fig. 2: Image with example edge-map ϕ. The white line denotes the exact edge-set S and the thick black line denotes the approximation $\tilde{S} = \{\phi(x) = 0\}$ to S. $\omega_{x_S}^{\perp}$ is the normal velocity vector of the set of trajectories Θ_{x_S} intersecting S at x_S.

III. CONSERVED REGULARIZER DENSITY

Our new approach is similar to anisotropic diffusion [13], [14] in that the prior $P(\nabla\phi)$ is required to assume smoothness along the edge-set S but not normal to it, and furthermore to assume smoothness in the domain $\Omega \diagdown S$. In contrast to [13], [14] our method does not rely on a point-wise eigenvalue analysis of the structure tensor since it computes this information implicitly. The main constraint we pose on $P(\nabla\phi)$ is that it should be conditionally independent on S

$$P(\nabla\phi|S) = P(\nabla\phi) \tag{6}$$

The rationale for the constraint is that we assume two configurations $\phi_{1,2}$ with different edge-sets $S_{1,2}$ to have the same probability. That is, we do not want to state any preference for one edge-set over another, irrespective of size and geometry.

III-A. Lie Groups and Conserved Quantities

We wish to make more precise the constraint on $P(\nabla\phi)$, eq. (6). We consider a set of arbitrary trajectories Θ_{x_S}, as shown in Figure 2, which are oriented at the points x_S in the direction of the vector field $\omega_{x_S}^{\perp}$, which is normal to S

$$\Theta_{x_S} = \left\{\theta_{x_S} : [0,1] \to \Omega | \theta_{x_S}(0) = x_S, \dot{\theta}_{x_S}(0) = \omega_{x_S}^{\perp}\right\} \tag{7}$$

Θ_{x_S} fully characterizes the edge-set S and

$$P(\nabla\phi|S) = P\left(\nabla\phi|\left\{\omega^{\perp}(x_S)\right\}\right) \tag{8}$$

Thus the constraint (6) translates via (8) to

$$P\left(\nabla\phi|\left\{\omega^{\perp}(x_S)\right\}\right) = P(\nabla\phi) \tag{9}$$

Ths latter constraint (9) is easier to fulfill in the context of Lie group theory if $P(\nabla\phi)$ belongs to the set of exponential distributions, such as the energy-density $\mathcal{E}(x)$

$$-\ln P(\nabla\phi) = \int_{\Omega} \mathcal{E}(x)\, dx \tag{10}$$

An n dimensional Lie group $\mathbb{G}$ over a domain Ω is imposed by a Lie algebra via the exponential map

$$\exp\left(\sum_i \omega_i(x) \cdot \mathfrak{g}_i\right) = g_\omega \in \mathbb{G} \quad \omega : \Omega \to \mathbb{R}^n \tag{11}$$

An infinitesimal Lie group $\mathbb{U}_r \subset \mathbb{G}$ is the group within the open ball U_r around unity

$$x'_j = x_j + \sum_i \frac{\delta x'_j}{\delta\omega_i} \cdot \omega_i(x) \quad \|\omega\| < r$$

$$\phi'(x') = \phi(x) + \sum_i \omega_i(x) \cdot (D(\mathfrak{g}_i) \circ \phi)(x) \tag{12}$$

where the representational operator D will be defined later in (17). From Noether's theorem [15], [16], given an integral $I = \int_Q \mathcal{E}(x,\phi)\, dx$ over a region $Q \subset \Omega$ enclosed by a subset $S_{\mathbb{G}} \subset S$. The change of I under the action (12) is equal to the divergence of n vector fields $W_a : \Omega \to \mathbb{R}^2$, $1 \le a \le n$

$$\delta I = I - I' = \sum_a \int_Q \mathrm{div} W_a(x) \cdot \omega_a(x)\, dx \tag{13}$$

If the value of the integral $I = \int_Q \mathcal{E}(x,\phi)\, dx$ remains constant under the action (12)

$$I = I' \tag{14}$$

then the vector fields W_a must be *divergence free*

$$\mathrm{div}\, W_a|_Q = 0 \tag{15}$$

where the W_a are conserved quantities. Using Gauss's law (15) translates to

$$W_a \cdot \omega^{\perp}|_{S_{\mathbb{G}}} = 0 \tag{16}$$

where $S_{\mathbb{G}}$ is a subset of S and $\omega^{\perp}$ is the normal component of ω (11) on $S_{\mathbb{G}}$. Thus the integral I is independent of $\omega^{\perp}$ and $S_{\mathbb{G}}$. The pertinent result here is that our prior constraint is fulfilled for $S_{\mathbb{G}}$

$$P(\nabla\phi|S) = P(\nabla\phi|S \backslash S_{\mathbb{G}})$$

For the rest of this paper we restrict ourselves to the infinitesimal translation group $\mathbb{T}$ over the domain Ω, which is a two dimensional Lie group with generators $\mathfrak{t}_1$ and $\mathfrak{t}_2$. The representation operators $D(\mathfrak{t}_i)$ are given by the partial derivatives

$$D(\mathfrak{t}_1) = \partial_x \quad D(\mathfrak{t}_2) = \partial_y \tag{17}$$

and the W_a by

$$W_a^i = \delta_{a,i} \cdot \mathcal{E}(x) \tag{18}$$

Given (17), then g_ω in (11) reduces to an element of the group $\mathbb{T}$. The conditional dependency of ϕ on S reduces via (13) to the form

$$I' = -\ln P(\nabla\phi|S) \tag{19}$$

$$= \int_\Omega \mathcal{E}(x)\,dx + \int_\Omega \partial_i\mathcal{E} \cdot \omega_i dx \tag{20}$$

If condition (14) is fulfilled then from Noether's theorem (15)

$$\partial_i \mathcal{E} = 0 \tag{21}$$

Now given (21) we see that $\nabla\phi$ is conditionally independent of the edge-set S

$$P(\nabla\phi|S) = P(\nabla\phi) \tag{22}$$

IV. STRUCTURE TENSOR

In this section we will introduce an energy-density $\mathcal{E}$ satisfying condition (21) in the absence of noise. Our method is based on the structure tensor (ST) A^σ introduced in [17], a well-known tool in image processing, defined as the 2 by 2 matrix

$$A^\sigma(\vec{x}) := \begin{pmatrix} \langle z_x^2 \rangle_{\vec{x}}^\sigma & \langle z_x \cdot z_y \rangle_{\vec{x}}^\sigma \\ \langle z_x \cdot z_y \rangle_{\vec{x}}^\sigma & \langle z_y^2 \rangle_{\vec{x}}^\sigma \end{pmatrix} \tag{23}$$

where we define

$$z_x = D(\mathfrak{t}_1)\,\phi \quad z_y = D(\mathfrak{t}_2)\,\phi$$
$$\langle f \rangle_{\vec{x}}^\sigma = (G_\sigma \star f)(x)$$

The convolution filter G_σ is a Gaussian filter with standard deviation σ. The eigenvalues a_1 and a_2 of A^σ characterize the local neighborhood as

1) $a_{1,2} = 0 \Rightarrow$ Constant neighborhood
2) $a_1 = 0$, $a_2 > 0 \Rightarrow$ Neighborhood with dominant orientation in a_2, constant in a_1
3) $a_{1,2} > 0$, $a_2 \gg a_1 \Rightarrow$ Neighborhood with dominant orientation in a_2, slowly varying in a_1
4) $a_{1,2} > 0$, $a_1 \approx a_2 \Rightarrow$ Neighborhood with no dominant orientation (noise, corners)

In order to have $\mathcal{E}$ discriminate between cases 1, 2 and 3, 4 we set

$$\mathcal{E}(x) = \mathcal{E}_{ST}(x) := \frac{\lambda_z}{2} \det\big(A^\sigma(x)\big) \tag{24}$$

avoiding the actual computation of a_1 and a_2, and ensuring rotation invariance.

First, for cases 1 and 2, $\mathcal{E}_{ST}$ vanishes and thus is trivially conserved.

Next, for case 3, the derivative tangential to the line of $\mathcal{E}_{ST}$ is non-zero, $\partial_y \mathcal{E}_{ST} \neq 0$. For the derivative normal to the line it is easily shown that

$$\partial_x \langle z_x^2 \rangle_{\vec{x}}^\sigma = 0 \tag{25}$$

at the discontinuity. Thus we have

$$\partial_x \mathcal{E}_{ST} = \frac{1}{2} \langle z_x^2 \rangle_{\vec{x}} \partial_x \left(\langle z_y^2 \rangle_{\vec{x}} \right) \neq 0 \tag{26}$$

meaning that the conservation of $\mathcal{E}_{ST}$ is broken only by $\langle z_y^2 \rangle_{\vec{x}}$. So regularizing ϕ in the tangential direction *alone* retains conservation of $\mathcal{E}_{ST}$ in *both* directions.

Finally for case 4 the constraint (21) cannot hold for any dimension, with the effect that any closed neighborhood $R \subset \Omega$ with $R \cap S = \{0\}$ condition (21) doesn't hold and by Gauss' Law we have

$$\mathcal{E}_{ST}|_{\partial R} \neq 0 \tag{27}$$

This means that $\mathcal{E}_{ST}$ penalizes this case.

At this point we are ready to estimate ϕ to see the effect of the prior $P(\nabla\phi)$ in the model. $P(\nabla\phi)$ in its present form is numerically difficult to handle since $\mathcal{E}_{ST}$ is quartic in ϕ. Our approach to this problem is to loosen the constraint $\vec{z} = \nabla\phi$ by defining a relaxed prior

$$-\ln P_R(\vec{z}|\phi) = \int_\Omega \mathcal{E}_R(x)\,dx$$
$$\mathcal{E}_R(x) = \frac{\lambda_\phi}{2}\left[(z_x - \partial_x\phi)^2 + (z_y - \partial_y\phi)^2\right] + \frac{\lambda_z}{2}\mathrm{Det}(A^\sigma) \tag{28}$$

This prior is Gaussian for the components of $\vec{z}$. P_R allows the development of an EM-like strategy to calculate a phasefield $\bar{\phi}$ given an initial phasefield ϕ^0:

1) Start with initial guess ϕ^0
2) E-Step: find MAP $\vec{z}^n$ from $P_R\left(\vec{z}|\phi^{n-1}\right)$
3) M-Step: set $\phi^n = \underset{\phi}{\operatorname{argmax}} \{P_R(\vec{z}^n|\phi)\}$
4) Repeat the E,M steps until $\left\|\phi^n - \phi^{n-1}\right\| < \epsilon$
5) Exit with result $\bar{\phi} = \phi^n$

This is essentially a diffusion algorithm which regularizes the phasefield ϕ^0. One example is shown in figure 3. ϕ^n resembles the constant line case 2, and it shows that the component z_y which breaks conservation of $\mathcal{E}_{ST}$ sets the direction of regularization.

We now use our relaxed prior $P_R(\vec{z}|\phi)$ as a substitute for the prior $P(\phi|\epsilon)$ in eq (4)

$$P(\vec{z},\phi|X) = P(X|\phi) \cdot P_R(\vec{z}|\phi) \cdot P(\phi) \sim \exp(-E)$$
$$E = \int_\Omega \left(\frac{\lambda}{2}\phi(x)^2 \|\nabla X\|^2 + \frac{1}{2}(\phi(x) - 1)^2 + \mathcal{E}_R(x) \right) dx \tag{29}$$

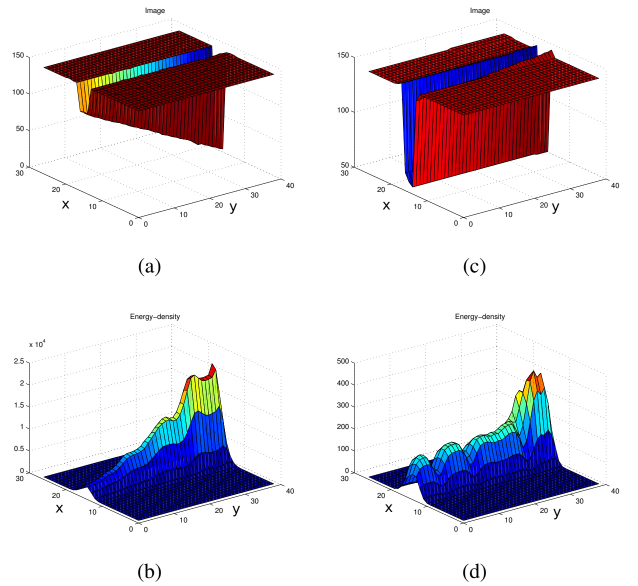


Fig. 3: We begin (a) with a synthetic initial image ϕ_0 and (b) its corresponding energy distribution $\mathcal{E}_{ST}(\phi_0)$. In contrast, (c) and (d) show the regularized $\bar{\phi}$ and $\mathcal{E}_R\left(\bar{\phi}\right)$, the result of regularization with $\mathcal{E}_{ST}$. The prior-based results demonstrate that the prior P_R penalizes the component of the gradient $\nabla\phi_0$ tangential to S while preserving the normal component.

Using this new posterior for the image X in figure 1a we calculate the MAP estimates $(\vec{z}^{\star}, \phi^{\star})$ with an algorithm similar to the aformentioned one with the difference that we set an initial guess for $\vec{z}$, $\vec{z}^0 = \mathbf{0}$. This reduces (29) to the original posterior in (4). We then minimize (29) alternately for ϕ^n and $\vec{z}^n$ until ϕ^n converges,

$$\lim_{n\to\infty} \phi^n = \phi^{\star}$$

Results for the image in figure 1a are shown in figure 4.

Relative to the method of Ambrosio and Tortorelli [10] in figure 1b, our proposed method very clearly reduces the noise in the edge space by smoothing in the tangential direction.

V. CONCLUSION AND FUTURE WORK

In this paper we addressed the problem of noise reduction on the edge-set S, governed by the phase-field ϕ in Ambrosio's and Tortorelli's approach to the Mumford-Shah posterior (1). We introduced the notion of conservation of an energy-density $\mathcal{E}_{ST}$ under the translation group T. This requirement was shown to be useful in the construction of a new prior $P(\nabla\phi)$. We required $\mathcal{E}_{ST}(x)$ to be conserved if the phase-field ϕ contained no noise at point x, and for conservation to be broken in the case of noise. As a density fulfilling these requirements we found the determinant of the structure tensor $A^{\sigma}(\vec{x})$ to be useful. Using an EM-algorithm

Fig. 4: MAP $\phi^{\star}$ of (29). Observe the significant improvement in edge-space smoothing relative to that of figure 1b.

we proved the effectiveness of our prior $P(\nabla\phi)$ on synthetic and real data.

Our framework is based on the operators in (17), however an extension to a more general Lie group $\mathbb{H}$ with generators $\mathfrak{h}_i$ is readily possible by replacing the operators $D(\mathfrak{t}_i)$ with $D(\mathfrak{h}_i)$, an approach similar to that in [18]). The idea then follows the same principles in section IV, constructing structure tensors $A_{\mathbb{G}_i}(\mathbf{x})$ for a set of N Lie groups $\mathbb{G}_i$. A possible generalization of the energy density in eq. (24) is the density

$$\mathcal{E}(\mathbf{x}) = \prod_{i=1}^{N} \mathrm{Det}\left(A_{\mathbb{G}_i}\right)(\mathbf{x}) \tag{30}$$

In theory the density (30) should preserve any edge $S_{\mathbb{G}_j}$ at the points $\mathbf{x}_S \in S_{\mathbb{G}_j}$ since the determinant of the corresponding structure tensor vanishes, $\mathrm{Det}\left(A_{\mathbb{G}_j}\right)(\mathbf{x}_S) = 0$. A thorough study of eq. (30) is planned for future experiments.

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