Hierarchical Sampling with Constraints

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Abstract. Reconstruction of porous media images is required in order
to study different properties of these material. Our research interest is
on generating samples from the posterior model in which low resolution
measurements are combined with a prior model. The reconstruction task
becomes intractable when the size of the samples increases, since it is
based on simulated annealing which is a slow convergence algorithm.
The hierarchical approaches have been applied to tackle this problem,
in the case of sampling from the prior model. However, in the posterior
sampling case, defining a suitable measurement model at each scale still
remains a challenging task. In this paper we define how we can incorpo-
rate the measurement in the hierarchical posterior model and then how
we generate samples from that model.

1 Introduction

An important class of heterogeneous materials are porous media, such as con-
crete, bone, wood, active carbon and glass, which contain complex microscopic
structures [1]. Studying different properties of these materials requires having
high resolution 2D images or 3D volumes. However, the 3D volumes, usually
obtained by Magnetic Resonance Imaging (MRI) are very low resolution and
noisy (such as the ones shown in Figure 1), and the 2D high resolution images
are obtained through a set of physical processes which usually affects the real
properties of the sample. Therefore, artificial samples of porous media are re-
quired to be generated through an image reconstruction task [10]. The recent
contributions in the reconstruction task are either only based on a prior model,
defined based on some statistical features of real high resolution training data,
and generating samples from that model [14],[2], [7], [13] [3], or based on a prior
model and the measurements, [9], [15], [8].

Our research is mainly focus on the case that the measurements, i.e. low reso-
lution real samples, are also considered in the reconstruction task. Based on this
idea we are considering a posterior sampling approach for porous media recon-
struction which has been introduced in [8]. In the posterior sampling approach
a statistical prior model is learned from 2D high resolution samples, and the low
resolution measurements, usually obtained by MRI, are fused with the prior to
form a posterior model. Then the artificial samples are generated by a posterior
sampling [4] approach. Reconstruction of large samples is challenging due to the
computational complexity of the sampling approach, since this approach is based

Sintered Glass Spheres

Carbonate Rock
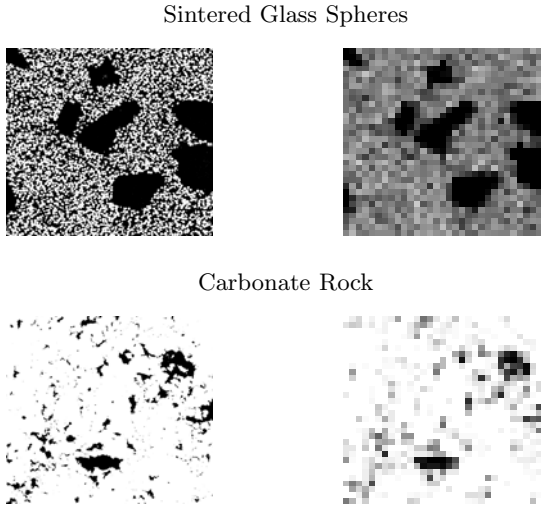
(a) High resolution samples (b) Low resolution measurement

Fig. 1. Two examples of porous media 2D samples with the corresponding low resolution measurement. The samples contain complex structures while the measurements only resolve the large-scale structures.

on simulated annealing, which is a slow convergence procedure. Moreover, this challenge turns out to be a barrier in 3D reconstruction of porous media due to the huge amount of data.

Hierarchical sampling approaches, proposed recently [2], [3], for porous media reconstruction, are able to manage the computational complexity issues. These approaches, developed mainly for binary random fields (porous media images are binary, containing pore and solid structures), decompose the reconstruction problem into several scales, and then choose different strategies to limit the number of data which are going to be reconstructed at each scale. However, these methods are based on image synthesis and prior sampling, therefore no measurement is involved in the sampling process. Here we propose a hierarchical approach for posterior sampling, i.e. when the measurements are also included in the reconstruction task. The main challenge here is how to incorporate the measurement with each scale in the hierarchical model. We propose a measurement model describing the relationship between the measurement and each scale, and then generate posterior samples based on hierarchical posterior sampling.

2   Hierarchical Model

The proposed reconstruction approach is based on a hierarchical posterior sampling [4], [2] and a Bayesian framework by which a prior model is combined with measurements. The prior model is obtained from high resolution training data

and the measurements are the low resolution samples usually obtained by MR imaging.

The hierarchical sampling approach is a top-down approach which starts at the coarsest scale and then the sample at each scale is mapped into the next scale, until the finest scale is reached. The hierarchical model is similar to the explicit hierarchical based models [6] in the sense that different Markov models are defined at each scale. However, the way that the measurements are incorporates in the hierarchy is different. In most of the explicit hierarchical models the measurements are estimated at each scale using wavelet or other heuristic multi-resolution decomposition methods [6]. Here, we do not estimate the measurement at each scale, rather we define a measurement model describing the relationship between the measurement and a simulated random field at each scale. Therefore the hierarchical posterior model consists of two parts: prior model and measurement model.

Simple image models such as correlation or standard deviation are not able to describe the complex structures of porous media. Gibbs probability distribution [12] are considered to be a suitable model to describe their characteristics [11]. The Gibbs probability distribution is defined as

$$p(X) = \frac{e^{-\frac{1}{T}E(X)}}{Z} \tag{1}$$

where E is the energy function, T is the temperature and Z is a normalization factor. The Gibbs posterior probability distribution for the non-hierarchical case is defined as

$$p(X|M) = \frac{e^{-\frac{1}{T}E(X|M)}}{Z_M} \tag{2}$$

where M is the measuremen. The posterior energy is

$$E(X|M) = E(X) + \alpha J(X; M) \tag{3}$$

where J is the constraint describing how the measurement is incorporated with the model, and α is a parameter balancing the contribution of the prior and measurement in the posterior model. For the hierarchical posterior model, a different prior energy, $E^{(k)}$, and constraint, $J^{(k)}$, are defined for each scale k.

The prior model is based on a ternary model, and defined at each scale [3]. The sample at the finest scale, X^n, is a binary image which demonstrate the reconstructed image, and at the coarsest scale, X^0, is in the same scale as the measurement scale. Variety of energy functions are reviewed and used to model the complex structures of porous media. [11], [1]. For examples, the histogram model [2], [3] which is a non-parametric model, keeps the probability of having different configurations within a neighborhood structure. For a 3×3 block as the neighborhood structure, there are 3^9 possible configurations when we are having a ternary random field. The prior energy for scale k when using the histogram model is defined as

$$E^k(X^k) = \sum_{c=1}^{3^9} \frac{|H^k(c) - h^k(X^k; c)|^2}{y^k(c) + \epsilon} \tag{4}$$

where H^k is the learned histogram distribution and h^k is the observed histogram distribution of a simulated random field, X^k, at scale k. The term y^k is the variance for each histogram entry, to account for sample variation at each scale. A small constant ϵ is introduced to avoid divisions by zero, especially in the comparatively common case of unobserved configurations k corresponding to $H^k(c) = 0$.

The training data at each scale is obtained by decimating the high resolution sample at the finest scale with different decimation factor. The level for each site at a coarse scale can be black (0), grey (0.5) or white (1), such that any site that is still black or white during decimation stays the same, while anything other than black or white is considered as grey (0.5).

3   Measurement Model

The measurement model describes how the information obtained from the measurement is included in the posterior energy function. In fact, the constraint J in (3) is considered as $J^k(X^k; M)$, at each scale. The measurement, M, is a low resolution observation, and can be described by a forward model $f_m(\cdot)$ such that for X at the finest scale

$$M = f_m(X) \tag{5}$$

The forward model f_m describes that each measured pixel, m_I, in M is a representative of a set of d sites in X, such that

$$m_I = \frac{1}{d} \sum_{j \in I} x_j^k \tag{6}$$

where x_j is a site in the binary random field X at the finest scale. In other words, each m_I corresponds to the average of grey values of a set of pixels at the finest scale. Therefore, since we are having a binary field at the finest scale (0 for black and 1 for white), each measured pixel corresponds to the fraction of white in the corresponding set of pixels at the finest scale.

At the coarse scales (when $k < n$), for the ternary random fields, the grey values means uncertain or undecided values. Using the prior model and the measurement model at the coarse scale, the grey values may turn to black or white, or even stay grey.

However, the relationship between the grey values and the measured pixels at the coarse scales is not pre-defined such as (6), rather it remains unclear. For example, a value 0.8 in the measurement means 80% of the pixels of the corresponding set of sites at the *finest* scale should be white, while at a given coarse scale, this value does not always mean that much white in the corresponding set of pixels, since some pixels may be grey at that scale. Therefore, we can not use the same forward model defined in (6) for the coarse scale. More specifically, the value 0.5 for a site at the coarse scale means a notion of uncertainty, and can not be counted as a real value in (6). As the examples shown in Figure 2, the value grey as 0.5 at the coarse scales does not necessary means half white and half black at the fine scale, e.g. we can see that some of the large white areas even with small portion of black, correspond to grey at the coarse scales.
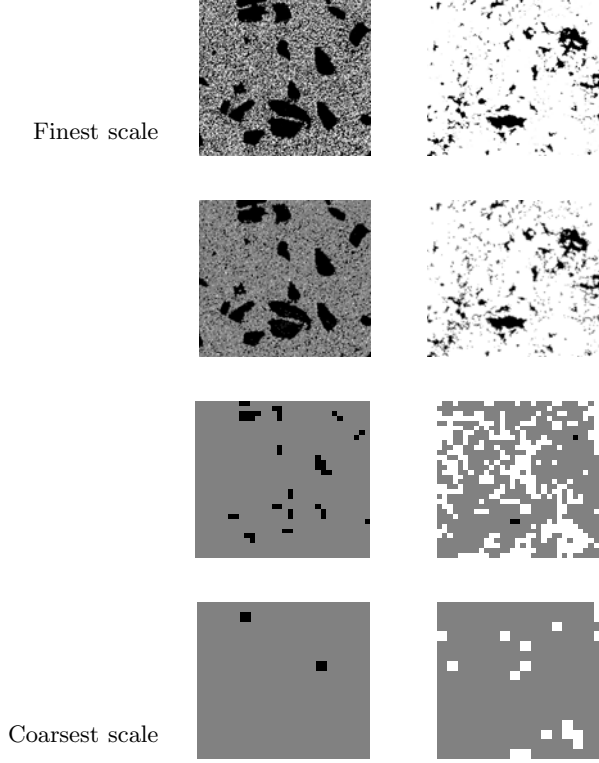
Fig. 2. Two examples of porous media at different scales. On the left, most of the grey at the coarse scales contain almost equal number of black and white at the fine scale , except near large black area. However on the right, the coarse-scale grey corresponds to mostly white at the fine scale, since the fraction of white at the finest scale is high. The grey values at the coarse scales does not necessary corresponds to an even distribution of black and white at the fine scale.

Before we introduce the measurement model, we define two forward models for the fraction of grey, G, and white, W, at a given scale k as

$$G^k = f_g^k(X^k), \qquad W^k = f_w^k(X^k). \tag{7}$$

The random fields G^k and W^k have the same resolution as the measurement M, and more specifically, a datum g_I in G^k (or w_I in W^k) is equal to the fraction of grey (or white) in the corresponding $2^k \times 2^k$ block of pixels in X^k. It is obvious that at the finest scale (i.e. when $k = n$), $f_w^k(X^k) = f_m^k(X^k)$ and $f_g^k(X^k) = 0$.

Since we do not expect to have any grey at the finest scale, the constraint J is defined as

$$J^n(X^n; M) = \|M - f_m(X^n)\|. \tag{8}$$

(a) Fraction of grey versus M (b) Fraction of grey versus $M - W$
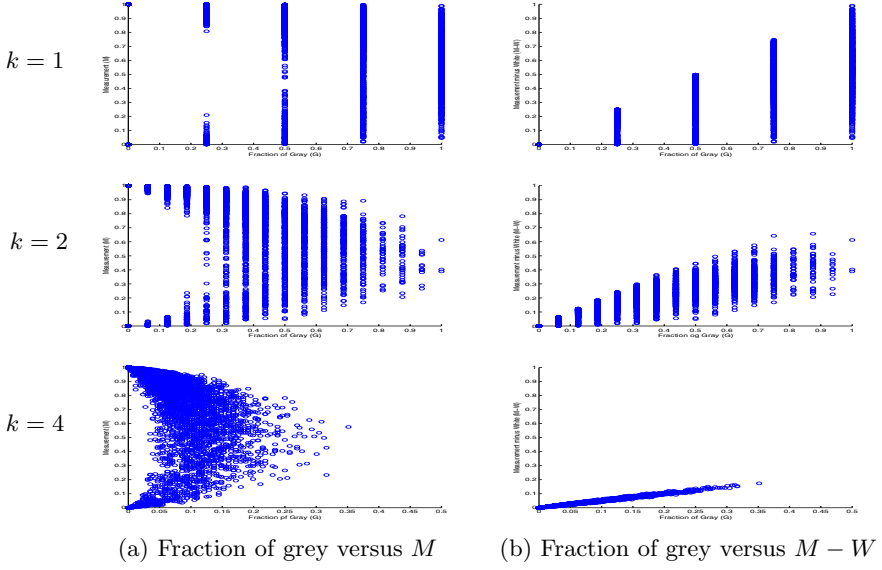
Fig. 3. (a) and (b) shows the scatter plot of fraction of grey versus the measurement and the fraction of grey versus measurement minus fraction of white, respectively for the Carbonate Rock data shown in Figure 1. Plots in (b) are simpler to be modeled than those in (a).

To define the constraint term for the scales above the finest, we have studied the relationship between the measurement and the corresponding g_Is for several pixels, for a set of high resolution training data. Figure 3 (a) shows the scatter plot of g_I versus m_I at different scales. As can been from this figure, there is a non-linear relationship between these two variables, though this relationship is hardly possible to be modeled.

On the other hand, according to the definition of the forward model f_m, given in (6), a given m_I in the measurement, in fact, shows the fraction of white, w_I in the corresponding set of pixels at the *finest* scale. Therefore, for an estimate at the finest scale, to obey the measurements, $r_I = m_I - w_I$ (for all I) should be zero (or very close to zero in case that the measurements are noisy), while at the coarse scale r_I can be greater than zero, and consequently the more grey we have, the greater r_I will be. Thus, we propose to consider the relationship between r_I and g_I, shown in figure 3 (b), instead to define the measurement model. This new relationship does not contain the nonlinearity shown in Figure 3 (a), thus we end up with a much simpler model, such that even at the fine scales, it can be easily defined based on a simple linear regression.

The results in Figure 3 (b) are obtained based on multiple runs and different training samples. As we go coarser and coarser the relationship between g_Is and r_Is gets more complicated, such that at the two coarsest scales below the measured scale, (when $k = 1$ and $k = 2$), it can not be described with a linear parametric model. For these scales we propose to consider a non-parametric model.
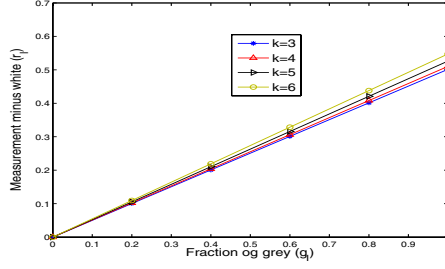
Fig. 4. The linear parametric model for $k > 2$, for Sintered glass spheres shown in Figure 1. Since this type of data has many regions with the same fraction of black and white, the slops turns to be almost 0.5 which means that at a given scale, half of the pixels corresponding to grey in G should turn to white.

3.1   The Linear Parametric Measurement Model

The parametric model is defined when $k > 2$. This model is characterized with two parameters: a^k and v^k. The parameter a^k is the slop of the line obtained by a linear regression of g_I and r_I. The variance v^k represents the average deviation of data from the estimated linear model. These parameters are learned from the high resolution training data. Figure 4 shows the estimated linear model for each scale

According to this model, the constraint term J^k in the hierarchical posterior energy function is defined as

$$J^k(X^k; M) = \frac{\|M - W^k - a^k \, G^k\|^2}{v^k + \epsilon}, \qquad 2 < k < n \qquad (9)$$

where $W^k = f_w^k(X^k)$ and $G^k = f_g^k(X^k)$ are the random fields corresponding to fraction of gray and white respectively, and ϵ is a small enough number to avoid dividing by zero in the model.

3.2   The Non-paramnetric Measurement Model

The non-parametric model is defined for the two coarsest scale below the measured scale, i.e when $k = 1$ and $k = 2$, since at these scales the relations between g_I and r_I can not be modeled with a simple parametric model. The non-parametric model is defined based on the conditional probability of having a certain fraction of grey given the measurement, or equivalently the residual of measurement from the fraction of white, r_I. This conditional probability can be modified according to Bays' rule, as

$$P(G^k = G^* | M - W^k = R^*) \; \propto \; P(M - W^k = R^* | G^k = G^*) \, P(G^k = G^*). \; (10)$$

where $k = 1, 2$. Then J^k is modeled as the logarithm of the conditional and marginal probabilities defined in (10), i.e.

$$J(X^k; M) = |log(P(M - W^k = R^* | G^k = G^*))| \; + \; |log(1 - P(G^k = G^*))| \; (11)$$
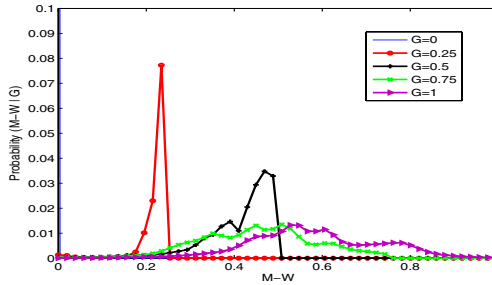
Fig. 5. The conditional probability distribution of $M - W$ given a certain fraction of grey, for one scale below the measured scale, $k = 1$. The probabilities are learned non-parametrically from the high resolution training data.
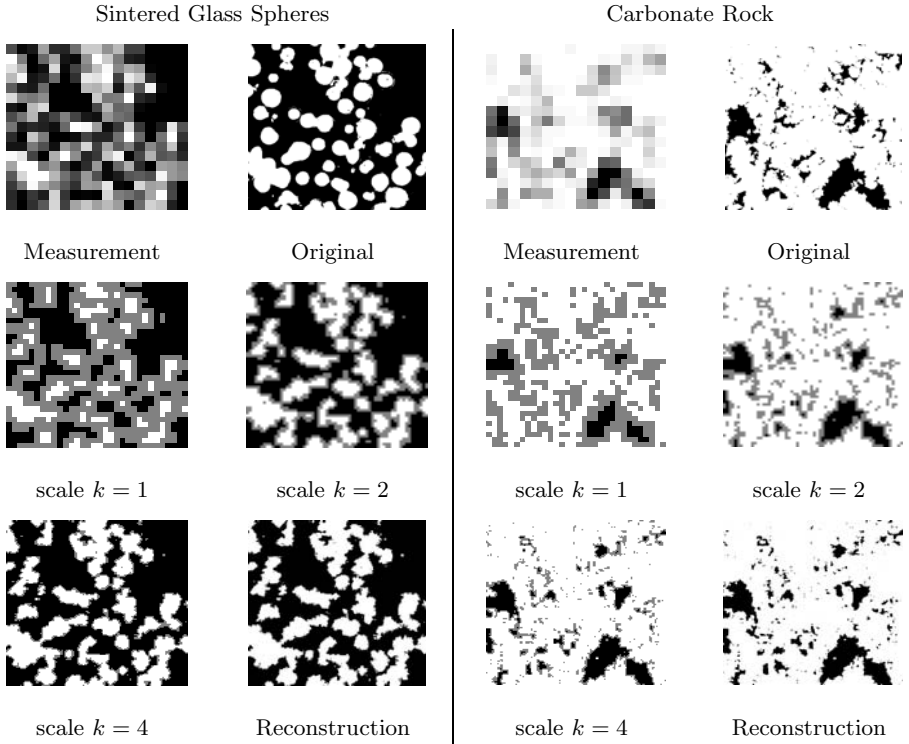
where $W^k = f_w^k(X^k)$ and $G^k = f_g^k(X^k)$. The conditional and marginal probabilities are learned non-parametrically from the training data.

More specifically, when $k = 1$, a site m_I in the measurement corresponds to a 2×2 block in X^1, and the fraction of grey, g_I, corresponding to that block can have 5 possible values: 0, 0.25, 0.5, 0.75 or 1. The conditional and marginal probabilities are then learned for each possible value of g_I. Figure 5 shows examples of the conditional probability distribution for different values of g_I at scale $k = 1$.

4   Hierarchical Sampling

The reconstruction task is done by Gibbs sampling from the posterior probability distribution [5]. The hierarchical sampling used here is a top-down approach based on Gibbs sampler along with simulated annealing, as proposed in [2],[3],[4]. We initialize at the measurement scale, $k = 0$, based on the measurement. At each scale the balck/white pixels from the previous coarse scale are frozen, while we only anneal over the grey pixels. At each scale a posterior sampling approach [8],[4] is used to generate the results such that the sample at each scale is consistent with the measurement according to the measurement model, while obeying the statistics defined by the prior model at that scale. Figure 6 shows the reconstruction based on the proposed modeling and sampling approach. The reconstructed image and the original image does not need to be exactly the same, since we are generating random samples from the model, rather they need to obey the same statistics.

The focus of this paper is on how to incorporate the measurement in the hierarchical model, and more specifically how to define the posterior energy function at each level of the hierarchy. The studies in [2] and [3] show a significant improvement in the time required to generate large reconstruction using hierarchical approaches, and it is also notable that the results in Figure 6 are obtained almost 5 times faster than the case of using non-hierachical approaches.

Sintered Glass Spheres   Carbonate Rock

Measurement   Original   Measurement   Original

scale $k = 1$ scale $k = 2$ scale $k = 1$ scale $k = 2$

scale $k = 4$ Reconstruction   scale $k = 4$ Reconstruction

Fig. 6. Reconstruction of two different porous media samples using the proposed hierarchical approach. The reconstructed images are 32 times higher in resolution than the measurements. The results does not need to be exactly the same as the original data, rather they are required to obey the same statistics.

5   Conclusion

Reconstructing binary random images of porous media becomes intractable when the size of the images increases. Due to the computational complexity of the non-hierachical porous media reconstruction approaches, we have proposed a hierarchical posterior model in which the measurement is incorporated with the prior model. Based on the relationship between the measurement and the fraction of grey at each scale, we define a parametric and non-parametric model. The parametric measurement model is used for the fine scales where the relationship can be described with a linear model. The non-parametric measurement model is defined for the coarse scales, where the relationship is difficult to be described with a simple linear model. By doing posterior sampling from the hierarchical model, we can generate results almost 5 times faster than the non-hierarchical case, as what has been shown in the recent studies on hierarchical *prior* sampling of binary mages.

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