MULTIPOLE-MOTIVATED REDUCED-STATE ESTIMATION

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ABSTRACT

This paper discusses efficient solutions to large-scale two-dimensional estimation problems, using reduced-state methods motivated by the multipole method of mathematical physics. The work is mainly exploratory, building on past efforts in multiscale statistical signal modeling and estimation. We will illustrate applications to the estimation of Markov random field textures, with the motivation and goal the estimation of remotely-sensed fields.

1. INTRODUCTION

The statistical estimation of large, global scale, two-dimensional remote sensing problems and even modestly-sized three-dimensional problems presents tremendous and pertinent challenges: heightened environmental awareness and concerns have led to an explosion in the quantity of remotely-sensed data, much of which contains irregular gaps and nonstationary underlying fields.

The origin of the difficulty in producing statistical estimates is simple. Methods such as nested dissection[4, 5] or multiscale estimation[1] are all based on recursive divide-and-conquer: a subset of the random field is found, such that conditioned on this subset the remaining portions of the field can be processed independently. For example, the four quadrants of a first-order Markov random field can be decorrelated by conditioning on the boundary pixels, shown in Figure 1. So whereas a single pixel can decorrelate the two halves of a one-dimensional process, a column of pixels is required for a 2D field, and a whole *plane* of pixels in three dimensions. Thus for an $n \times n \times ...$ hypercube of voxels in d dimensions, the computational effort to

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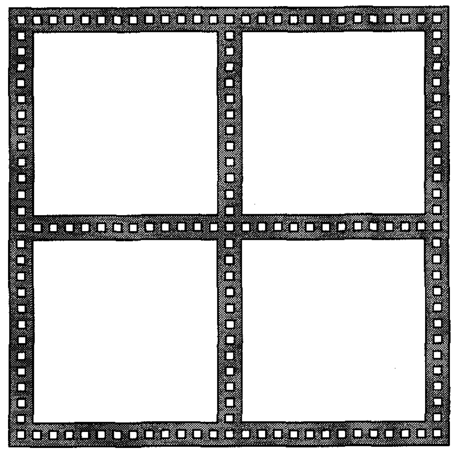

Figure 1: Densely sampled boundaries which conditionally decorrelate the four quadrants of a first-order Markov random field.

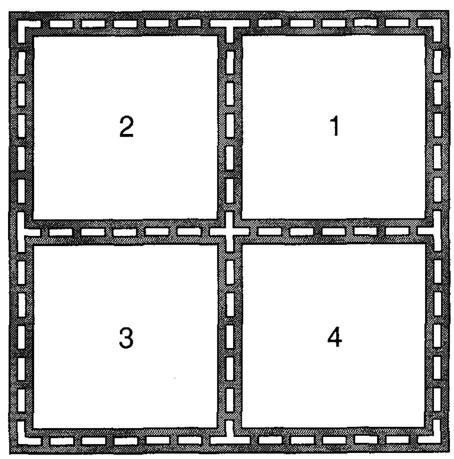

Figure 2: A reduced-state approximation to Figure 1.

solve the estimation problem is

$$\mathcal{O}(n^{(d-1)^3}) \quad (1)$$
$$= \mathcal{O}(n^{(2d-3)}) \text{ per pixel} \quad (2)$$

which very rapidly becomes infeasible for large n and even modest d.

2. MULTISCALE ESTIMATION

The multiscale statistical estimation method[1, 7] has, at its core, the following statistical model:

$$x(s) = A(s)x(s\bar{\gamma}) + B(s)w(s) \quad (3)$$

where s is an index on a tree with parent $s\bar{\gamma}$, A and B are deterministic matrices, and w is a white-noise process. This equation is essentially a restatement of the conditional decorrelation discussed in the Introduction: the whiteness of w implies that the state $x(s\bar{\gamma})$ must conditionally decorrelate all states connected to $s\bar{\gamma}$. A simple approach[2] to reducing the state dimension is to subsample the state boundary (Figure 2), however this does not alter the asymptotic computational complexity, nor does it render three-dimensional problems tractable.

The key problem with the state representation in Figure 2 is that it attempts to decorrelate too much: in terms of estimating quadrant "2", keeping details of the distant part of quadrant "4" is largely irrelevant; that is, the reduced state of Figure 5 will perform very nearly as well. Although we now require four such reduced models (one for each quadrant), the state dimension needs to be reduced by only a factor of $4^{1/3}$ for the computational effort of each model to be less than one fourth of that for Figure 2, giving an overall performance increase. Figure 7 gives one illustration of the approach: we measured a "tree" texture (the Markov random field of Figure 6) at 0dB SNR and computed estimates based on Figure 2 (the specific state assignments based on [8]), and again based on four multiscale trees using Figure 5; the two approaches differ, in this example, by only 1% in MSE and are indistinguishable visually.

A second illustration is shown in Figures 3 and 4. Figure 3 shows the estimates for the top-left quadrant, using a detailed state (sampling every pixel along horizontal boundaries); the results for the four quadrants are patched together in Figure 4. There are no artifacts or discontinuities at the patch boundaries.

For presentation purposes, we have limited ourselves to promoting a four-quadrant decomposition which achieves modest reductions in computational complexity. However there are two very important considerations which further motivate this approach:

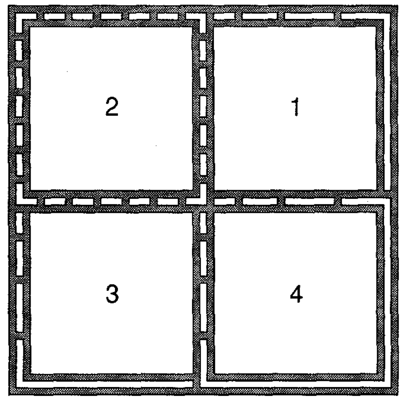

Figure 5: A further reduction in state from Figure 2, appropriate for estimating the top-left quadrant.

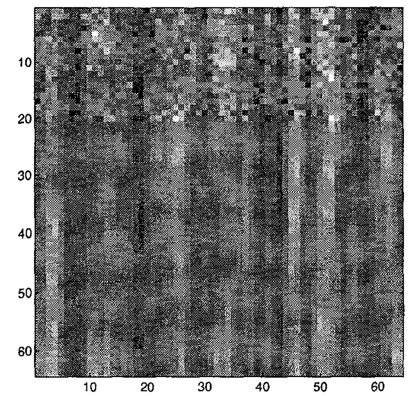

Figure 6: Original sample path of "tree" random field; the whole field is observed at 0dB SNR, a portion of which is shown in the top of the image.

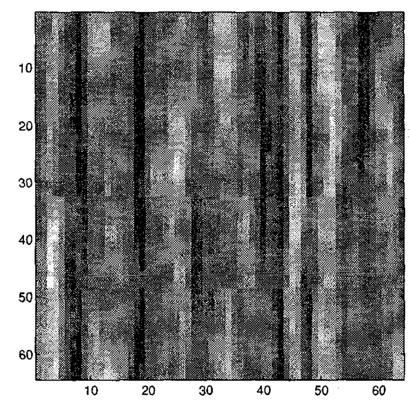

Figure 7: Random field estimates based on a reduced-order multiscale model based on either Figure 2 or 3 (both give visually identical results).

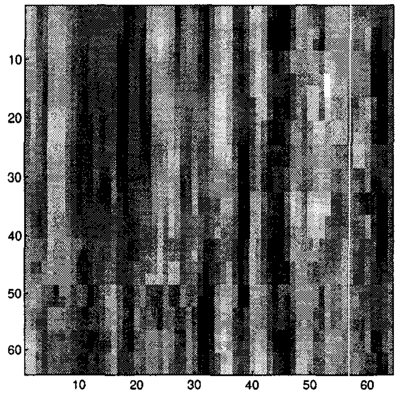

Figure 3: Estimates from one tree, designed to estimate the top-left quadrant.

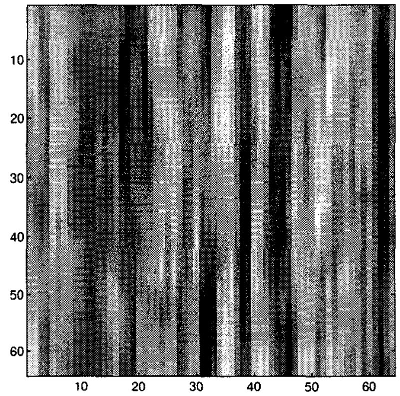

Figure 4: The combined estimates, patched together from separate estimates in each quadrant. No seams or artifacts are visible along the patch boundaries.

1. Further subdivision: Figure 5 really illustrates only the first step in state reduction; depending on the nature of the prior model, further gains may be realized by dividing the domain more finely. Furthermore, and much more significantly, if we design q different sets of states to estimate the domain using q trees, then a *great* part of the work done by the q trees is duplicated or redundant, and opens the possibility for much greater reductions in complexity.

2. Numerical conditioning: A reduction in state dimension can lead to substantial improvements in the conditioning of the estimation problem. For example, in a related oceanographic surface-temperature estimation problem[3], the root-node covariance for a 256×256 pixel domain has a condition number of $5 \cdot 10^{17}$, which is rather poor, even at double-precision. If we modify the state to concentrate on one sixteenth of the domain, then the root-node covariance for the entire domain has condition $3 \cdot 10^{10}$, a substantial improvement.

3. MULTIPOLE-MOTIVATED ESTIMATION

A philosophically very different approach is suggested by the multipole algorithm[6], originally used to solve for the potentials of very large gravitational and electromagnetic systems. The specific multipole algorithm and associated mathematical bounds do not appear to apply in the estimation context, however the essence of the approach is very appealing. Consider the regions shown in Figure 8; the multipole algorithm performs two separate aggregations:

1. In determining the potential at point "A", all of the points in "C" act similarly and so their effects are grouped.

2. The effect of "C" on other points near "A" (e.g., "B") is similar to that at "A", so its influence can be grouped.

We can begin to exploit this philosophy in the estimation context as follows. Figure 9 shows a set of 13 regions, centered about the single element $\bar{x}$ to be estimated. Each region R_i is designed to satisfy

$$\frac{P(\bar{x}|\sum_{x\in R_i} x)}{P(\bar{x}|\{x \in R_i\})} \leq \tau \tag{4}$$

That is, the rise in estimation uncertainty due to averaging the elements in a region is constrained by τ (here set to 1.2). Each pixel in the random field can then be estimated by solving a problem of fixed difficulty – 13-dimensional – regardless of the size of the overall field. Figure 10 shows the result of using the regions in Figure 9 to estimate the texture in Figure 6.

This multipole method has appealing features, however challenges remain: the definition of the regions needs to be made more systematic and rigorous. Secondly, and more significantly, although the estimation problem to be solved at each pixel is now simple, we are now required to *set up* a problem, possibly involving many measurements, for each pixel to be estimated. If the measurements are fairly dense, then the brute-force averaging of measurements into regions would already lead to $\mathcal{O}(n^d)$ complexity per pixel, so a more insightful approach is needed.

4. CONCLUSIONS

This paper has discussed the state reduction principles being investigated in our research, specifically to weaken the need to keep large state vectors that conditionally decorrelate a random field, by adopting a multipole-like philosophy. The multipole approach offers the possibility of an algorithm capable of statistical estimation at a constant complexity per pixel, even for fields possessing long-range correlations.

5. REFERENCES

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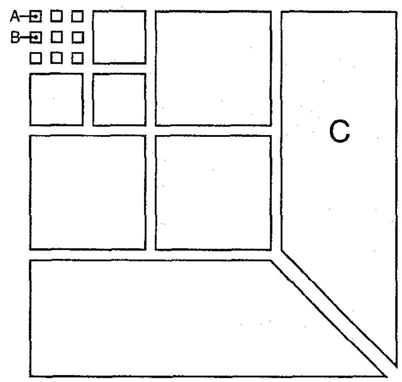

Figure 8: A hierarchical, multipole-like grid.

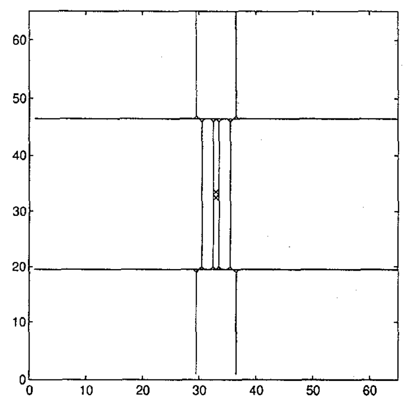

Figure 9: The reduced-state decomposition determined for the "tree" random field of Fig-6.

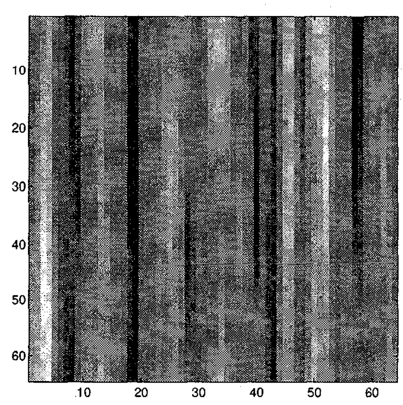

Figure 10: Estimates produced by the hierarchical decomposition of Fig-9 based on dense measurements.