# MULTI-SCALE 3D REPRESENTATION VIA VOLUMETRIC QUASI-RANDOM SCALE SPACE

*Akshaya Mishra, Alexander Wong, Paul Fieguth, and David A. Clausi*

Department of Systems Design Engineering, University of Waterloo, Waterloo, Canada

## ABSTRACT

A novel nonlinear volumetric scale-space framework is proposed for multi-scale volumetric data representation. The problem is formulated as a Bayesian least-squares estimator, and a quasi-random density estimation approach is introduced for estimating the posterior distribution between consecutive volumetric scale space realizations. Experimental results using both synthetic and real MR volumetric data demonstrate the effectiveness of the proposed scale-space framework for three-dimensional representation with significantly better structural separation and localization across all scales when compared to existing volumetric scale-space frameworks such as volumetric anisotropic diffusion and volumetric linear Gaussian scale-space, especially under scenarios with high noise levels.

***Index Terms***— Scale space, random sampling, volumetric representation, multi-scale

## 1. INTRODUCTION

The physical world is highly complex in nature and consists of a wide range of structures and phenomena that have different semantics and significance at different scales. Therefore, to study the world in a systematic and analytical manner, as well as solve physical problems that have significance across multiple scales, it is intuitive that a multi-scale approach to data representation is desired for the purpose of modeling and analyzing complex data. The decomposition and analysis of volumetric data at different scales has important applications in a variety of fields such as medical image analysis, 3D reconstruction and visualization, and 3D tracking.

A powerful approach for multi-scale data decomposition and representation is scale space theory [1], where the multi-scale structural characteristics of complex data are handled by decomposing the data into a single-parameter family of data representations, with a gradual decrease in fine scale structures between successive scales. Scale space theory has proven to be highly effective for many different computer vision applications such as noise compensation [2, 3], edge identification [2, 4, 5], and segmentation [6].

Existing volumetric scale space frameworks can be divided into two main groups: i) linear and ii) nonlinear scale space frameworks. First proposed by Witkin [1] and Koen-derink and Van Doorn [7], a linear scale space framework has high computational efficiency, making it a good candidate for fast multi-scale volumetric representation. However, the representations produced under linear frameworks exhibit poor structural localization, as well as undesirable structure merging at coarse scales, particularly in the case of multi-object data. To address these issues, nonlinear scale space frameworks [2, 3, 4, 5] model the structural decomposition problem using a generalized diffusion framework and encourages diffusion along structures with similar characteristics to improve structural localization. While considerable advancements have been made in nonlinear scale space, significant structural delocalization and structure merging continues to persist for complex data at coarser scales, leading to unsatisfactory multi-scale representations of complex data, particularly in situations characterized by high noise levels.
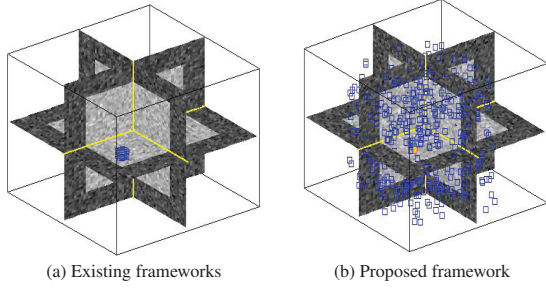
Recently, a nonlinear scale space framework was introduced based on quasi-random density estimation theory [8], where highly relevant data samples are drawn in a stochastic manner to construct statistically robust scale space representations of the data that provide strong structural localization and, importantly, low sensitivity to noise. However, this framework was developed and validated entirely in two dimensions, and no investigation has been conducted for extending this framework to higher-dimensional data decomposition and representation. The main contribution of this work is an extension upon the fundamental theory behind quasi-random nonlinear scale space theory for robust multi-scale volumetric data decomposition and representation.

## 2. PROBLEM FORMULATION

Let $X$ be a set of sites in a discrete lattice $\mathscr{L}$ upon which the volumetric data is defined and $\underline{x} \in X$ be a site in $\mathscr{L}$, as defined by three-vector $\underline{x} = (x, y, z)$. Let the observed data $I = \{I(\underline{x}) | \underline{x} \in X\}$, volumetric gradient $G_i = \{G_i(\underline{x}) | \underline{x} \in X\}$, volumetric scale space representation $L_i = \{L_i(\underline{x}) | \underline{x} \in X\}$, and residual inter-scale structure $C_i = \{C_i(\underline{x}) | \underline{x} \in X\}$ be random fields on $X$. The volumetric scale space decomposition can be expressed mathematically as

$$L_{i-1}(\underline{x}) = L_i(\underline{x}) + C_i(\underline{x}), \tag{1}$$

where $L_0(\underline{x}) = I(\underline{x})$. As such, the volumetric scale space construction process can be treated as an inverse problem, and

(a) Existing frameworks      (b) Proposed framework

**Fig. 1**. (a) In existing linear and nonlinear volumetric scale space frameworks, all samples within a local volumetric neighborhood are used, regardless of relevance. (b) In the proposed volumetric quasi-random nonlinear scale space framework, only samples with high relevancy from across the volumetric data are used to achieve improved volumetric scale space representations.

can be solved using a general Bayesian least-squares estimation approach:

$$\hat{L}_i\left(\underline{x}\right) = \arg_{\hat{L}_i} \min \left\{ E\left( \left( \hat{L}_i\left(\underline{x}\right) - L_i\left(\underline{x}\right) \right)^2 \Big| L_{i-1}\left(\underline{x}\right) \right) \right\}.$$
(2)

Based on (2), the analytical solution for $\hat{L}_i\left(\underline{x}\right)$ is given by [9]

$$\hat{L}_i\left(\underline{x}\right) = \underbrace{\int L_i\left(\underline{x}\right) p\left(L_i\left(\underline{x}\right)|L_{i-1}\left(\underline{x}\right)\right) dL_i\left(\underline{x}\right)}_{E(L_i(\underline{x})|L_{i-1}(\underline{x}))}.$$
(3)

Unfortunately, the conditional mean of $\hat{L}_i\left(\underline{x}\right)$ can be a highly complicated and nonlinear function of $L_{i-1}\left(\underline{x}\right)$, and is often not possible to be solved in an analytical manner. To work around this issue, a volumetric quasi-random density estimation approach is proposed to estimate the conditional mean $\hat{L}_i\left(\underline{x}\right)$ in a robust manner by utilizing only highly relevant samples from across $L_{i-1}\left(\underline{x}\right)$. This volumetric quasi-random density estimation strategy is fundamentally different from that used for existing linear and nonlinear volumetric scale space frameworks, where all samples within a local volumetric neighborhood are used, as illustrated in Fig. 1.

### 3. VOLUMETRIC QUASI-RANDOM DENSITY ESTIMATION

The volumetric quasi-random density estimation strategy can be described as follows. To enable the drawing of low discrepancy samples for $p\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right)$, a set of $n$ samples is drawn from a Sobol quasi-random sequence [10] with respect to $\underline{x}$. The quasi-random approach is designed to bias towards those samples with high relevancy for estimating $p\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right)$, therefore the distribution $p\left(L_i(\underline{x})\right)$ is first constructed to allow for the identification of samples with a high likelihood of being realizations of $p\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right)$.



(a) Input image      (b) Density function
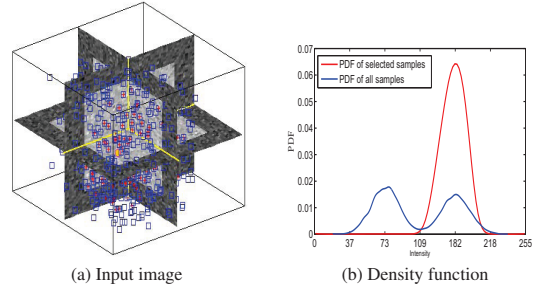
**Fig. 2**. A synthetic volume, left, where the orange pixel (at center) represents the estimation site, blue markers represent random samples, and red markers represent the selected samples for density estimation. The corresponding probability distribution $p(L_{i-1}(\underline{x}))$ of the orange pixel is shown in (b). The PDF based on *all* samples (shown in blue) is undesirable as they do not well-represent the relevant samples, whereas the PDF taken over the selected samples provides a better representation of relevant samples.

An example probability distribution $p\left(L_{i-1}(\underline{x})\right)$ in a sample volume at a particular scale is shown in Fig. 2(b).

    To identify those samples having a high relevancy, a Gaussian mixture model based on $p\left(L_{i-1}(\underline{x})\right)$ is constructed as follows. The local maxima of $p\left(L_{i-1}(\underline{x})\right)$ are detected, denoted as $L^\kappa = \{L^1_{i-1}(\underline{x}), L^2_{i-1}(\underline{x}), \cdots, L^k_{i-1}(\underline{x})\}$ for $k$ maxima. The individual Gaussian distributions within the mixture model are set as having means of $L^\kappa$ and variances of $\sigma^2_{L_{i-1}}$ (usually the noise variance at scale $i$) respectively. The realizable sample set $\Omega$ can then be determined by identifying the Gaussian distribution to which $L_{i-1}(\underline{x})$ belongs,

$$L^\gamma = max\left(L^\kappa\right), \text{ where } L_{i-1}(\underline{x}) - \sigma_{L_{i-1}} \le L^\kappa \le L_{i-1}(\underline{x}) + \sigma_{L_{i-1}},$$
(4)

and accepting all samples within $\sigma_{L_{i-1}}$ of $L^\gamma$:

$$\Omega = \left\{ \underline{x}_j \ni |L_{i-1}(\underline{x}_j) - L^\gamma| < \sigma_{L_{i-1}} \right\}.$$
(5)

Finally, given $\Omega$, the estimated posterior distribution $\hat{p}\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right)$ can be computed as

$$\hat{p}\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right) = \frac{p^*\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right)}{\int_0^1 p^*\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right) dL_i(\underline{x})},$$
(6)

where the distribution $p^*$ is used as a measure of sample relevancy for estimating $L_i(\underline{x})$. In this paper, $p^*$ is defined as

$$p^*\left(L_i(\underline{x})|L_{i-1}(\underline{x})\right) =$$

$$\frac{1}{\sqrt{2\pi}\sigma_{L_i}} \sum_{k=\Omega} f_1\left(k\right) f_2\left(k\right) f_3\left(k\right) \exp\left( -\frac{1}{2}\left( \frac{L_i - L_{i-1}(\underline{x}_k)}{\sigma_{L_{i-1}}} \right)^2 \right),$$
(7)

where $f_1\left(k\right)$, $f_2\left(k\right)$, and $f_3\left(k\right)$ are objective functions that assess sample relevance based on volumetric intensity, volumetric gradient, and volumetric spatial offset respectively:

$$f_1\left(k\right) = \exp\left( -\frac{1}{\rho_{I_{i-1}}}(L_{i-1}(\underline{x}) - L_{i-1}(\underline{x}_k))^2 \right),$$
(8)

$$f_2(k) = \exp\left(-\frac{1}{\rho_{G_{i-1}}}(G_{i-1}(\underline{x}) - G_{i-1}(\underline{x}_k))^2\right), \quad (9)$$

and

$$f_3(k) = \exp\left(-\frac{1}{\rho_{V_{i-1}}}\|(\underline{x}) - (\underline{x}_k)\|_2\right), \quad (10)$$

where $G$ represents the volumetric gradient. The terms $\rho_{I_i}$, $\rho_{G_i}$ and $\rho_{V_i}$ are regularization constants for scale $i$, where the spatial parameter $\rho_{V_i}$ is user-specified, and $\rho_{I_i}$, $\rho_{G_i}$ are defined as the median over local standard-deviations over a sliding window:

$$\rho_{I_i} = median(\sigma_{I_i}(j, k)), \quad (11)$$

$$\rho_{G_i} = median(\sigma_{G_i}(j, k)). \quad (12)$$

## 4. EXPERIMENTAL RESULTS

To study the noise sensitivity of the proposed volumetric scale space framework, which will be referred to as VQRSS, a synthetic volumetric data set was contaminated by additive Gaussian noise with a standard deviation of $\sigma = 40\%$ of the data dynamic range. For comparison purposes, a linear Gaussian volumetric scale space framework based on that proposed in [1] (GS) and a nonlinear volumetric scale space framework based on that proposed in [4] (PM) are also performed.

The scale space representations of the test volumetric data at three different scales constructed using the tested volumetric scale space frameworks are shown in Fig. 3. The scales shown for each tested framework were chosen such that they have similarly scaled structures. For GS, the scale space representations are shown for scales $i = 2, 4, 10$. For PM, the scale space representations are shown for scales $i = 4, 10, 20$. For VQRSS, the scale space representations are shown for $i = 2, 4, 8$.

While all tested frameworks were able to produce scale space representations with monotonically decreasing fine scale structures as scale increased, the scale space representations produced using VQRSS visually exhibit significantly better structural localization at all scales when compared to GS and PM. This is most noticeable at the coarsest scale, where the structural characteristics of the largest box is noticeably degraded (i.e., giving a rounded appearance) for both GS and PM, while VQRSS produces a representation that preserves the structural characteristics of the largest box. Furthermore, both GS and PM exhibit undesirable structure merging at medium scale, which does not appear in VQRSS. Finally, VQRSS is noticeably less sensitive than GS or PM to the presence of noise across all scales.

In the second set of experiments, the volumetric scale space frameworks were applied to real clinical MR volumetric data to study its effectiveness at decomposing complex real-world structures. The volumetric scale space representations constructed using the tested scale space frameworks
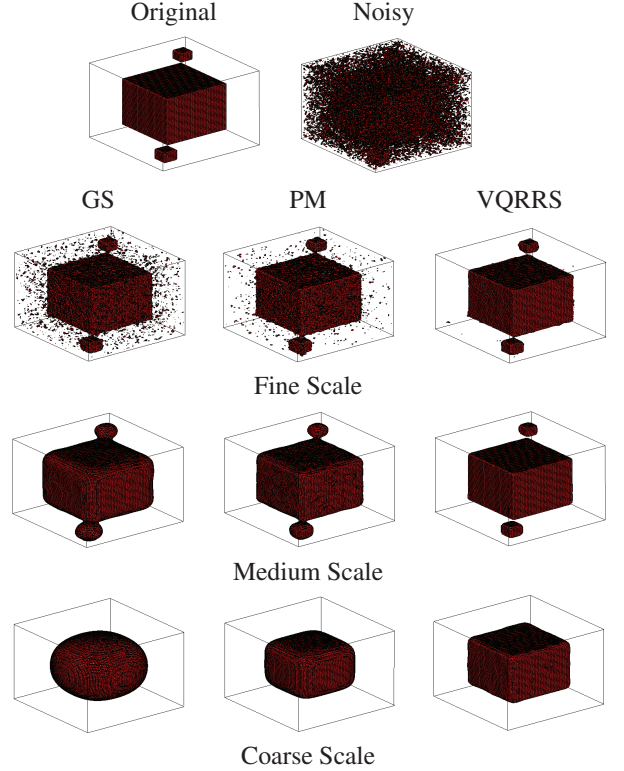


Fig. 3. Scale space representations for a synthetic volumetric data set contaminated by additive Gaussian noise with a standard deviation of $\sigma = 40\%$ of the dynamic range of the data. The scale space representations produced using VQRSS visually exhibit significantly better structural localization at all scales when compared to GS [1] and PM [4].

are shown in Fig. 4, with a particular slice from the volumetric scale space representations shown at three scales in Fig. 5. As with the first set of experiments, the volumetric scale space representations constructed using VQRSS provide noticeably superior structural localization at all scales. In particular, compare the representations at the coarse and fine scales in Fig. 5; whereas the GS and PM both introduce significant amounts of blurring, the VQRSS clearly produces piecewise-constant structures, with the fine details removed but with little to no blurring.

## 5. CONCLUSIONS

A nonlinear volumetric scale space framework based on quasi-random scale space theory was proposed for the purpose of multi-scale volumetric data decomposition and representation. Experiments using synthetic volumetric data and real MR volumetric data were performed using the proposed framework, and it was shown to provide improved structural separation and localization capabilities when compared to existing volumetric scale space frameworks. Future work involves investigating the potential of extending the proposed

Original       GS

PM       VQRSS
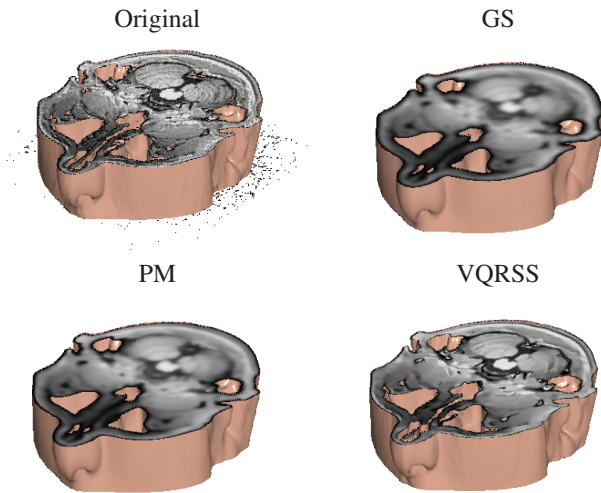
**Fig. 4**. The volumetric scale space representations (at medium scale) constructed for the clinical MR data. The volumetric scale space representations constructed using VQRSS provide noticeably superior structural localization at all scales.
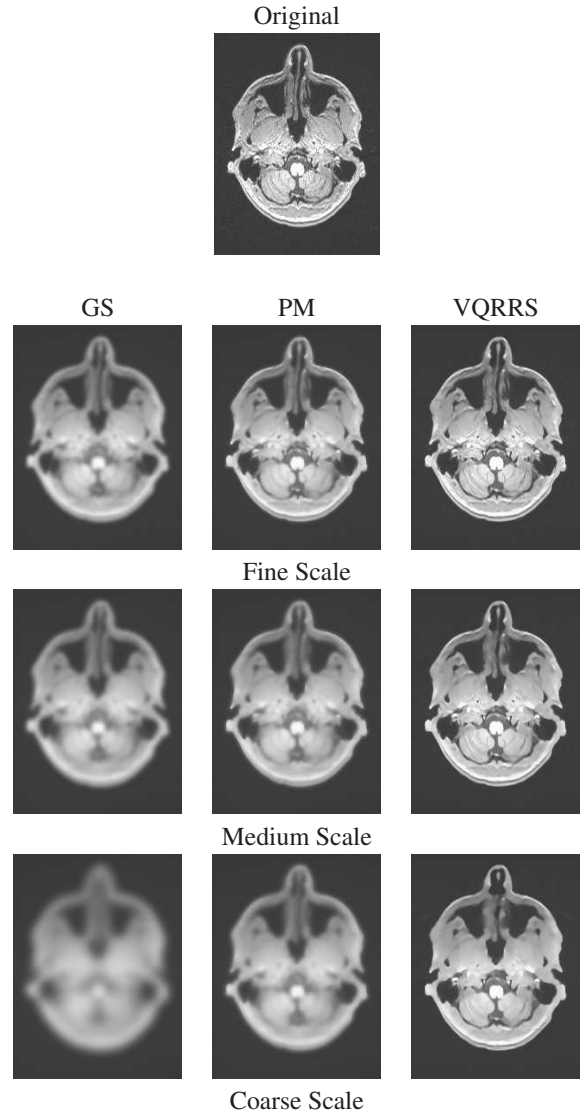
volumetric scale scale framework for decomposing and analyzing spatial-temporal data such as videos, which holds another set of challenges related to the temporal aspects.

## Acknowledgment

**Fig. 5**. A sample slice from the volumetric scale space representations shown in Fig. 4.

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