All Investors are Risk-averse
Expected Utility Maximizers

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Contributions

1. In any behavioral setting respecting First-order Stochastic Dominance, investors only care about the distribution of final wealth (law-invariant preferences).

2. In any such setting, the optimal portfolio is also the optimum for a risk-averse Expected Utility maximizer.

3. Given a distribution $F$ of terminal wealth, we construct a utility function such that the optimal solution to

$$\max_{X_T \mid \text{budget}=\omega_0} E[U(X_T)]$$

has the cdf $F$.

4. Use this utility to infer risk aversion.

5. Decreasing Absolute Risk Aversion (DARA) can be directly related to properties of the distribution of final wealth and of the financial market in which the agent invests.
FSD implies Law-invariance

Consider an investor with **fixed horizon** and objective $V(\cdot)$.

**Theorem**

Preferences $V(\cdot)$ are non-decreasing and law-invariant if and only if $V(\cdot)$ satisfies first-order stochastic dominance.

- **Law-invariant** preferences
  
  \[ X_T \sim Y_T \Rightarrow V(X_T) = V(Y_T) \]

- **Increasing** preferences
  
  \[ X_T \succeq Y_T \text{a.s.} \Rightarrow V(X_T) \succeq V(Y_T) \]

- **first-order stochastic dominance (FSD)**
  
  \[ X_T \sim F_X, Y_T \sim F_Y, \forall x, F_X(x) \leq F_Y(x) \Rightarrow V(X_T) \succeq V(Y_T) \]
Main Assumptions

- Given a portfolio with final payoff $X_T$ (consumption only at time $T$).
- $P$ ("physical measure"). The initial value of $X_T$ is given by
  \[ c(X_T) = \mathbb{E}_P[\xi_T X_T]. \]
  where $\xi_T$ is called the pricing kernel, state-price process, deflator, stochastic discount factor...
- All market participants agree on $\xi_T$ and $\xi_T$ is continuously distributed.
- Preferences satisfy FSD.
- Another approach: $\xi_T$ is a Radon-Nikodym derivative. Let $Q$ be a "risk-neutral measure" such that
  \[ \xi_T = e^{-rT} \left( \frac{dQ}{dP} \right)_T, \quad c(X_T) = \mathbb{E}_Q[e^{-rT} X_T]. \]
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Definition: A payoff is cost-efficient (Dybvig (1988), Bernard et al. (2011)) if any other payoff that generates the same distribution under $P$ costs at least as much.

Let $X_T$ with cdf $F$. $X_T$ is cost-efficient if it solves

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

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The a.s. unique optimal solution to (1) is $X_T^* = F^{-1}(1 - F_{\xi_T}(\xi_T))$.

Consider an investor with preferences respecting FSD and final wealth $X_T$ at a fixed horizon.

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Theorem 1:

Optimal payoffs must be cost-efficient.
Theorem 2:

An optimal payoff $X_T$ with a continuous increasing distribution $F$ also corresponds to the optimum of an expected utility investor for

$$U(x) = \int_0^x F_{\xi_T}^{-1}(1 - F(y))dy$$

where $F_{\xi_T}$ is the cdf of $\xi_T$ and budget $= E[\xi_T F^{-1}(1 - F_{\xi_T}(\xi_T))]$. The utility function $U$ is $C^1$, strictly concave and increasing.

- When the optimal portfolio in a behavioral setting respecting FSD is continuously distributed, then it can be obtained by maximum expected (concave) utility.
- All distributions can be approximated by continuous distributions. Therefore all investors appear to be approximately risk averse...
Optimal Portfolio and Cost-efficiency

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Generalization

We can show that all distributions can be the optimum of an expected utility optimization with a “generalized concave utility”.

**Definition: Generalized concave utility function**

A generalized concave utility function $\tilde{U} : \mathbb{R} \to \mathbb{R}$ is defined as

$$\tilde{U}(x) := \begin{cases} 
U(x) & \text{for } x \in (a, b), \\
-\infty & \text{for } x < a, \\
U(a^+) & \text{for } x = a, \\
U(b^-) & \text{for } x \geq b,
\end{cases}$$

where $U(x)$ is concave and strictly increasing and $(a, b) \subset \mathbb{R}$. 
General Distribution

Let $F$ be

- a **continuous** distribution on $(a, b)$
- a **discrete** distribution on $(m, M)$
- a **mixed** distribution with $F = pF^d + (1 - p)F^c$, $0 < p < 1$
  and $F^d$ (resp. $F^c$) is a discrete (resp. continuous) distribution.

Let $X^*_T$ be the cost-efficient payoff for this cdf $F$. Assume its cost, $\omega_0$, is finite. Then $X^*_T$ is also an optimal solution to the following expected utility maximization problem

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} E \left[ \tilde{U}(X_T) \right]$$

where $\tilde{U} : \mathbb{R} \to \overline{\mathbb{R}}$ is a generalized utility function given explicitly in the paper.
Illustration in the Black-Scholes model.

Under the physical measure $P$,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P, \quad \frac{dB_t}{B_t} = rdt$$

Then

$$\xi_T = e^{-rT} \left( \frac{dQ}{dP} \right)_T = a \left( \frac{S_T}{S_0} \right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$, $\theta = \frac{\mu - r}{\sigma}$ and $b = \frac{\mu - r}{\sigma^2}$. 
Power utility (CRRA) & the LogNormal Distribution

The utility function explaining the Lognormal distribution $\mathcal{LN}(A, B^2)$ writes as

$$U(x) = \begin{cases} 
    a \left( x^{1 - \frac{\theta \sqrt{T}}{B}} \right), & \frac{\theta \sqrt{T}}{B} \neq 1, \\
    a \log(x), & \frac{\theta \sqrt{T}}{B} = 1,
\end{cases}$$  \hspace{1cm} (2)

where $a = \exp\left( \frac{A\theta \sqrt{T}}{B} - rT - \frac{\theta^2 T}{2} \right)$. This is a CRRA utility function with relative risk aversion $\frac{\theta \sqrt{T}}{B}$. 
Explaining the Demand for Capital Guarantee Products

\[ Y_T = \max(G, S_T) \]

where \( S_T \) is the stock price and \( G \) the guarantee.

\( S_T \sim \mathcal{L}\mathcal{N}(M_T, \Sigma^2_T) \).

The utility function is then given by

\[
\tilde{U}(x) = \begin{cases} 
-\infty & x < G, \\
a \left( x \frac{1 - \theta \sqrt{T}}{\Sigma_T} - G \frac{1 - \theta \sqrt{T}}{\Sigma_T} \right) & x \geq G, \frac{\theta \sqrt{T}}{\Sigma_T} \neq 1, \\
\log(x) & x \geq G, \frac{\theta \sqrt{T}}{\Sigma_T} = 1,
\end{cases}
\]

with \( a = \exp\left( \frac{M_T \theta \sqrt{T}}{\Sigma_T} - rT - \frac{\theta^2 T}{2} \right) \).

- The mass point is explained by a utility which is infinitely negative for any level of wealth below the guaranteed level.
- The CRRA utility above this guaranteed level ensures the optimality of a Lognormal distribution above the guarantee.
Yaari’s Dual Theory of Choice Model

Final wealth $X_T$. Objective function to maximize

$$\mathbb{H}_w [X_T] = \int_0^\infty w (1 - F(x)) \, dx,$$

where the (distortion) function $w : [0, 1] \to [0, 1]$ is non-decreasing with $w(0) = 0$ and $w(1) = 1$. Then, the optimal payoff is

$$X^*_T = b \mathbf{1}_{\xi_T \leq c}$$

where $b > 0$ is given to fulfill the budget constraint.

We find that the utility function is given by

$$U(x) = \begin{cases} 
-\infty & x < 0 \\
f(x - c) & 0 \leq x \leq b \\
f(b - c) & x > b 
\end{cases}$$

where $f > 0$ is constant.
Exponential utility & the Normal Distribution

The exponential utility investor maximizes expected utility of final wealth where the utility function is given by

\[ U(x) = -\exp(-\gamma x), \]

where \( \gamma \) is the constant absolute risk aversion parameter. The optimal wealth obtained with an initial budget \( x_0 \) is given by

\[ X_T^* = e^{rT}x_0 + \frac{T}{\gamma} \theta^2 - \frac{\theta}{\gamma \sigma} \left( \mu - \frac{\sigma^2}{2} \right) T + \frac{\theta}{\gamma \sigma} \ln \left( \frac{S_T}{S_0} \right), \]

where \( \theta = \frac{\mu - r}{\sigma} \) is the instantaneous Sharpe ratio for the risky asset \( S \). Its cdf \( F_{Exp} \) corresponds to the cdf of a normal distribution with mean \( e^{rT}x_0 + \frac{T}{\gamma} \theta^2 \) and variance \( \frac{\theta^2 T}{\gamma^2} \).
more natural for an investor to describe her target distribution than her utility (Goldstein, Johnson and Sharpe (2008) discuss how to estimate the distribution at retirement using a questionnaire).

From the investment choice, get the distribution and find the corresponding utility $U$. $\Rightarrow$ Inferring preferences from the target final distribution

$\Rightarrow$ Inferring risk-aversion. The Arrow-Pratt measure for absolute risk aversion can be computed from a twice differentiable utility function $U$ as $A(x) = -\frac{U''(x)}{U'(x)}$.

Always possible to approximate by a twice differentiable utility function...
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Theorem (Arrow-Pratt Coefficient)

Consider an investor who wants a terminal absolutely continuous cdf $F$ (with density $f$). For $0 < p < 1$, let $x = F^{-1}(p)$. The Arrow-Pratt coefficient for absolute risk aversion is given as

$$A(x) = \frac{f(F^{-1}(p))}{g(G^{-1}(p))},$$

where $g$ and $G$ denote respectively the density and cdf of $-\log(\xi_T)$.

Theorem (Distributional characterization of DARA)

An investor who targets some terminal cdf $F$ has DARA iff

$$x \mapsto F^{-1}(G(x)) \text{ is strictly convex}.$$
Consider an investor who targets some cdf $F$ for his terminal wealth. In a Black-Scholes market the investor has decreasing absolute risk aversion if and only if for all $p \in (0, 1)$

$$\frac{f(F^{-1}(p))}{\phi(\Phi^{-1}(p))}$$

is decreasing,

or equivalently,

$$F^{-1}(\Phi(x))$$

is strictly convex,

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and the cdf of a standard normally distributed random variable.
Some comments

- In the Black Scholes setting, DARA if and only if the target distribution $F$ is fatter than a normal one.
- In a general market setting, DARA if and only if the target distribution $F$ is fatter than the cdf of $-\log(\xi_T)$.
- Sufficient property for DARA:
  - logconvexity of $1 - F$
  - decreasing hazard function ($h(x) := \frac{f(x)}{1-F(x)}$)
- Many cdf seem to be DARA even when they do not have decreasing hazard rate function. ex: Gamma, LogNormal, Gumbel distribution.
Conclusions & Future Work

▶ **Inferring** preferences and risk-aversion from the choice of distribution of terminal wealth.

▶ **Understanding** the interaction between changes in the financial market, wealth level and utility on optimal terminal consumption for an agent with given preferences.

▶ **FSD** or law-invariant behavioral settings **cannot explain** all decisions. One needs to look at state-dependent preferences to explain investment decisions such as
  - Buying protection...
  - Investing in highly path-dependent derivatives...

*Do not hesitate to contact me to get updated working papers!*
References