Multifractal Volatility: Theory, Forecasting, and Pricing

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University of British Columbia

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Multifractal Volatility
THEORY, FORECASTING, and PRICING

“Multifractal volatility modeling is a major advance, and this book is a milestone in the modern literature on financial econometrics.”
— From the Foreword by John Y. Campbell, Martin and Carole Olshan Professor of Economics, Harvard University, and former President of the American Finance Association

“Calvet and Fisher have fashioned the definitive treatment of multifractal models of return volatility. . . . I highly recommend their book.”
— Darrell Duffie, Dean Witter Distinguished Professor of Finance, Stanford University, CA

“This starkly original work defines a key part of the research frontier . . . it is of immediate practical relevance for asset management, asset pricing and risk management.”
— Francis X. Diebold, J.M. Cohen Professor of Economics, University of Pennsylvania

“A compelling read for financial theorists and practitioners.”
— Peter C. B. Phillips, Sterling Professor of Economics & Statistics, Yale University, CT

“I have always been intrigued by the multifractal approach pioneered by Calvet and Fisher. This book does a wonderful job in gathering together all of the fundamental ideas and results in a coherent framework, and I highly recommend it . . . ”
— Tim Bollerslev, Juanita and Citron Kreps Professor of Economics, Duke University, NC

“The methods and models expounded in Calvet and Fisher’s book should be part of the toolkit of researchers interested in understanding and characterizing the stochastic nature of volatility fluctuations . . . .”
— Lars P. Hansen, Livingston Distinguished Service Professor, University of Chicago, IL

Laurent Calvet and Adlai Fisher show in this book that a simple class of models efficiently captures seemingly disparate aspects of financial market returns. Inspired by earlier uses of multifractals in the natural sciences, the authors construct multifrequency regime-switching models that are convenient to estimate, provide excellent volatility forecasts, and easily integrate into asset pricing applications.

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Adlai J. Fisher is an Associate Professor and the A. E. Hall Chair in Finance at the Sauder School of Business, University of British Columbia, Canada.

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Properties of Financial Data

- **Foreign Exchange**
  - Thick tails
  - Volatility Persistence
  - Volatility comovement across markets

- **Equity**
  - Skewness
  - Jumps
  - Volatility high after down markets (leverage effect/ volatility feedback)

- **Options**
  - Smile / smirk \(\rightarrow\) (thick tails and volatility asymmetry)
  - Volatility term-structure and smile decay slowly
Time Scales in Financial Markets

• **High Frequency**
  – Daily / intraday: macro news, internet bulletin boards, weather (Roll, 1984), analyst reports, liquidity

• **Medium Term**
  – Monthly, quarterly, business cycle range (Fama and French, 1989)

• **Long-run**
  – Demographics, technology (Pastor and Veronesi, 2005), natural resource uncertainty, consumption growth (Bansal and Yaron, 2004)
Standard Approaches

• **Thick-tailed conditional returns (e.g., Student-t, jumps)**
  – Unpredictable high-frequency shocks

• **ARCH / GARCH / SV**
  – Good one-step-ahead volatility predictors
  – Capture medium-run volatility dynamics

• **The long-run**
  – Fractional Integration (FIGARCH),
  – Component Models (Engle and Lee, 1989; Heston, 1993)
  – Markov-switching (Hamilton, 1989)

Typically viewed as unrelated modelling choices
Multifractal Approach

Volatility and Returns

Arbitrarily many frequencies with 4 parameters
Applications: 10 frequencies and over 1,000 states
Durations range from minutes to decades
Closed form likelihood
Improves on standard models in- and out-of-sample
Integrates easily into asset pricing applications
OUTLINE

1 – Modelling multifrequency volatility

2 – Volatility comovement

3 – Pricing multifrequency risk
1 – MULTIFREQUENCY MODEL

L. Calvet and A. Fisher

MARKOV-SWITCHING MULTIFRACTAL (MSM)

Volatility components with highly heterogeneous durations
Parsimonious, tractable, good performance
MSM Definition

\[ x_t = \sigma(M_t) \varepsilon_t \quad \sigma(M_t) = \bar{\sigma} \left( M_{1,t} \ldots M_{\bar{k},t} \right)^{1/2} \]

- **Independent dynamics:**
  \[ M_{k,t} \xleftarrow{\gamma_k} \text{Draw } M_{k,t+1} \text{ from } M \]
  \[ M_{k,t} \xrightarrow{1-\gamma_k} M_{k,t+1} = M_{k,t} \]

- **Multipliers:** binomial \{m_0, 2-m_0\}, equal probability

- **Frequencies:**
  \[ \gamma_k = 1 - (1 - \gamma_{\bar{k}})^{b_{k-\bar{k}}} \approx \gamma_{\bar{k}} b^{k-\bar{k}} \]

Four parameters

Arbitrary number of frequencies
$M = m_0$ or $2 - m_0$ with equal probability

Volatility $\sigma(M_t)$
SIMULATION

Dollar-Mark (1973-1996)

Multifrequency Model
PROPERTIES

• Multifrequency volatility persistence

• Parsimonious

• Thick tails

• Convenient parameter estimation and forecasting

• Out-of-sample volatility forecasts and in-sample measures of fit significantly improve on standard models.
TRACTABILITY OF MSM

A special Markov-switching model

Finite State Space

State vector $M_t$ belongs to finite state space \{m^1,...,m^d\}

Transition matrix $A$

Bayesian Updating

Conditional distribution \[ \Pi_t = (\Pi_{t1},...,\Pi_{td}) \quad \Pi_{t+1} = f(\Pi_t; r_{t+1}) \]

Closed-Form Likelihood

\[ L(r_1,...,r_T) \]

Multistep Forecasting

Given $\Pi_t$, future states have probability $\Pi_t A^n$
### Maximum Likelihood Estimation of Binomial MSM

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m_0$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\gamma}_k$</th>
<th>$\hat{b}$</th>
<th>$\ln L$</th>
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<td>(0.016)</td>
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<td>(0.021)</td>
<td>(0.026)</td>
<td>(0.76)</td>
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<td>7</td>
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<td>0.547</td>
<td>0.932</td>
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<td>(0.025)</td>
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<td>(0.035)</td>
<td>(0.025)</td>
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<td>(0.035)</td>
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<td>2.70</td>
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</tbody>
</table>

*Deutsche Mark / US Dollar*


- Increase in likelihood from $k=1$ to $k=2$ is large by any model selection criterion
- Constant number of parameters as number of frequencies increases
- Models with 7 to 10 frequencies dominate
## In-Sample Comparison

<table>
<thead>
<tr>
<th></th>
<th>No. of Parameters</th>
<th>$\ln L$</th>
<th>BIC</th>
<th>BIC $p$-value vs. Multifractal</th>
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<tbody>
<tr>
<td></td>
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<td>Vuong (1989)</td>
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<td><strong>Deutsche Mark / US Dollar</strong></td>
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<td>1.7830</td>
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<tr>
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<td>0.140</td>
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<td><strong>Japanese Yen / US Dollar</strong></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>Binomial Multifractal</td>
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<tr>
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<td><strong>British Pound / US Dollar</strong></td>
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<tr>
<td>Binomial Multifractal</td>
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<td><strong>Canadian Dollar / US Dollar</strong></td>
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<td></td>
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<td>Binomial Multifractal</td>
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<td>GARCH</td>
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<td>MS-GARCH</td>
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<td>-73.51</td>
<td>0.0322</td>
<td>0.092</td>
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</table>
OUT-OF-SAMPLE ANALYSIS

• Estimate MSM(10)

• Assess forecasting accuracy on out of sample data

• Realized volatility
  \[ RV_{t,n} = \sum_{i=0}^{n-1} r_{t-i}^2 \]

• Out-of-sample \( R^2 = 1 - \frac{\sum_t (RV_{t,n} - E_{t-n}(RV_{t,n}))^2}{\sum_t (RV_{t,n} - RV)^2} \)
## Volatility Forecasts

<table>
<thead>
<tr>
<th>_horizon (Days)</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
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<td><strong>A. Restricted $R^2$</strong></td>
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<tr>
<td><strong>Deutsche Mark / US Dollar</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Binomial Multifractal</td>
<td>0.041</td>
<td>0.124</td>
<td>0.160</td>
<td>0.135</td>
<td>0.038</td>
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<td>GARCH</td>
<td>0.035</td>
<td>0.069</td>
<td>0.033</td>
<td>-0.147</td>
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<td>0.039</td>
<td>0.072</td>
<td>0.030</td>
<td>-0.180</td>
<td>-1.137</td>
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<tr>
<td><strong>Japanese Yen / US Dollar</strong></td>
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<td></td>
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<tr>
<td>Binomial Multifractal</td>
<td>0.053</td>
<td>0.113</td>
<td>0.142</td>
<td>0.205</td>
<td>0.213</td>
</tr>
<tr>
<td>GARCH</td>
<td>0.048</td>
<td>0.054</td>
<td>0.011</td>
<td>-0.024</td>
<td>-0.358</td>
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<tr>
<td>MS-GARCH</td>
<td>0.048</td>
<td>0.044</td>
<td>-0.009</td>
<td>-0.067</td>
<td>-0.569</td>
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<td><strong>British Pound / US Dollar</strong></td>
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</tr>
<tr>
<td>Binomial Multifractal</td>
<td>0.057</td>
<td>0.165</td>
<td>0.235</td>
<td>0.250</td>
<td>0.273</td>
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<tr>
<td>GARCH</td>
<td>0.076</td>
<td>0.191</td>
<td>0.244</td>
<td>0.188</td>
<td>-0.026</td>
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<tr>
<td>MS-GARCH</td>
<td>0.072</td>
<td>0.165</td>
<td>0.238</td>
<td>0.203</td>
<td>0.038</td>
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<tr>
<td><strong>Canadian Dollar / US Dollar</strong></td>
<td></td>
<td></td>
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<tr>
<td>Binomial Multifractal</td>
<td>0.051</td>
<td>0.172</td>
<td>0.221</td>
<td>0.217</td>
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<td>GARCH</td>
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<td>0.154</td>
<td>0.205</td>
<td>0.204</td>
<td>0.070</td>
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<td>MS-GARCH</td>
<td>0.055</td>
<td>0.181</td>
<td>0.229</td>
<td>0.199</td>
<td>0.036</td>
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Results confirmed in CFT (2006), Lux (2008), and Bacry, Kozhemyak, and Muzy (2008).
## Forecast Summary – p-values against MSM

<table>
<thead>
<tr>
<th>B. MSE Test vs. Multifractal (p-value)</th>
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<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
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<tr>
<td><strong>Deutsche Mark / US Dollar</strong></td>
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<td>GARCH</td>
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<td>GARCH</td>
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<td>GARCH</td>
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<td>0.565</td>
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# 2 – VOLATILITY COMOVEMENT


## Correlation of Volatility Components

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<th>DM1</th>
<th>DM2</th>
<th>DM3</th>
<th>DM4</th>
<th>DM5</th>
<th>DM6</th>
<th>DM7</th>
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<tr>
<td>UK1</td>
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<td>0.979</td>
<td>0.603</td>
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<td>0.022</td>
<td>0.022</td>
<td>0.009</td>
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<td>UK2</td>
<td>0.717</td>
<td>0.739</td>
<td>0.620</td>
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<td>UK3</td>
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<td>0.618</td>
<td>0.596</td>
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<td>UK4</td>
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<td>0.079</td>
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<td>0.030</td>
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<td>0.034</td>
<td>0.076</td>
<td>0.168</td>
<td>0.387</td>
<td>0.503</td>
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</table>
MULTIVARIATE MSM

Two financial series \( \alpha \) and \( \beta \)

\[
M_{k,t} = \begin{bmatrix} M_{k,t}^\alpha \\ M_{k,t}^\beta \end{bmatrix} \in \mathbb{R}^2, \quad k \in \{1, \ldots, \bar{k}\}
\]

\[
r_t^\alpha = (M_{1,t}^\alpha \ldots M_{\bar{k},t}^\alpha)^{1/2} \varepsilon_t^\alpha
\]

\[
r_t^\beta = (M_{1,t}^\beta \ldots M_{\bar{k},t}^\beta)^{1/2} \varepsilon_t^\beta
\]

\[
\begin{bmatrix} \varepsilon_t^\alpha \\ \varepsilon_t^\beta \end{bmatrix} = \text{IID } \mathcal{N}(0, \Sigma)
\]

Arrivals in \( M_{kt}^\alpha \) and \( M_{kt}^\beta \) are correlated

Drawn from bivariate binomial:

\[
\begin{array}{c|cc}
    & m_0^\beta & 2 - m_0^\beta \\
\hline
m_0^\alpha & p & 1/2 - p \\
2 - m_0^\alpha & 1/2 - p & p
\end{array}
\]
VALUE-AT-RISK

One-day failure rate

<table>
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<th>Bivariate MSM</th>
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<tr>
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</table>

This table displays the frequency of returns that exceed the VaR forecasted by the model. Bivariate MSM uses 5 components. For quantile $p\%$ the number reported is the frequency of portfolio returns below quantile $p$ predicted by the model. If the VaR forecast is correct, the observed failure rate should be close to the prediction. Boldface numbers are statistically different from $p$ at the 1% level.
3 – Pricing Multifrequency Risk

- Idea: When fundamentals (dividends, earnings, consumption) have multifrequency risks, the equilibrium stock price will include endogenous responses to changes in state variables.

- Volatility feedback: Prices fall when fundamental volatility increases.
  - Overall contribution of endogenous prices responses is 10-40 times larger in multifrequency economy than in single frequency benchmark.

- Learning about volatility:
  - Generates endogenous skewness.
U.S. EQUITY INDEX

Daily excess returns on US aggregate equity
1926-2003: 20,765 observations
Markov-Switching Exchange Economy

Dividends: \( \Delta d_t = \mu_d (M_t) - \sigma_d (M_t)/2 + \sigma_d (M_t) \varepsilon_{d,t} \)

\( M_t \): first order Markov vector, MSM

Consumption: \( \Delta c_t = g_c + \sigma_c \varepsilon_{c,t} \)

\( \varepsilon_{c,t} , \varepsilon_{d,t} \sim N(0,1) \), correlation \( \rho \)

Preferences: Epstein-Zin, risk-aversion \( \alpha \), EIS \( \psi \)

Equilibrium: Stock prices, returns, constant \( r_f \)

In equilibrium, P/D ratio driven by the volatility components (feedback)
Each component has impact inversely related to its frequency
EMPIRICAL RESULTS

- **Calibration**
  - Consumption parameters \((g_c, \delta_c, \bar{n}_{cd})\)
  - \(\vartheta\) and \(\ddot{\alpha}\) appear only in \(r_f\) \(\text{Set } r_f = 1\%\)
  - Dividend growth: \(\bar{r}_d - r_f = 1.2\%, \quad \bar{\sigma}_d = 11\% \text{ per year}\)
  - Long-run P/D = 25

- **Maximum Likelihood Estimation**
  - 3 free parameters \((m_0, \ddot{\alpha}_k, b)\)
  - Daily US excess stock returns
## ML ESTIMATION

*Postwar (1952 - 2003)*

<table>
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<tr>
<th>$k$</th>
<th>$m_0$ (d)</th>
<th>$\gamma_k$ (d)</th>
<th>$b$ (d)</th>
<th>$\ln L$ (d)</th>
<th>mean (%/d)</th>
<th>s.d. (%/d)</th>
<th>skew (d)</th>
<th>kurt (d)</th>
<th>LFY (a)</th>
<th>FB (%)</th>
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<td>84.8</td>
<td>18.2</td>
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</tr>
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</table>

### Feedback

\[ \text{Feedback} = \frac{\text{Var}(r_t)}{\text{Var}(\Delta d_t)} - 1 \]

### Annual equity premium

Annual equity premium $= 4.2\%$
VOLATILITY FEEDBACK

- Estimation on 1926-2003 sample
  Feedback = 40%

- Campbell and Hentschel (JFE, 1992)
  Based on skewed unifrequency process (QGARCH).
  Multifrequency economy outperforms Campbell and Hentschel in sample.

  Feedback
  CH : 1% - 2%
  Multifrequency economy: 30% - 40%
Learning Equilibrium

- Noisy signals: \( M_{t+1} + \sigma \delta Z_{t+1}, \quad Z_{t+1} \sim N(0, I) \)

- P:D linear in investor beliefs: \( Q(\Pi_t) = \sum_{j=1}^{d} \Pi_t^j Q(m^j) \)

Learning

Investors learn abruptly about volatility increases, gradually about decreases.

Equilibrium

Fewer large positive returns than under full information

Endogenous skewness

Tradeoff between skewness and kurtosis

Learning about Volatility is Asymmetric
## Calibrated Results

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<th>Signal Standard Deviation $\sigma_\delta$</th>
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<tr>
<td>$E[r_t]$</td>
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<tr>
<td>$Var[r_t]^{1/2}$</td>
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<tr>
<td>$Skew[r_t]$</td>
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<tr>
<td>$Kurt[r_t]$</td>
<td>133.2</td>
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<td>Feedback</td>
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### A. Full Sample

### B. Postwar

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<td>0.796</td>
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<td>$Skew[r_t]$</td>
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<td>-0.097</td>
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<td>-0.804</td>
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<td>29.4</td>
<td>29.4</td>
<td>29.1</td>
<td>29.2</td>
<td>29.2</td>
</tr>
</tbody>
</table>
Skewness / Kurtosis Tradeoff

- **σ_δ = 0**
- **σ_δ = 0.5**
- **σ_δ = 5**

Graph showing the relationship between skewness and kurtosis for different values of σ_δ.
LONG-RUN
CONSUMPTION RISK

- **IID consumption**
  Consumption parameters \((g_c, \sigma_c, \rho_{c,d})\) of Bansal and Yaron (2004)
  Implied risk-aversion \(\alpha \approx 35\),
  comparable to Lettau Ludvigson and Wachter (2006)

- **Long-run / multifrequency consumption risk**
  Add switches in dividend drift, consumption drift, and consumption volatility
  Generate reasonable equity premium with \(\alpha = 10\)
  Dividend volatility feedback > 20%
4 – Continuous-time MSM and Endogenous Jump-diffusions

Use equilibrium valuation to generate a parsimonious model of multifrequency price jumps

“Multifrequency Jump-Diffusions: An Equilibrium Approach”
JUMP-DIFFUSIONS IN FINANCE

Option pricing

• Stock price follows exogenous jump-diffusion (Merton, 1976)

• Statistical refinements of price process:
  - Stochastic volatility (Bakshi, Cao and Chen, 1997; Bates, 2000)
  - Infinite number of jumps in a finite time interval (Carr, Géman, Madan and Yor, 2002)
  - Exogenous correlation between price jumps and volatility (Duffie, Pan and Singleton, 2000; Carr and Wu, 2003)

Equilibrium

• Exogenous jumps in endowment process
  - Equity premium (Liu, Pan and Wang, 2005)
Equilibrium Specification

Representative Agent

Expected isoelastic utility

\[ E_0 \int_0^{+\infty} e^{-\delta t} u(c_t) dt \]

\[ u'(c) = e^{-\alpha} \]

Observes state \( M_t \) and receives consumption flow

**Endowment**

\[ \frac{dC_t}{C_t} = g_c(M_t) dt + \sigma_c(M_t) dZ_C(t) \]

**Dividend**

\[ \frac{dD_t}{D_t} = g_d(M_t) dt + \sigma_d(M_t) dZ_D(t) \]

The drift and volatility of each process are deterministic functions of \( M_t \)

\[ \begin{bmatrix} Z_C(t) \\ Z_D(t) \end{bmatrix} \]: Brownian with zero drift and covariance matrix

\[
\begin{pmatrix}
1 & \rho_{C,D} \\
\rho_{C,D} & 1
\end{pmatrix}
\]
The log price follows the jump-diffusion

\[ p_t = d_t + q(M_t) \]

where \( d_t \) is the log dividend, and \( q(M_t) \) is the log of the P/D ratio.

\[ q(M_t) = \ln \mathbb{E}_t \left( \int_0^{+\infty} e^{-\int_0^s \left[r_f(M_{t+h}) - g_D(M_{t+h}) + \alpha \sigma_C(M_{t+h}) \sigma_D(M_{t+h}) \rho_{C,D}\right] ds} \right) \]

Endogenous volatility feedback
Many small jumps, some moderate jumps, a few large jumps
Volatility and price jumps endogenously correlated
CONCLUSION

• **MSM**
  Parsimoniously specifies shocks of heterogeneous durations
  Captures persistence and high variability of financial volatility
  Performs well in- and out-of-sample

• **Tractable Multifrequency Equilibrium**
  Feedback increases with likelihood and number of frequencies
  Information quality generates an endogenous trade-off between skewness and kurtosis

• **Jump-Diffusions**
  Price jumps endogenously driven by volatility changes
  Endogenous jump size: small jumps common, rare large jumps
ADDITIONAL SLIDES
INFINITY OF FREQUENCIES

Fixed parameters \((m_0, \sigma_D, \gamma_1, b)\)

When \(\bar{k} \rightarrow \infty\), fundamentals include components of increasing frequency

Volatility \(\sigma_{D,k}(M_t) = \bar{\sigma}_D (M_{1,t} \Delta M_{\bar{k},t})^{1/2}\) degenerate when \(\bar{k} \rightarrow \infty\)

Time deformation \(\theta_{\bar{k}}(t) = \int_0^t \sigma^2_{D,k}(M_s) ds\)

\(\{\theta_{\bar{k}}(t)\}_{\bar{k}}\) is a positive martingale with bounded expectation

\(\downarrow\)

Sequence \(\{\theta_{\bar{k}}(t)\}_{\bar{k}}\) converges to a random variable
LIMITING DIVIDEND PROCESS

If $E(M^2) < b$, the sequence of time-deformations converges to a limit $\theta_{\infty}$, which has continuous sample paths.

Local Hölder exponent: $|X(t + \Delta t) - X(t)| \sim C_t(\Delta t)^{\beta(t)}$

<table>
<thead>
<tr>
<th></th>
<th>$\beta(t)$</th>
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<tbody>
<tr>
<td>Continuous Itô processes</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Traditional Jump diffusion</td>
<td>0 or $\frac{1}{2}$</td>
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<tr>
<td>Multifractal $\theta_{\infty}$</td>
<td>Continuum</td>
</tr>
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</table>
If $C_t = D_t$, $\alpha \leq 1$ and $\rho = \delta - (1 - \alpha)g_D > 0$,
the log-price weakly converges to
\[
d_t + q_\infty(t),
\]
where $q_\infty(t) = \ln \mathbb{E} \left\{ \int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2} [\theta_\infty(t+s) - \theta_\infty(t)]} ds \bigg| M_t \right\}.$

The limiting price process is a multifractal jump-diffusion
with countably many frequencies and infinite activity.
Continuous-time MSM

\[ \sigma_D(M_t) = \sigma_D(M_{1,t} \ldots M_{k,t})^{1/2} \]

COMPONENTS

Draw \( M_{k,t+dt} \) from distribution \( M \)

\[ M_{k,t} \xleftarrow{\gamma_k dt} M_{k,t+dt} = M_{k,t} \]

Distribution: \( M \geq 0, \mathbb{E} M = 1 \)

FREQUENCIES

\[ \gamma_k = \gamma_1 b^{k-1} \]

Four parameters \((m_0, \bar{\sigma}, \gamma_1, b)\)
CONSTRUCTION

\[ M = m_0 \text{ or } 2 - m_0 \] with equal probability

Volatility \( \sigma(M_t) \)