Multi-Factor Affine Extension of Heston Model with Stochastic Interest Rates for Pricing of FX, Inflation, Equity and Volatility Derivatives

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Affine Diffusion Models

Canonical form of affine diffusion SDEs
"Discounted characteristic function" for affine diffusion models
Pricing of European Options

Affine Extended Heston Model with Hull-White Interest Rates

Original Heston’93 model
Model for one asset with displaced SV and Gaussian IRs
  System of SDEs
  Forward vs. Futures prices and Convexity Adjustment
  Price of European Options
  Price of Variance Swaps

Multi-asset model
Suppose the interest rates, log-prices of commodities, equity or inflation indices, logarithms of FX rates, stochastic variances, integrated stochastic variances and other financial variables are represented via the vector of state variables $X(t) \in \mathbb{R}^n$.

Assume that the domestic short interest rate (nominal interest rate for the Jarrow and Yildirim (2003) inflation model) is represented as an affine function of the state variables

$$r(t) = \rho_0(t) + \rho_1 \cdot X(t)$$

(1)

with possibly time-dependent $\rho_0(t) \in \mathbb{R}$ (in order to fit into the initial term structures of the interest rates) and constant $\rho_1 \in \mathbb{R}^n$. 
We fix a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\) and filtration \(\mathcal{F}_t\) for the equivalent martingale measure \(\mathbb{Q}\), and suppose the risk neutral dynamics of the state variables \(X(t)\) under the measure \(\mathbb{Q}\) is defined by the following Markovian process

\[
dX(t) = \mu(t, X(t))dt + \sigma(X(t))dW,
\]

where the drift and covariance matrix are affine in state variables:

\[
\mu(t, x) = K_0(t) + K_1 x, \quad K_0(t) \in \mathbb{R}^n, \quad K_1 \in \mathbb{R}^{n \times n} \tag{3}
\]

\[
\sigma(x)\sigma(x)^T = H_0 + H_1 \cdot x, \quad H_0 \in \mathbb{R}^{n \times n}, \quad H_1 \in \mathbb{R}^{n \times n \times n} \tag{4}
\]

Vector \(W(t) \in \mathbb{R}^n\) is a standard \(\mathbb{Q}\)-Brownian motion with independent components. Coefficient \(K_0(t)\) is time-dependent (including equations for the stochastic variances) to provide consistency with the interest rate dynamics and allow for the exact fit into the initial commodity/equity/FX forward price curves and variance swap price term structures. Coefficients \(K_1, H_0\) and \(H_1\) are constant.
Dai and Singleton (2000) showed that under some non-degeneracy conditions and a possible reordering of indices, it is sufficient for affinity of the diffusions (2) that the volatility matrix $\sigma(X)$ is of the following canonical form

$$\sigma(X) = \Sigma \begin{pmatrix} \sqrt{v_1(X)} & 0 & \ldots & 0 \\ 0 & \sqrt{v_2(X)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{v_n(X)} \end{pmatrix},$$

(5)

where $\Sigma$ is a constant matrix in $\mathbb{R}^{n \times n}$ and $v_j(X)$ are affine functions with constant coefficients

$$v_j(X) = \chi_j + \lambda_j \cdot X, \quad \chi_j \in \mathbb{R}, \quad \lambda_j \in \mathbb{R}^n$$

(6)
However, Cheridito, Filipovic and Kimmel (2007) and Collin-Dufresne, Goldstein and Jones (2008) showed that the Dai and Singleton canonical form is not the most general for the number of state variables more than three. We will use the most general CDGJ08 canonical form based on the rectangular volatility matrix with the number of Wiener processes possibly greater than the number of state variables, constant matrix $\Sigma \in \mathbb{R}^{n \times m}$ ($n \leq m$), and $k \geq 0$ Gaussian and $m - k$ square root components:

$$
\sigma(X) = \Sigma \begin{pmatrix}
1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & \sqrt{X_{k+1}} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \vdots & 0 & 0 & \ldots & \sqrt{X_m}
\end{pmatrix}, \quad (7)
$$
Main result of the seminal Duffie, Pan and Singleton (2000) paper is the derivation of a closed-form "extended transform" for affine jump-diffusion models. It provides semi-analytical solutions for a variety of option pricing problems including the Heston (1993) model. For general affine diffusions, we re-write the “extended transform” in a more natural for the Heston model form of a “discounted characteristic function”:

\[
\phi(u, X_t, T, t) = \mathbb{E}\left(\exp\left(-\delta \int_t^T r(X_s) ds\right) e^{iu \cdot X_T} \bigg| \mathcal{F}_t\right) \tag{8}
\]

Here, we combine together a definition of the “discounted characteristic function” and regular multivariate characteristic function (needed, for example, for pricing futures) using a flag \(\delta = 1\) and \(\delta = 0\) correspondingly.
Proposition

Under the same technical regularity conditions as in DPS00:

$$\phi(u, X_t, T, t) = e^{A(\tau, u) + B(\tau, u) \cdot X_t},$$  \hspace{1cm} (9)

where $\tau = T - t$ (i.e. $t = T - \tau$) and for a fixed $u \in \mathbb{C}^n$ the vector-functions $B(\tau) = B(\tau, u)$ and the function $A(\tau) = A(\tau, u)$ satisfy the following complex-valued ODEs

$$\dot{B}(\tau) = -\delta \rho_1 + K_1^T B(\tau) + \frac{1}{2} B(\tau)^T H_1 B(\tau)$$  \hspace{1cm} (10)

$B(0) = iu$  \hspace{1cm} (11)

$$\dot{A}(\tau) = -\delta \rho_0 (T - \tau) + K_0 (T - \tau) \cdot B(\tau) + \frac{1}{2} B(\tau)^T H_0 B(\tau)$$  \hspace{1cm} (12)

$A(0) = 0$  \hspace{1cm} (13)
Equation (10) is a Riccati equation with constant complex coefficients. It has closed-form solutions for all extensions of the Heston model considered below. The function $A(\tau, u)$ is calculated from (12) by simple integration of a known function and it has a close form, at least, for a piece-wise constant coefficient $K_0(t)$ and standard interpolation of $\rho_0(t)$. Let $S(t)$ be the price of the underlying equity index or FX rate and $X_j(t) = \ln S(t)$ for some fixed index $j$. Denote by $\epsilon_j \in \mathbb{R}^n$ a vector with $j$-th element equal to one and all other elements equal to zero. The price $C$ of a European call option with maturity $T$ and strike $K$ on the equity index or FX rate $S$ is presented by

$$C = G_{\epsilon_j, -\epsilon_j}(-\ln(K); X_t, T, t) - KG_{0, -\epsilon_j}(-\ln(K); X_t, T, t) \quad (14)$$
Here for the vectors $a, b \in \mathbb{R}^n$ the “discounted probability distribution function” of $b \cdot X_T$ conditional on $X_t$

$$G_{a,b}(y; X_t, T, t) = \mathbb{E}\left( \exp \left( - \int_t^T r(X_s) ds \right) e^{a \cdot X_T} 1_{b \cdot X_T \leq y} \right) \quad (15)$$

(with $\delta = 1$) is given in the “Heston” form as:

$$G_{a,b}(y; x, T, t) = \frac{\phi(-ia, x, T, t)}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{ivy} \phi(-ia - vb, x, T, t)}{iv} \right] dv \quad (16)$$

The price of the corresponding put option is found from the put–call parity.
The Heston (1993) model defines the risk neutral dynamics of the underlying asset $S$ and the stochastic variance $V$ (with constant risk-free interest rate $r$) in terms of the state variables $X_1 = V$ and $X_2 = \ln(S)$ as:

$$
\begin{align*}
    dX_1 &= \kappa(\theta - X_1)dt + \eta\sqrt{X_1}dW_1 + 0\sqrt{X_1}dW_2 \\
    dX_2 &= (r - \frac{1}{2}X_1)dt + a_{21}\sqrt{X_1}dW_1 + a_{22}\sqrt{X_1}dW_2
\end{align*}
$$

(17)

Here, $W_1$ and $W_2$ are independent under measure $\mathbb{Q}$ Brownian motions, and $a_{21} = \rho_{sv}, a_{22} = \sqrt{1 - \rho_{sv}^2}$. Positive parameters $\kappa$, $\theta$, and $\eta$ represent the mean-reversion speed, mean-reversion level, and volatility for the stochastic variance $V(t)$. The correlation coefficient $|\rho_{sv}| \leq 1$ controls skewness of the log-price distribution. The stochastic variance $V(t)$ follows a well-known Feller square-root process, therefore it is always non-negative.
A distribution of $V(t)$ at time $t > 0$ conditional on the initial value $V^0$ is a non-central chi-square distribution with $d > 0$ degrees of freedom and noncentrality parameter $\lambda$ (Cox, Ingersoll and Ross (1985)):

$$P(V(t) \leq y \mid V(0) = V^0) = F_{\chi_d^2}(\lambda)\left(\frac{4\theta y}{\eta^2(1 - e^{-\kappa t})}\right) = F_{\chi_d^2}(\lambda)(x),$$

$$d = \frac{4\theta \kappa}{\eta^2}, \quad \lambda = \frac{4\theta e^{-\kappa t}}{\eta^2(1 - e^{-\kappa t})} V^0$$

This distribution has a PDF of the form:

$$f_{\chi_d^2}(\lambda)(x) = \frac{1}{2} e^{-\left(x + \lambda\right)/2} \left(\frac{x}{\lambda}\right)^{d/4 - 1/2} I_{d/2-1}\left(\sqrt{\lambda}x\right), \quad x > 0,$$

where $I_\nu(z)$ is a modified Bessel function of the first kind with power behavior at $z = 0$ resulting in the following power asymptotics for the PDF of $V(t)$:

$$f_{\chi_d^2}(\lambda)(y) \sim C y^{\frac{2\theta \kappa}{\eta^2} - 1}, \quad y \to 0+$$

(18)
It is also known that $V(t)$ has a stationary gamma $\Gamma(\alpha, \beta)$ distribution on $(0, \infty)$ with a PDF of the form

$$f_{\Gamma(\alpha,\beta)}(y) = \frac{1}{\Gamma(\alpha)(\beta)^{\alpha}} y^{\alpha-1} e^{-y/\beta} \quad (19)$$

$$\alpha = \frac{2\theta\kappa}{\eta^2}, \quad \beta = \frac{\eta^2}{2\kappa} \quad (20)$$

and mean equal to $\theta = \alpha\beta$. This stationary PDF has the same asymptotics at zero as the one in (18). The following well-known property of the stochastic variance process is very important for the equivalent measure transformation and “extended affine price of volatility risk” considered below: if $\alpha \geq 1$ then $V(t)$ cannot reach zero, i.e. 0 is an unattainable boundary. Otherwise, 0 is an attainable boundary, which is strongly reflecting in the sense that the length of time when $V(t) = 0$ has zero Lebesgue measure (effectively meaning that when $V(t)$ hits zero it immediately leaves the origin).
The Heston'93 model has the volatility matrix $\sigma^H(X)$ of the following form

$$\sigma^H(X) = \begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \sqrt{X_1} & 0 \\ 0 & \sqrt{X_1} \end{pmatrix}$$

confirming that it is an affine model. Now, let us consider, for example, a one-factor Vasicek (1977) model for the interest rate. The following seemingly simple extension of the Heston model with the stochastic interest rate correlated with the asset ($\text{Corr}(dW_r, dW_s) \neq 0$) is not affine and analytically tractable:

$$dr = \kappa_r(\theta_r - r)dt + \sigma_r dW_r$$
$$dV = \kappa_v(\theta_v - V)dt + \eta \sqrt{V}dW_v$$
$$dX_2 = (r - \frac{1}{2}V)dt + \sqrt{V}dW_s$$
The exponentially affine Heston characteristic function of the $X_2(T)$ conditional on the initial values $X_2(t)$ and $X_1(t)$ is

$$
\phi(T, t, S_t, V_t, u) = \mathbb{E}(e^{iuX_2(T)} | S(t), V(t)) = \exp \left( iur\tau + A(\tau, u) + B_1(\tau, u)X_1(t) + B_2(\tau, u)X_2(t) \right), \tag{21} \n$$

$$
B_2(\tau, u) = iu \tag{22} \n$$

$$
B_1(\tau, u) = \frac{\zeta(u) - d(u)}{2\gamma} \frac{e^{-d(u)\tau} - 1}{G(u)e^{-d(u)\tau} - 1} \tag{23} \n$$

$$
A(\tau, u) = \frac{\kappa \theta}{2\gamma} \left[ (\zeta(u) - d(u))\tau - 2 \ln \frac{G(u)e^{-d(u)\tau} - 1}{G(u) - 1} \right] \tag{24} \n$$

$$
G(u) = \frac{(\zeta(u) - d(u))}{(\zeta(u) + d(u))} \tag{25} \n$$

$$
d(u) = \sqrt{\zeta(u)^2 - 4\gamma \xi(u)} \tag{26} \n$$

$$
\zeta(u) = \kappa - i\eta \rho_{SV} u, \quad \xi(u) = \frac{1}{2} iu(u - 1), \quad \gamma = \frac{1}{2} \eta^2 \tag{27} \n$$
Functions $B_1(\tau, u)$ and $A(\tau, u)$ were written in the original Heston (1993) paper in the different from (23)–(24) form:

$$\tilde{B}_1(\tau, u) = \frac{\zeta(u) + d(u)}{2\gamma} \frac{e^{d(u)\tau} - 1}{g(u)e^{d(u)\tau} - 1}$$  \hspace{1cm} (28)$$

$$\tilde{A}(\tau, u) = \frac{\kappa \theta}{2\gamma} \left[ (\zeta(u) + d(u))\tau - 2 \ln \frac{g(u)e^{d(u)\tau} - 1}{g(u) - 1} \right]$$  \hspace{1cm} (29)$$

with $g(u) = 1/G(u)$. The prices of European options are represented in terms of the integrals of the complex-valued functions (22)–(24). Though both formulations (23)–(24) and (28)–(29) are algebraically equivalent, it is known that in general (29) has discontinuities when the principal branch of the complex logarithm from standard software packages is used during numerical integration. It was proven in Lord and Kahl (2008) that formula (24) does not have discontinuities.
Based on the closed-form solution for the characteristic function (21), Heston derived a semi-analytical formula for the price of a European call option with the strike $K$ and time to maturity $\tau$ as:

$$C(S_t, K, T, t, V_t) = S_t P_1 - e^{-r\tau} K P_2$$  

(30)

Here two Fourier transforms $P_j = P_j(S_t, K, T, t, V_t)$ are calculated for the values of parameters $b_j = 2 - j$, $j = 1, 2$, as follows:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ e^{iv \ln (S_t/K) + r\tau} + A(\tau, v - ib_j) + B_1^H(\tau, v - ib_j) V_t \right] \frac{1}{iv} dv$$

(31)
Let \( r(t) \) be the domestic risk free short interest rate, \( P(t, T) \) - the prices at time \( t \) of domestic zero-coupon bonds with maturities \( T \geq t \), \( f^r(t, T) \) - the corresponding instantaneous forward rates defined as \( f^r(t, T) = -\frac{\partial \ln P(t, T)}{\partial T} \), and \( Y(t, T) \) - the corresponding cumulative yields defined as \( Y(t, T) = - \ln P(t, T) \). From the no-arbitrage condition, \( r(t) = f^r(t, t) \). We assume that the initial \( (t = 0) \) instantaneous forward rate curve \( f^r(0, T) \) (or, equivalently, the initial cumulative yield curve \( Y(0, T) \) or zero-coupon bond price curve \( P(0, T) \)) is given and \( r(0) = f^r(0, 0) \). In the additive two-factor Hull-White (HW2F) model the risk-neutral dynamics of the short rate \( r(t) \) consistent with the initial interest rate term structure \( f^r(0, T) \) is defined as an affine function of the Gaussian state variables \( X_1(t) \) and \( X_2(t) \):

\[
\begin{align*}
    r(t) &= \rho_0^r(t) + X_1(t) + X_2(t) = \rho_0^r(t) + \rho_1^r \cdot X(t)
\end{align*}
\] (32)
Let $H(\kappa, t) = (1 - e^{-\kappa t})/\kappa$. The function $\rho^r_0(t)$ is

$$\rho^r_0(t) = f^r(0, t) + \frac{\sigma^2_1}{2}H(\kappa_1, t)^2 + \frac{\sigma^2_2}{2}H(\kappa_2, t)^2 + \rho^r_{x_1, x_2}\sigma_1\sigma_2H(\kappa_1, t)H(\kappa_2, t)$$  \hspace{1cm} (33)

The dynamics of $X_1(t)$ and $X_2(t)$ under the domestic equivalent martingale measure $\mathbb{Q}$ is defined as $(X_1(0) = X_2(0) = 0)$:

$$dX_1(t) = -\kappa_1 X_1(t)dt + \sigma_1(1\ dW_1 + 0\ dW_2)$$
$$dX_2(t) = -\kappa_2 X_2(t)dt + \sigma_2(a_{21}dW_1 + a_{22}dW_2)$$ \hspace{1cm} (34)

Here, $\sigma_1$ and $\sigma_2$ are the risk factor volatilities, the correlation coefficient $|\rho^r_{x_1, x_2}| \leq 1$, the coefficients $a_{21} = \rho^r_{x_1, x_2}$,

$$a_{22} = \sqrt{1 - a^2_{21}}$$, and positive parameters $\kappa_1$ and $\kappa_2$ are the mean-reversion speeds.
Let us consider the risk neutral dynamics of one asset (equity index, spot commodity price, FX rate or inflation index), $S(t)$, described by the Levin (2008) affine “Extended displaced stochastic volatility Heston” (EDSVH) model with HW2F domestic (or nominal) interest rates correlated with the asset and HW2F stochastic dividend yield (convenience yield, foreign interest rate or real interest rate for inflation model) also correlated with the asset and domestic (or nominal) interest rates. Suppose the initial value of the asset, $S(0)$, and the initial dividend yield (convenience yield, foreign interest rate or real interest rate) curve $f^q(0, T)$ are known. The HW2F stochastic dividend yield (convenience yield, foreign interest rate or real interest rate for inflation model), $q(t)$, is represented via the state variables $X_3(t)$ and $X_4(t)$ as:

$$q(t) = \rho^q_0(t) + X_3(t) + X_4(t) = \rho^q_0(t) + \rho^q_1 \cdot X(t) \quad (35)$$
The function $\rho_0^q(t)$ is

$$
\rho_0^q(t) = f^q(0, t) + \frac{\sigma_3^2}{2} H(\kappa_3, t)^2 + \frac{\sigma_4^2}{2} H(\kappa_4, t)^2 \\
+ \rho_{x_3,x_4}^q \sigma_3 \sigma_4 H(\kappa_3, t) H(\kappa_4, t)
$$

(36)

Let the rest of state variables are: stochastic variance $X_6 = V$, integral of the stochastic variance $X_5 = I = \int_0^t V(s)ds$, and logarithm of the underlying $X_7 = \ln S$. Suppose the mean-reversion level of the Heston stochastic variance is time-dependent deterministic function, specifically, piece-wise constant function. Let for $N \geq 1$ time intervals $\triangle_j = (t_j, t_{j+1}]$, $0 = t_0 < t_1 < \cdots < t_N = \infty$, this piece-wise constant mean-reversion level is $\theta_6(t) = \theta_6^j$, $\forall t \in \triangle_j$. Also, denote $\theta_6^{N-1} = \theta_6^\infty$. 
The system of seven SDEs for the affine EDSVH–HW2F model is:

\[
\begin{align*}
    dX_1 &= -\kappa_1 X_1 \, dt + \sigma_1 a_{11} dW_1 \\
    dX_2 &= -\kappa_2 X_2 \, dt + \sigma_2 \sum_{j=1}^{2} a_{2j} dW_j \\
    dX_3 &= \kappa_3 (\theta_3 - X_3) \, dt + \sigma_3 \sum_{j=1}^{3} a_{3j} dW_j \\
    dX_4 &= \kappa_4 (\theta_4 - X_4) \, dt + \sigma_4 \sum_{j=1}^{4} a_{4j} dW_j \\
    dX_5 &= X_6 \, dt \\
    dX_6 &= \kappa_6 (\theta_6 (t) - X_6) \, dt + \sigma_6 \tilde{a}_{66} \sqrt{X_6} \, dW_6
\end{align*}
\]
\[ \text{d}X_7 = \left[ r(t) - q(t) - \frac{\sigma_7^2}{2} \left( \sum_{j=1}^{5} a_{7j}^2 + (\tilde{a}_{76}^2 + \tilde{a}_{77}^2) X_6 \right) \right] \, \text{d}t \\
+ \sigma_7 \left( \sum_{j=1}^{4} a_{7j} \, \text{d}W_j + a_{75} \, \text{d}W_5 + \sqrt{X_6} (\tilde{a}_{76} \, \text{d}W_6 + \tilde{a}_{77} \, \text{d}W_7) \right) \]

The initial conditions are

\[ X_1(0) = \cdots = X_5(0) = 0, \quad X_6(0) = V_0, \quad X_7(0) = \ln S_0 \]

For the equity/commodity models \( \theta_3 = \theta_4 = 0 \). For the FX (inflation) models \( \theta_3 \) and \( \theta_4 \) represent the price of FX (inflation) risk for the foreign (real) interest rates:

\[ \theta_j = -\frac{c_{x_j, x_7}}{\kappa_j}, \quad c_{x_j, x_7} = \sigma_j \sigma_7 \sum_{k=1}^{j} a_{jk} a_{7k}, \quad j = 3, 4 \]
Denote $a_{66} = \tilde{a}_{66} \sqrt{\theta}_6^\infty$, $a_{76} = \tilde{a}_{76} \sqrt{\theta}_6^\infty$, $a_{77} = \tilde{a}_{77} \sqrt{\theta}_6^\infty$. Coefficients $a_{ij}$ related to the Cholesky decomposition of the correlation matrix are subjected to the normalization constraints

$$a_{11} = 1, \quad \sum_{l=1}^{k} a_{kl}^2 = 1, \quad k = 2, \ldots, 4, \quad a_{66} = 1, \quad \sum_{l=1}^{7} a_{7l}^2 = 1$$

The coefficient $a_{75} \geq 0$ represents a “displacement” of the stochastic volatility and defines the lowest bound for the variance of the underlying. Let $\omega_s$ and $\omega_v$ be two non-negative weights corresponding to the constant and stochastic portions of the “total” variance $\sigma^2_7 (\omega_s + \omega_v X_6 / \theta_6^\infty)$: $\omega_s = \sum_{l=1}^{5} a_{7l}^2$, $\omega_v = a_{76}^2 + a_{77}^2$. It is obvious that $0 \leq \omega_s \leq 1$, $0 \leq \omega_v \leq 1$, $\omega_s + \omega_v = 1$. 
It is obvious that the drift of the EDSVH-HW2F model is affine in state variables. The instantaneous covariance matrix is also affine and given by \( C(X) = \Sigma(X) \Sigma(X)^T = \text{diag}(\sigma) R(X) \text{diag}(\sigma) \) with \( R(X) \) of the form

\[
\begin{bmatrix}
1 & \rho_{x_1,x_2} & \rho_{x_1,x_3} & \rho_{x_1,x_4} & 0 & 0 & \rho_{x_1,s} \\
\rho_{x_1,x_2} & 1 & \rho_{x_2,x_3} & \rho_{x_2,x_4} & 0 & 0 & \rho_{x_2,s} \\
\rho_{x_1,x_3} & \rho_{x_2,x_3} & 1 & \rho_{x_3,x_4} & 0 & 0 & \rho_{x_3,s} \\
\rho_{x_1,x_4} & \rho_{x_2,x_4} & \rho_{x_3,x_4} & 1 & 0 & 0 & \rho_{x_4,s} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rho_{x_1,s} & \rho_{x_2,s} & \rho_{x_3,s} & \rho_{x_4,s} & 0 & \rho_{s,v} X_6/\theta_6^\infty & (\omega_S + \omega_v X_6/\theta_6^\infty)
\end{bmatrix}
\]
Let us define the following variables $\delta_{jk} = \delta_j \delta_k$ and $s_{jk} = s_j s_k$ with

$$
\bar{\delta}_j = \begin{cases} 
\delta, & j = 1, 2 \\
0, & j = 3, 4
\end{cases}, \quad s_j = \begin{cases} 
1, & j = 1, 2 \\
-1, & j = 3, 4
\end{cases}
$$

The functions $B_j (\tau)$ and $A(\tau)$ in the exponentially affine representation of the “discounted characteristic function” (DCF) $\phi(u, X_t, T, t)$ satisfy the following system of Riccati ODEs:

$$
\dot{B}_j (\tau) = -\bar{\delta}_j - \kappa_j B_j (\tau) + s_j B_7 (\tau), \quad j = 1, \ldots, 4
$$

$$
\dot{B}_5 (\tau) = \dot{B}_7 (\tau) = 0
$$

$$
\dot{B}_6 (\tau) = B_5 (\tau) - \kappa_6 B_6 (\tau) - \frac{\sigma_6^2}{2\theta_6^\infty} \omega_v B_7 (\tau)
$$

$$
+ \frac{1}{2\theta_6^\infty} \left[ \sigma_6^2 B_6^2 (\tau) + 2\sigma_6 \sigma_7 \rho_{s,v} B_6 (\tau) B_7 (\tau) + \sigma_7^2 \omega_v B_7^2 (\tau) \right]
$$
\[
\dot{A}(\tau) = \left[ \rho_0 r (T - \tau) - \rho_0 q (T - \tau) \right] B_7(\tau) - \delta \rho_0 r (T - \tau) \\
+ \kappa_6 \theta_6 (T - \tau) B_6(\tau) + \frac{1}{2} \sigma_7^2 \omega_s (B_7(\tau) - 1) B_7(\tau) \\
- c_{x_3,s} B_3(\tau) - c_{x_4,s} B_4(\tau) \\
+ \frac{1}{2} \sum_{j,k=1}^{4} \sigma_j \sigma_k \rho_{x_j,x_k} B_j(\tau) B_k(\tau) + \sigma_7 B_7(\tau) \sum_{j=1}^{4} \sigma_j \rho_{x_j,s} B_j(\tau) 
\]  

Introduce the notation

\[
Y_t^T = Y(0, T) - Y(0, t) = \int_t^T f^r(0, s) \, ds \\
Q_t^T = Q(0, T) - Q(0, t) = \int_t^T f^q(0, s) \, ds 
\]
Let $U_r(\tau)$ and $U_q(\tau)$ represent the variance of $\int_0^\tau (X_1 + X_2) \, ds$ and $\int_0^\tau (X_3 + X_4) \, ds$ respectively. Then

$$\int_t^T \rho_0^r(s) \, ds = Y_t^T + \frac{1}{2} U_{t,T}^r, \quad \int_t^T \rho_0^q(s) \, ds = Q_t^T + \frac{1}{2} U_{t,T}^q$$

$$U_{t,T}^r = U_r(T) - U_r(t), \quad U_{t,T}^q = U_q(T) - U_q(t)$$

Let $U_j^S(\kappa_j, \tau, u_7) = s_j (iu_7 - \delta_j) U_1(\kappa_j, \tau), \; j = 1, \ldots, 4,$

$$U_r(\tau) = \sigma_1^2 U_{12}(\kappa_1, \kappa_1) + \sigma_2^2 U_{12}(\kappa_2, \kappa_2) + 2\rho_{x_1,x_2}\sigma_1\sigma_2 U_{12}(\kappa_1, \kappa_2)$$

$$U_q(\tau) = \sigma_3^2 U_{12}(\kappa_3, \kappa_3) + \sigma_4^2 U_{12}(\kappa_4, \kappa_4) + 2\rho_{x_3,x_4}\sigma_3\sigma_4 U_{12}(\kappa_3, \kappa_4)$$

$$U_{rq}(\tau) = \rho_{x_1,x_3}\sigma_1\sigma_3 U_{12}(\kappa_1, \kappa_3) + \rho_{x_1,x_4}\sigma_1\sigma_4 U_{12}(\kappa_1, \kappa_4)$$

$$+ \rho_{x_2,x_3}\sigma_2\sigma_3 U_{12}(\kappa_2, \kappa_3) + \rho_{x_2,x_4}\sigma_2\sigma_4 U_{12}(\kappa_2, \kappa_4)$$

$$U_{12}(\kappa_1, \kappa_2) = [\tau - H(\kappa_1, \tau) - H(\kappa_2, \tau) + H(\kappa_1 + \kappa_2, \tau)] / (\kappa_1\kappa_2)$$

$$U_1(\kappa, \tau) = [\tau - H(\kappa, \tau)] / \kappa$$
Under this notation, a domestic zero-coupon bond price is

\[ P(t, T) = \mathbb{E} \left[ \exp \left\{ - \int_t^T r(s) \, ds \right\} \right| \mathcal{F}_t \] = \exp \left\{ - Y_t^T + \frac{1}{2} \left( U_r(\tau) - U_{r,t}^T \right) - H(\kappa_1, \tau) X_1(t) - H(\kappa_2, \tau) X_2(t) \right\} \]

A foreign zero-coupon bond price (for FX model) or integrated dividend yield factor (for equity model) is

\[ Q(t, T) = \mathbb{E} \left[ \exp \left\{ - \int_t^T q(s) \, ds \right\} \right| \mathcal{F}_t \] = \exp \left\{ - Q_t^T + \frac{1}{2} \left( U_q(\tau) - U_{q,t}^T \right) - H(\kappa_3, \tau) X_3(t) - H(\kappa_4, \tau) X_4(t) \right\} \]
There exists a closed-form solution of the Riccati ODEs for the multivariate DCF. We will only present a marginal DCF for the underlying that is important for the European option pricing (i.e., \( \phi(u, X_t, T, t) \)) evaluated for \( u_1 = \cdots = u_6 = 0 \) and \( \delta = 1 \). Consider the following parameters that generalize the Heston model parameters (25)–(27)

\[
\eta = \frac{\sigma_6}{\sqrt{\theta_6}}, \quad \nu_s = \frac{\sigma_7}{\sqrt{\theta_6}}, \quad \kappa_v = \kappa_6, \quad \omega = \nu_s^2 \omega_v, \quad \gamma = \frac{\eta^2}{2} \\
\zeta^L = \kappa_v - i \nu_s \eta \rho_{s,v} u_7, \quad \xi^L = 0.5 \left( -i \omega u_7 - \omega u_7^2 \right) \\
d^L = \sqrt{(\zeta^L)^2 - 4 \gamma \xi^L}, \quad \lambda^\pm = 0.5 \left( \zeta^L \pm d^L \right), \quad G^L = \lambda^- / \lambda^+
\]

Then the coefficient of the underlying’s DCF for the EDSVH-HW2F model are as follows:
\[ B_j(\tau, u_7) = s_j(iu_7 - \delta_j)H(\kappa_j, \tau), \ j = 1, \ldots, 4, \]
\[ B_5(\tau, u_7) = 0, \quad B_7(\tau, u_7) = iu_7, \tag{50} \]
\[ B_6(\tau, u_7) = \frac{\zeta^L(u_7) - d^L(u_7)}{2\gamma} \frac{e^{-d^L(u_7)\tau} - 1}{G^L(u_7)e^{-d^L(u_7)\tau} - 1} \]
\[ A(\tau, u_7) = (iu_7 - 1) \left( Y_t^T + \frac{1}{2} U_{r,T}^r \right) - iu_7 \left( Q_t^T + \frac{1}{2} U_{q,T}^q \right) \]
\[ - c_{x_3,x_7} U_3^S - c_{x_4,x_7} U_4^S + \frac{1}{2} \sigma_7^2 \omega_s (iu_7 - 1) iu_7 \tau \]
\[ + \frac{1}{2} (U_r + U_q) + U_{rq} + iu_7 \sigma_7 \sum_{j=1}^4 \sigma_j \rho_{x_j,s} U_j^S + A^L(u_7) \tag{51} \]
\[ A^L(u_7) = \frac{\kappa_\nu}{2\gamma} \sum_{l=0}^{m-k} \theta_6^{m-l} \left[ (\zeta^L - d^S) h_l - 2 \ln \frac{G^L e^{-d^L\tau_{l+1}} - 1}{G^L e^{-d^L\tau_l} - 1} \right] \tag{52} \]
Forward price is given by

\[ F(t, T) = \frac{\mathbb{E} \left[ e^{-\int_{t}^{T} r_s ds} S_T \mid \mathcal{F}_t \right]}{P(t, T)} \]

\[ = \frac{\phi(u_1 = \cdots = 0, u_7 = -i, \delta = 1)}{P(t, T)} = \frac{S_t Q(t, T)}{P(t, T)} \exp \{-\alpha^q(\tau)\}, \]

\[ \alpha^q(\tau) = (\sigma_3 \sigma_7 \rho_{x_3,s} - c_{x_3,x_7}) U_1(\kappa_3, \tau) + (\sigma_4 \sigma_7 \rho_{x_4,s} - c_{x_4,x_7}) U_1(\kappa_4, \tau) \]

Futures price is given by

\[ G(t, T) = \mathbb{E} [S_T \mid \mathcal{F}_t] = \phi(u_1 = \cdots = 0, u_7 = -i, \delta = 0) \]

\[ = F(t, T) \exp \{U_r(\tau) - U_{rq}(\tau) + \alpha^r(\tau)\}, \]

\[ \alpha^r(\tau) = \sigma_1 \sigma_7 \rho_{x_1,s} U_1(\kappa_1, \tau) + \sigma_2 \sigma_7 \rho_{x_2,s} U_1(\kappa_2, \tau) \]
European call option price in EDSVH-HW2F model is:

\[
Call = S_t Q(t, T) e^{-\alpha q} P_1^{EDSVH} - P(t, T) K P_2^{EDSVH}
\]

\[
= P(t, T) \left[ F(t, T) P_1^{EDSVH} - K P_2^{EDSVH} \right]
\]

(55)

Here functions \( P_j^{EDSVH} \) are calculated for \( b_j = 2 - j, j = 1, 2, \) as

\[
P_j^{EDSVH} = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ \frac{1}{i v} \exp \left\{ i v \left( \ln \frac{F(t, T)}{K} \pm \frac{1}{2} U^{TOT}(\tau) \right) \right\} \right] \cdot \exp \left\{ -v^2 \frac{1}{2} U^{TOT} + A^L(\tau, v - ib_j) + B_6(\tau, v - ib_j) V_t \right\} \text{d}v,
\]

\[
U^{TOT}(\tau) = U_r(\tau) - 2U_{rq}(\tau) + U_q(\tau) + \sigma_q^2 \omega_s \tau + 2\sigma_7 \sum_{j=1}^{4} s_j \sigma_j \rho_{x_j} s U_j(\kappa_j, \tau)
\]

\[
Put = Call - P(t, T) [ F(t, T) - K ]
\]
The price of variance swap in EDSVH-HW2F model is:

\[ VarSwap = \omega_s \sigma_7^2 + \omega_v \nu_s^2 \frac{1}{T} \mathbb{E} \left[ \int_0^T V(s) \, ds \right] \]  

(56)

\[
\frac{1}{T} \mathbb{E} \left[ \int_0^T V(s) \, ds \right] = \frac{1}{iT} \partial \phi \left( u_1 = \cdots = u_4 = 0, u_5, u_6 = u_7 = 0, \delta = 0, x, T, 0 \right) \bigg|_{u_5=0}
\]

\[
= \frac{1}{T} \left[ \frac{1 - e^{-\kappa_\nu T}}{\kappa_\nu} V_0 + \sum_{l=0}^{m-k} \theta_6^{m-l} \left( h_l + \frac{e^{-\kappa_\nu \tau_{l+1}} - e^{-\kappa_\nu \tau_l}}{\kappa_\nu} \right) \right]
\]
Example for HW2F interest rate and two equity indices with HW1F dividend yields

\[ r(t) = \rho_0^r(t) + X_1(t) + X_2(t) \]
\[ q^1(t) = \rho_0^{q^1}(t) + X_3(t), \quad q^2(t) = \rho_0^{q^2}(t) + X_4(t) \]
\[ X_5(t) = V(t), \quad X_6(t) = \ln(S^1(t)), \quad X_7(t) = \ln(S^2(t)) \]

System of SDEs:

\[ dX_1 = -\kappa_1 X_1 dt + \sigma_1 a_{11} dW_1 \]
\[ dX_2 = -\kappa_2 X_2 dt + \sigma_2 \sum_{j=1}^{2} a_{2j} dW_j \]
\[ dX_3 = \kappa_3 (\theta_3 - X_3) dt + \sigma_3 \sum_{j=1}^{3} a_{3j} dW_j \]
\[ dX_4 = \kappa_4 (\theta_4 - X_4) dt + \sigma_4 \sum_{j=1}^{4} a_{4j} dW_j \]
Outline
Affine Diffusion Models
Affine Extended Heston Model with Hull-White Interest Rates

Original Heston'93 model
Model for one asset with displaced SV and Gaussian IRs
Multi-asset model

\[
dX_5 = \kappa_5 (\theta_5 (t) - X_5) \, dt + \sigma_5 \tilde{a}_{55} \sqrt{X_5} \, dW_5
\]

\[
dX_6 = \left[ r(t) - q^1(t) - \frac{\sigma_6^2}{2} \left( \sum_{j=1}^{5} a_6^2 j + X_5 \sum_{j=5}^{6} \tilde{a}_{6j}^2 + a_{68}^2 \right) \right] dt
\]

\[
+ \sigma_6 \left( \sum_{j=1}^{4} a_{6j} \, dW_j + \sqrt{X_5} \sum_{j=5}^{7} \tilde{a}_{6j} \, dW_j + \sum_{j=8}^{8} a_{6j} \, dW_j \right)
\]

\[
dX_7 = \left[ r(t) - q^2(t) - \frac{\sigma_7^2}{2} \left( \sum_{j=1}^{4} a_7^2 j + X_5 \sum_{j=5}^{7} \tilde{a}_{7j}^2 + \sum_{j=8}^{9} a_{7j}^2 \right) \right] dt
\]

\[
+ \sigma_7 \left( \sum_{j=1}^{4} a_{7j} \, dW_j + \sqrt{X_5} \sum_{j=5}^{7} \tilde{a}_{7j} \, dW_j + \sum_{j=8}^{9} a_{7j} \, dW_j \right)
\]