Pricing of Guaranteed Minimum Benefits in Variable Annuities

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Agenda

1. Introduction and Motivation
2. Financial and Insurance Market Model
3. Pricing of Variable Annuities
4. Model Calibration
5. Example
Introduction and Motivation
What are Variable Annuities

- **Variable Annuities** (VA) are (deferred), fund-linked annuity and insurance products allowing guaranteed payments and participation in the financial markets at the same time.

- Examples for guaranteed payments include
  - minimum interest rate guarantees
  - ratchets

- Variable annuities are often referred to as GMxB, **Guaranteed Minimum Benefits** of type x:
  - GMDB (Death)
  - GMAB (Accumulation)
  - GMIB (Income)
  - GMWB (Withdrawal)
Markets for Variable Annuities

- **Motivation**
  - Increasing life expectancy
  - Reduction of the state retirement pensions in several countries

- **Consequences**
  - VA as a major success story in the North American insurance market
  - Rapid growth of VA business in Japan - from $1.3 billion in 2001 to more than $140 billion in 2008
  - Europe as the latest market for Variable Annuities
European VA Market

There is significantly more non-public activity

Source: Milliman
Existing Literature

Most of the papers in the academic literature differentiate in: guarantees priced, financial and insurance models, consideration of policyholder behavior, pricing methods

- [Milevsky and Posner 2001] GMDB
- [van Haastrecht et al. 2009] GMAB
- [Milevsky and Salisbury 2006], [Dai et al. 2008] GMWB

**Our contribution:** Derivation of explicit solutions for the prices of some of the VA products currently offered on the market in a hybrid model for insurance and market risks.
Financial Market Model
Financial Market Model
Notation and definitions

- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Furthermore, we assume
- the adapted short rate process $r$, with the money-market account described as
  \[ M_t = \exp \left( \int_0^t r_s ds \right), \]
- a risk-neutral measure $\mathbb{Q}$ under which discounted security $S$ is a $\mathbb{Q}$-martingale:
  \[ S_t = \mathbb{E}_Q \left[ e^{-\int_t^T r_s ds} S_T | \mathcal{F}_t \right]. \]
- Financial market under $\mathbb{Q}$ is described via Hull-White-Black-Scholes hybrid model with time-dependent volatility (HWBS$^{tdv}$)
  \[ dr_t = (\theta_r(t) - a_r t) dt + \sigma_r dW^r_t, \]
  \[ dY_t = \left( r_t - \frac{1}{2} \sigma^2_Y(t) \right) dt + \sigma_Y(t) dW^Y_t, \]
  where $Y = \ln \left( \frac{S_t}{S_0} \right)$ and $dW^r_t dW^Y_t = \rho dt$. 

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Financial Market Model
Equity as a numeraire

- Let $Q^S$ denote the equity price measure, with equity price $S$ used as a numeraire. The corresponding Radon-Nikodym derivative (e.g. see [Geman et al. 1996]) is given by

$$
\frac{dQ^S}{dQ} = \exp \left[ -\frac{1}{2} \int_0^T \sigma_Y^2(t) dt + \int_0^T \sigma_Y(t) dW_Y^Y \right],
$$

- Using multi-dimensional version of Girsanov’s theorem (e.g. see [Oksendal 2005]) we can rewrite the dynamics under $Q^S$:

$$
\begin{align*}
    dr_t &= \left( \theta_r(t) - a_r r_t + \sigma_r \sigma_Y(t) \rho \right) dt + \sigma_r dW_t^{r,Q^S}, \\
    dY_t &= \left( r_t + \frac{1}{2} \sigma_Y^2(t) \right) dt + \sigma_Y(t) dW_t^{Y,Q^S}.
\end{align*}
$$

- It can be shown that, conditional on the current filtration $\mathcal{F}_0$, both $r_T$ and $Y_T$ are normally distributed with the corresponding moments $\mu_{r_T}, \sigma_{r_T}$ and $\mu_{Y_T}, \sigma_{Y_T}$. 
Insurance Model
Notation and definitions

- Random lifetime of a person aged $x$ at $t = 0$ is modeled as a stopping time $\tau(x)$ of a counting process $N_t(x + t)$ with corresponding mortality intensity $\lambda_t(x + t)$.

- Introduce two subfiltrations of $\mathcal{F}$ by $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ and $\mathcal{H} = (\mathcal{H}_t)_{t \geq 0}$
  \[ \mathcal{G}_t = \sigma(\lambda_s(x + s) : s \leq t), \quad \mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau(x) \leq s\}} : s \leq t). \]

- **Definition 1.** Survival probability is defined as a probability that a person at the age of $x + t$ at time $t$ survives at least up to time $T$:
  \[ p_t(x + t, T|\mathcal{G}_t) := \mathbb{P}(\tau(x) > T|\mathcal{G}_t \vee \mathcal{H}_t), \]
  \[ p_t(x + t) := p_t(x + t, t + 1|\mathcal{G}_t) \text{ - is called one-year survival probability.} \]

- For the survival probability measured at time $t$ of a person at the age of $x + t$ at time $t$ it holds that
  \[ p_t(x + t, T|\mathcal{G}_t) = \mathbb{E} \left[ e^{-\int_t^T \lambda_s(x + s)ds} \mid \mathcal{G}_t \vee \mathcal{H}_t \right]. \]
Insurance Model
Mortality improvement ratio

- Compare the mortality intensity at time $0$ with mortality intensity at time $t$
- Introduce the **mortality improvement ratio** as
  \[
  \xi_t(x + t) = \frac{\lambda_t(x + t)}{\lambda_0(x + t)}
  \]

Sample path for the mortality improvement ratio
Insurance Model
Mortality improvement ratio

- We model $\xi_t$ as an extended Vasicek process
  \[ d\xi_t = k(e^{-\gamma t} - \xi_t)dt + \sigma dW_t. \]

- Initial mortality intensity is described by the Gompertz model
  \[ \lambda_0(x + t) = bc^{x+t} \]
  and is calibrated to the current life table.

- Future mortality intensity can be calculated as
  \[ \lambda_t(x + t) = \lambda_0(x + t) \cdot \xi_t. \]

- Survival probabilities can be expressed as
  \[ p_t(x + t, T|G_t) = C_\lambda(t, T)e^{-D_\lambda(t,T)\lambda_t(x+t)}, \]
  where $C_\lambda(t, T)$ and $D_\lambda(t, T)$ satisfy two ordinary differential equations.
Guaranteed Minimum Income Benefit

Definition

- Let
  - $P$ - single premium
  - $T$ - end of the accumulation period
  - $A_t$ - account value at time $t$
  - $G^x_t$ - guaranteed amount at time $t$ for the corresponding GMxB

- GMIB provides a policyholder who is alive at the end of the accumulation period $T$ with a benefit $V^I_T$, defined as

  $$V^I_T = \mathbb{1}_{\tau > T} \max(A_T, G^I_T \cdot g \cdot a_n(T)),$$

  where $g$ is a guaranteed annuitization rate, $a_n(T)$ is a price at time $T$ of a $n$-year annuity paying one unit each year starting from $T$. 

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Guaranteed Minimum Income Benefit
Definition

- The following options for $G^I_T$ are quite common on the market:
  - No guarantee: $G^I_T = 0$
  - Return of premium: $G^I_T = P$
  - Roll-up: $G^I_T = P e^{\delta T}$, where $\delta$ is a continuously compounded roll-up rate
  - Ratchet: $G^I_T = \max_{t_i < T} A_{t_i}$

- We consider a pure equity fund underlying the policyholder’s account, i.e.
  $$dA_t = A_t \frac{dS_t}{S_t}, \quad A_0 = P.$$

- The time 0 fair value of GMIB can be written as
  $$V^I_0 = \mathbb{E}_Q \left[ e^{-\int_0^T r_s \, ds} \mathbb{1}_{\{\tau > T\}} \max(A_T, G^I_T \cdot g \cdot a_n(T)) \right].$$
Guaranteed Minimum Income Benefit

Theorem

Analytical expression for $V^I_0$ can be derived:

$$V^I_0 = P \cdot C(0, T)e^{-D(0, T)\lambda_0} \left( 1 + e^{\delta T} g \sum_{i=1}^{n} \left[F_i N(h^1_i) - K_i N(h^2_i)\right] \right),$$

where

$$F_i = e^{M_i + \frac{1}{2}V_i},$$

$$h^1_i = \frac{ln \left( \frac{F_i}{K_i} \right) + \frac{1}{2}V_i}{\sqrt{V_i}},$$

$$h^2_i = h^1_i - \sqrt{V_i},$$

$$M_i = ln(\tilde{C}_i) = ln \left( C(T, t_i)C_r(T, t_i)e^{-D(0, T)\mu_{T_r} - D(T, t_i)\mu_{T_r'} - \mu_{T_r}} \right),$$

$$V_i = \tilde{D}_i^2 = D^2_{\lambda} \sigma^2_{\lambda} + D^2_{r} \sigma^2_{r} + \sigma^2_{Y} + 2D_r \sigma_{r} \sigma_{Y} \rho_{r,Y_T}. $$

$K_i$ is defined as $K_i := \tilde{C}_i e^{-\tilde{D}_i x^*}$, where $x^*$ is a solution of

$$\sum_{i=1}^{n} \tilde{C}_i e^{-\tilde{D}_i x^*} = K.$$
Model Calibration
Insurance Model Calibration

Data

- Initial mortality table (Source: Federal Statistical Office of Germany)

- Mortality improvement ratio (Source: Federal Statistical Office of Germany)
Insurance Model Calibration

Algorithm

- Estimate parameters of the Gompertz model from the initial mortality intensity via least square method

- Estimate parameters of the mortality improvement ratio process via maximum likelihood method. Corresponding log-likelihood function is given by:

\[
\mathcal{L}(k, \gamma, \sigma) = \sum_{i=1}^{n} \ln(f(\xi_i|\xi_{i-1}; k, \gamma, \sigma))
\]

\[
= \frac{n}{2} \ln(2\pi) - n \ln \hat{\sigma}
\]

\[
- \frac{1}{2\hat{\sigma}} \sum_{i=1}^{n} \left( \xi_i - \xi_{i-1} e^{-k \cdot \Delta} - \frac{k}{k - \gamma} e^{-\gamma t_i} \cdot \left( 1 - e^{(\gamma-k) \cdot \Delta} \right) \right)^2,
\]

where

\[
\hat{\sigma} = \sigma \sqrt{\frac{1 - e^{-2k \cdot \Delta}}{2k}}
\]
Financial Model Calibration

Data

- Interest rate data: deposit rates, swaps, swaptions (Source: Bloomberg)

- Equity data: implied volatilities term structure (Source: Bloomberg)
Financial Model Calibration
Algorithm

- Estimate $\theta_t$ based on the current term structure of the interest rates
- Calibrate Hull-White model to the observed prices for the European swaptions
- Estimate instantaneous volatility from the term structure of the implied volatility
Example

5
Setup

- Type of the guarantee: single premium GMIB
- Guaranteed annuitization rate: 7.5%
- Roll-up rate: 2%
- Maturity of the guarantee: 10 years
- Maturity of the guaranteed annuity: 20 years
- Policyholder: male, 55 year old
- Mortality: German mortality table for 2007/2009
- Financial Model: HWBS calibrated to the market data as of 30/04/2012 (Market data for calibration: VSTOXX, EUR swap based yield curve and swaptions)
Sensitivities to Financial Market Parameters

- Changes in implied volatility

- Changes in interest rates
Sensitivities to Insurance Market Parameters

- Changes in the underlying mortality table

- Resulting GMIB prices
Sensitivities to Product Parameters

- Changes in annuitization rates

- Changes in roll-up rates
Conclusion & Further Research

- HWBS$^{dv}$ for the financial market
- 2-step approach for stochastic mortality modeling
- Explicit expression for GMIB
- Calibration of the presented hybrid model
- Analyze other types of guarantees (GMWB, GMDB)
- Analyze additional guarantee riders (ratchets, resets)
- Incorporate policyholder behavior risk
- Advance the underlying model
- ...

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Thank you for your attention.
Bibliography


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