Comonotonic Measures of Multivariate Risks

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Mathematical Finance (forthcoming)

WatRISQ
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• How to evaluate risk from the point of view of a regulator?

• Notions of risk measures, convex, coherent, comonotonic risk measures.

• How to measure the aggregate risk of losses that are not perfectly substitutable?

• Interpretation, computation, estimation.
Measuring risk

• A risk $X$ is a random variable (whose realization is a loss). A risk measure $\rho$ is a functional, that to a risk $X$ associates the real number $\rho(X)$.
  
  – Example: Value-at-Risk
    $$\text{VaR}_\alpha(X) = Q_X(\alpha) = \inf\{x : \mathbb{P}(X \leq x) \geq \alpha\}.$$ 

• Some properties one can require of risk measures:
  
  – **Law invariance:** If $X =_d Y$ then $\rho(X) = \rho(Y)$: only the loss distribution matters, not the states of the world when the losses occur.
  
  – **Monotonicity:** If $X \geq Y$, then $\rho(X) \geq \rho(Y)$: the risk measure increases if losses increase in all states of the world.
Properties one can require of risk measures (continued):

- **Convexity**: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$

- **Positive homogeneity**: If $\lambda > 0$, $\rho(\lambda X) = \lambda \rho(X)$ (level proportionality).

- **Sub-additivity**: $\rho(X + Y) \leq \rho(X) + \rho(Y)$: the measure of risk cannot be artificially reduced by segmenting the risk.

- **Comonotonicity**: If $X$ and $Y$ are comonotonic, then
  
  \[ \rho(X + Y) = \rho(X) + \rho(Y) \]
Seminal references

• “Dual theory of choice under risk” Yaari - Econometrica 1989
• “Ordering risks” Wang and Young - Insurance: Mathematics and Economics 1998
• “Coherent measures of risk” Artzner, Delbaen, Eber, Heath - Mathematical Finance, 1999
• “On law invariant coherent risk measures” Kusuoka - Advances in Mathematical Economics, 2001
• “Vector valued coherent risk measures” Jouini, Meddeb, Touzi - Finance and Stochastics 2004
• Rüschendorf et al. - multiple references
Plan of the talk

- Comonotonicity
- Functional representation of comonotonic measures
- Comonotonicity revisited: the concept of strong coherence
- Multivariate risks
- Computation and estimation of comonotonic measures
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Comonotonicity

• $X$ and $Y$ are comonotonic if there exists a risk $Z$ such that $X = T_X(Z)$ and $Y = T_Y(Z)$, $T_X, T_Y$ increasing functions.

• Example:
  - If $X(\omega_i) = x_i$ and $Y(\omega_i) = y_i$, $i = 1 \ldots, n$, with $x_1 \leq \ldots \leq x_n$ and $y_1 \leq \ldots \leq y_n$, then $X$ and $Y$ are comonotonic.
  - By the simple rearrangement inequality,

$$\sum_{i=1}^{n} x_i y_i = \max \left\{ \sum_{i=1}^{n} x_i y_{\sigma(i)} : \sigma \text{ permutation} \right\}.$$

• General characterization: $X$ and $Y$ are comonotonic iff

$$\mathbb{E}[XY] = \sup \left\{ \mathbb{E}[X\tilde{Y}] : \tilde{Y} =_d Y \right\}.$$
Comonotonicity (continued)

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- \( X \) and \( Y \) are comonotonic risks.
- \( \tilde{Y} \) has the same distribution as \( Y \) but \( X \) and \( \tilde{Y} \) are not comonotonic.
- Note that \( \mathbb{E}[XY] = 2 > -2 = \mathbb{E}[X\tilde{Y}] \).
Plan of the talk

- Comonotonicity
- **Functional representation of comonotonic measures**
- Comonotonicity revisited: the concept of strong coherence
- Multivariate risks
- Computation and estimation of comonotonic measures
Yaari-Kusuoka representation theorem

- $\rho$ is law invariant, monotonic and comonotonic iff there exists a nonnegative function $\phi$ such that

$$\rho(X) = \int_0^1 \phi(u)Q_X(u)du = \mathbb{E}[\phi(U)Q_X(U)].$$

Hence $\rho(X)$ is a weighted sum of quantiles of $X$:

$$Q_X(u) = \inf\{x : \mathbb{P}(X \leq x) \geq u\}$$

- If in addition $\rho$ is convex, then $\phi$ is non decreasing, hence $\rho$ gives larger weights to high quantiles (large losses) than low quantiles (small losses).

  - Ex: **Conditional Value-at-Risk** or **Expected Shortfall**

  $$\text{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_{\alpha}^{1} Q_X(u)du.$$
Figure 1: Comparison of value-at-risk and expected shortfall.
Yaari-Kusuoka representation (continued)

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- $\rho(X) = -1 \times \int_{0}^{1/2} \phi(u)du + 1 \times \int_{1/2}^{1} \phi(u)du$
- $\rho(Y) = \rho(\tilde{Y}) = -2 \times \int_{0}^{1/2} \phi(u)du + 2 \times \int_{1/2}^{1} \phi(u)du$
- If $\rho$ is sub-additive, $\phi$ is non decreasing and $\rho(Y) \geq \rho(X)$ (larger weights for large positive losses).
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• Functional representation of comonotonic measures
• Comonotonicity revisited: the concept of strong coherence
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Why impose comonotonicity?

• How can comonotonicity be justified from the point of view of the regulator?

• The definition of comonotonicity appears to rely on the natural ordering of the real line. How can we measure the multi-dimensional risks?
Strong coherence

- A regulator imposes a risk measurement rule to two institutions. Institution $i$ knows the distribution of its own risk $X_i$ and must report $\rho(X_i)$.

- The regulator only receives $\rho(X_1)$ and $\rho(X_2)$, and measures aggregate risk with the sum $\rho(X_1) + \rho(X_2)$.

- The regulator knows nothing of the joint distribution of $(X_1, X_2)$. So as not to underestimate aggregate risk, the regulator chooses $\rho$ in order that $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ irrespective of the joint distribution.

- To avoid unnecessary conservativeness,

  $$\rho(X_1) + \rho(X_2) = \sup\{\rho(X_1 + \tilde{X}_2) : \tilde{X}_2 \overset{d}{=} X_2\} :$$

  which we call strong coherence.
Comonotonicity and strong coherence

If the risk measure is \textit{convex} and l.s.c. then:

\textbf{Strong Coherence} \iff \textbf{Comonotonicity} + Law invariance

Hence the concept of strong coherence allows

- interpretation (with the point of view of the regulator),
- simplification (two properties for the price of one)
- generalization (the property is independent of dimension).
Properties and interpretations

A strongly coherent convex l.s.c. risk measure satisfies

- Positive homogeneity \( (\rho(2X) \leq \sup_{\tilde{X}=dX} \rho(X + \tilde{X}) = 2\rho(X)) \)
- Law invariance \( (\sup_{\tilde{X}=dX} \rho(X + 0) = \rho(X) + \rho(0) = \rho(X)) \)
- No worst case diversification effect:
  \[
  \sup_{\tilde{Y}=dY} \rho((X + \tilde{Y})/2) = (\rho(X) + \rho(Y))/2
  \]
- No premium to conglomerates
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Multivariate risks

Measuring the risk of a vector of non substitutable risks?

- Risks specified in different currencies
- Risks with different maturities
- Monetary and non monetary risks...

In that case, a risk is a random vector $X$, and a risk measure $\rho$ is a functional that to a risk $X$ associates a real number $\rho(X)$.

Using the strong coherency property, we generalize

- the Yaari-Kusuoka representation theorem,
- the concept of comonotonicity.
Generalized Yaari-Kusuoka representation theorem

- A risk measure $\rho$ is **comonotonic and law invariant** iff there exist a random vector $U$ and a non negative function $\phi$ such that
  \[
  \rho(X) = \mathbb{E}[\langle Q_X(U) \cdot \phi(U) \rangle],
  \]
  where $Q_X$ is the generalized quantile of random vector $X$ relative to $U$.

- $\rho$ is **convex l.s.c. and strongly coherent** iff there exists a random vector $U$ such that
  \[
  \rho(X) = \mathbb{E}[\langle Q_X(U) \cdot U \rangle] = \sup\{\mathbb{E}[\langle X \cdot \tilde{U} \rangle]; \tilde{U} =_d U\}.
  \]
Multivariate quantiles

• The quantile function of a random variable $X$ is an *increasing rearrangement* of $X$. Hence the quantile $Q_X(U)$ of $X$ is comonotonic with the uniform random variable $U$ on $[0, 1]$.

  – $Q_X$ is the only **increasing function** such that

    \[ \mathbb{E}[Q_X(U)U] = \sup\{\mathbb{E}[\tilde{X}U]; \tilde{X} =_d X\}. \]

• Similarly, the generalized quantile $Q_X(U)$ of a random vector $X$ is a rearrangement of $X$ relative to a random vector $U$.

  – $Q_X$ is the essentially unique **gradient of a convex function** such that

    \[ \mathbb{E}[\langle Q_X(U) \cdot U \rangle] = \sup\{\mathbb{E}[\langle \tilde{X} \cdot U \rangle]; \tilde{X} =_d X\}. \]
Figure 2: The quantile of a random variable is an increasing rearrangement.
Generalization of comonotonicity

To preserve the equivalence

**Strong Coherence)** ⇐⇒ **Coomonotonicity** + Law invariance

the suitable generalization of the concept of comonotonicity between two random vectors \(X\) and \(Y\) is the following:

\(X\) and \(Y\) are \(\mu\)-comonotonic if \(X = Q_X(U)\) and \(Y = Q_Y(U)\), namely if \(X\) and \(Y\) can be simultaneously rearranged relative to a reference random vector \(U\) with distribution \(\mu\).
The equidistribution class of $U$ is the circle, and two $\mu$-comonotone random vectors $X$ and $Y$ have the same $L^2$ projection $\tilde{U}$ on the equidistribution class of $U$ with distribution $\mu$.

Figure 3: The equidistribution class of $U$ is the circle, and two $\mu$-comonotone random vectors $X$ and $Y$ have the same $L^2$ projection $\tilde{U}$ on the equidistribution class of $U$ with distribution $\mu$. 
Elements of proof

• Since $\rho$ is convex l.s.c., by the Fenchel-Moreau conjugation theorem,

$$\rho(X) = \rho^{**}(X) = \sup_{Y \in L^q_d} \{\mathbb{E}(X \cdot Y) - \rho^*(Y)\}.$$ 

• Since $\rho$ is positively homogeneous,

$$\rho^*(Y) = \sup_{X \in L^p_d} \{\mathbb{E}(X \cdot Y) - \rho(X)\} = 0 \text{ or } + \infty.$$ 

So

$$\rho(X) = \sup_{Y \in \mathcal{D}} \mathbb{E}(X \cdot Y) \quad \text{with} \quad \mathcal{D} = \{Y \in L^q_d : \rho^*(Y) = 0\}$$
• Since $\rho$ is law invariant

$$
\rho(X) = \sup_{\tilde{X} \sim X} \rho(\tilde{X}) = \sup_{\tilde{X} \sim X} \sup_{Y \in \mathcal{D}} \mathbb{E}(Y \cdot \tilde{X}) = \sup_{Y \in \mathcal{D}} \sup_{\tilde{X} \sim X} \mathbb{E}(Y \cdot \tilde{X}) = \sup_{Y \in \mathcal{D}} \rho_Y(X).
$$

where $\rho_Y(X)$ is a maximum correlation measure.

• For all $Y \in \mathcal{D}$, $-\rho(-X) \leq \mathbb{E}(Y \cdot X) \leq \rho(X)$, so $\mathcal{D}$ is bounded in $L^q$ hence weakly compact (Banach-Alaoglu).

• The function $Y \mapsto \rho_Y(X)$ is u.s.c., hence the set

$$
\mathcal{M}(\rho, X) = \left\{ Y \in \mathcal{D} : \rho_Y(X) = \sup_{Y' \in \mathcal{D}} \rho_{Y'}(X) \right\}
$$

is non empty and weakly closed.

• By weak compactness of $\mathcal{D}$, for a portfolio $(X_1, \ldots, X_N)$, there
exists $Y_0$ such that
\[
\frac{1}{N} \sum_{i=1}^{N} \rho_{Y_0}(X_i) = \sup_{Y \in \mathcal{D}} \left( \frac{1}{N} \sum_{i=1}^{N} \rho_Y(X_i) \right)
\]

by the converse
\[
= \sup_{Y \in \mathcal{D}} \sup_{\tilde{X}_i \sim X_i} \rho_Y \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{X}_i \right)
\]

\[
= \sup_{\tilde{X}_i \sim X_i} \sup_{Y \in \mathcal{D}} \rho_Y \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{X}_i \right)
\]

by strong coherence
\[
= \frac{1}{N} \sum_{i=1}^{N} \rho(X_i)
\]
Hence

\[ \frac{1}{N} \sum_{i=1}^{N} \rho_{Y_0}(X_i) = \frac{1}{N} \sum_{i=1}^{N} \sup_{Y \in \mathcal{D}} \rho_Y(X_i) \]

so that for all \( i = 1, \ldots, N \),

\[ \rho_{Y_0}(X_i) = \sup_{Y \in \mathcal{D}} \rho_Y(X_i) \]

• We have shown that all finite intersections \( \bigcap_{i=1}^{N} \mathcal{M}(\rho, X_i) \) are non empty on the weakly compact set \( \mathcal{D} \), hence

\[ \bigcap_{X \in L^p_d} \mathcal{M}(\rho, X) \neq \emptyset \]

and \( \rho \) is a maximum correlation measure.
• **Conversely**, let $\rho_\mu$, be a maximum correlation measure and $(X_1, \ldots, X_N)$ be a $\mu$-comonotonic vector.

• By Brenier’s Theorem, there exist $Y \sim \mu$ and l.s.c. convex functions $f_i$ such that $X_i \in \partial f_i(Y)$ for all $i = 1, \ldots, N$.

• $\sum_{i=1}^N f_i$ is also convex l.s.c., so $\sum_{i=1}^N X_i \in \partial \sum_{i=1}^N f_i(Y)$ is also $\mu$-comonotonic with the $X_i$’s.

• Hence

$$\sup_{\tilde{X}_i \sim X_i} \rho_\mu \left( \frac{1}{N} \sum_{i=1}^N \tilde{X}_i \right) = \sup_{\tilde{Y} \sim \mu} \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N X_i \right) \cdot \tilde{Y} \right]$$

$$= \frac{1}{N} \sum_{i=1}^N \mathbb{E}[X_i \cdot Y] = \frac{1}{N} \sum_{i=1}^N \sup_{\tilde{Y} \sim \mu} \mathbb{E}[X_i \cdot \tilde{Y}] = \frac{1}{N} \sum_{i=1}^N \rho_\mu(X_i)$$

and the result follows by convexity and homogeneity of $\rho_\mu$. $\square$
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Example: Gaussian risk measure

Suppose the baseline $U$ has a standard normal distribution, and $X$ and $Y$ are risks with distribution $N(0, \Sigma_X)$ and $N(0, \Sigma_Y)$ respectively.

- The generalized quantile of $X$ relative to $U$ is $Q_X(U) = \Sigma_X^{1/2} U$.

- $X$ and $Y$ are $U$-comonotonic when $E(X \cdot Y) = \Sigma_X^{1/2} \Sigma_Y^{1/2}$ $(X$ and $Y$ have the same “orientation”).

- The $U$-comonotonic risk measure is $\rho(X) = \text{tr}(\Sigma_X^{1/2})$ $(\rho$ is the trace norm).
Example: Special case of Gaussian risk measure

In $\mathbb{R}^2$, if $X$ is normal with covariance

$$\Sigma_X = \begin{pmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix},$$

then the comonotonic risk measure with standard normal baseline is

$$\rho(X) = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \sigma_2 \sqrt{1 - \rho^2}}.$$
Example: Multivariate expected shortfall

Consider the Bernoulli random vector $U = (1/\alpha, \ldots, 1/\alpha)'$ with probability $\alpha$ and 0 with probability $1 - \alpha$ as a baseline risk.

The comonotonic risk measure relative to baseline $U$ is

$$
\rho(X) = \mathbb{E} \left[ \left( \sum_{i=1}^{d} X_i \right) 1\{\sum_{i=1}^{d} X_i \geq c\} \right],
$$

where $c$ is determined by $\mathbb{P}(\sum_{i=1}^{d} X_i \geq c) = \alpha$.

Hence $\rho$ is the $\alpha$ expected shortfall for $\sum_{i=1}^{d} X_i$. 
Estimation of risk measures

Consider the case where the baseline risk $U$ has distribution $\mu$ on $[0, 1]^d$ and where a discrete estimate of the distribution of $X$ is available, typically the empirical distribution from a sample of realizations $(X_1, \ldots, X_n)$.

The empirical quantile function satisfies

- $\hat{Q}_X(U) \in \{X_1, \ldots, X_n\}$
- $\mu(\hat{Q}_X^{-1}(\{X_k\})) = 1/n$, for each $k = 1, \ldots, n$
- $\hat{Q}_X$ is the gradient of a convex function $V : \mathbb{R}^d \to \mathbb{R}$. 
Estimation of risk measures (continued)

The solution for the “potential” $V$ is

$$V(u) = \max_k \{ \langle u, X_k \rangle - w_k \},$$

where $w = (w_1, \ldots, w_n)'$ minimizes the convex function

$$w \mapsto \int V(u) d\mu(u) + \sum_{k=1}^n w_k/n.$$
Figure 4: Mapping the uniform to a discrete distribution in dimension $d = 2$. Graphical representation of the potential $V(u)$ for 27 observations.