The Tutte Polynomial and Applications

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Overview

- Background Graph Theory
- Tutte Polynomial and Flow Polynomial
- Applications

Background

Graph

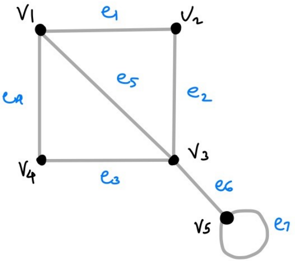
Definition

A *graph* $G = (V, E)$ consists of a set of vertices, V and a set of edges denoted as two element subsets of V, E. For $u, v \in V(G)$ and (u, v) an edge we sometimes denote this uv.

Definition

The *order of a graph G* is the number of vertices it has, written as $|G|$. And so an empty graph, $(\varnothing, \varnothing)$ is a graph of order 0.

Example



$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

$|G| = 5$

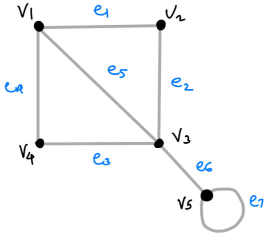
Adjaceny and Degree

Definition

Two vertices, x and y are *adjacent* or neighbours if xy is an edge of the graph.

Definition

The *degree of a vertex* v is the number of edges incident to v. This is equal to the number of neighbours of v, written as $deg(v)$.



$deg(v_1) = 3$
$deg(v_2) = 2$
$deg(v_3) = 4$
$deg(v_4) = 2$
$deg(v_5) = 2$

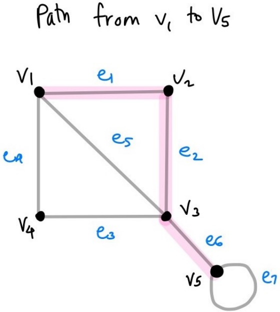
Path and Cycle

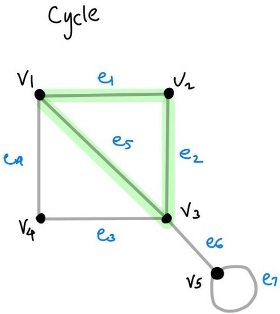
Definition

A *path* is a non-empty graph $P = (V, E)$, where $V = \{x_0, x_1, ..., x_k\}$ and $E = \{x_0x_1, x_1x_2, ...x_{k-1}x_k\}$, where all the x_i are distinct.

Definition

A *cycle* is a path where $x_0 = x_k$ and every other vertex is distinct.





Connected, Loops, and Bridges

Definition

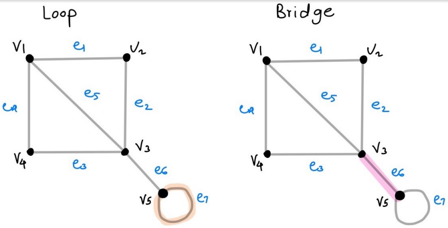
A graph G is *connected* if there is path for every $u, v \in V$, that is there is a path that connects every vertex to another.

Definition

A *bridge* is an edge which when removed will disconnect the graph.

Definition

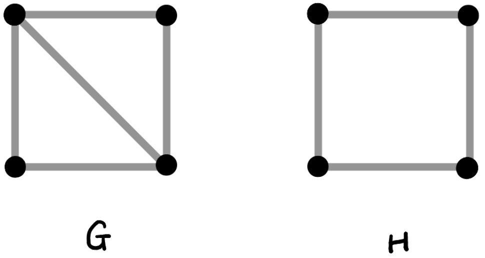
An edge e is a *loop* if $e = xx$ for some vertex x.



Subgraph

Definition

A *subgraph* H of G is the graph that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.



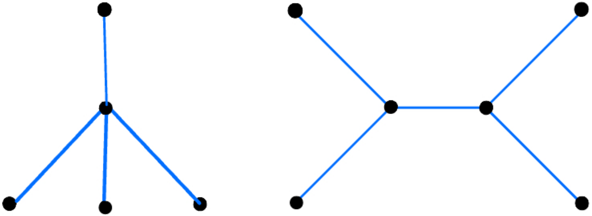
Tree & Forest

Definition

A connected graph containing no cycles is a *tree*.

Definition

A *forest* is a graph whose components are trees.



Spanning Tree & Maximal Spanning Forest

Definition

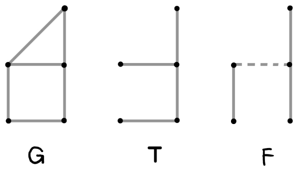
If H is a subgraph of G and $V(H) = V(G)$, H is *spanning*.

Definition

A *spanning tree* of G is a spanning subgraph that is a tree.

Definition

A *maximal spanning forest* is a subgraph that is a forest with maximal possible edges.



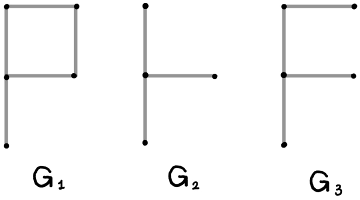
Vertex Deletion and Edge Deletion

Definition

For a graph G the *deletion* of $v \in V(G)$ or $G - v$ is the graph obtained from G by removing v and all edges incident to v.

Definition

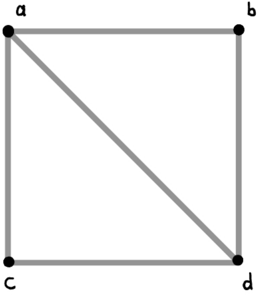
For a graph G the *deletion* of $e \in E(G)$ or $G - e$ is the graph obtained from G by removing v and all edges incident to v.

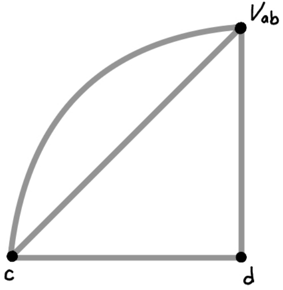


Edge Contraction

Definition

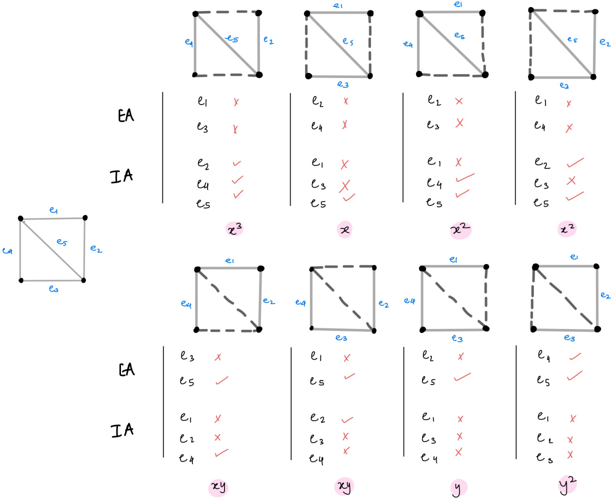
For a graph G the *contraction* of $e = uv \in E(G)$ results in a new graph $G/e = (V - v - u + w, E - e)$ where w is adjacent to all neighbors of u and v.





Tutte Polynomial

Example



$$T(G) = x^3 + 2x^2 + x + 2xy + y + y^2$$

Activity Definitions

Definition

Let G be a graph and F a maximal spanning forest of G. If $e \notin E(F)$ then $F \cup \{e\}$ contains a cycle. This is the *fundamental cycle* of e with respect to F. If $e \in F$ and T is the tree in F that contains e then $F - e$ disconnects T. The *fundamental bond* of e with respect to F is the set of all edges in G that reconnect T.

Definition

Let G be a graph and F a maximal spanning forest of G and $\overline{E} = (e_1, e_2, \ldots, e_m)$ a total ordering on the edges of G then

- $e_i \in F$ is *internally active* with respect to F if e_i is maximal in its fundamental bond. (inactive otherwise)
- $e_i \notin F$ is *externally active* with respect to F if e_i is maximal in its fundamental cycle. (inactive otherwise)

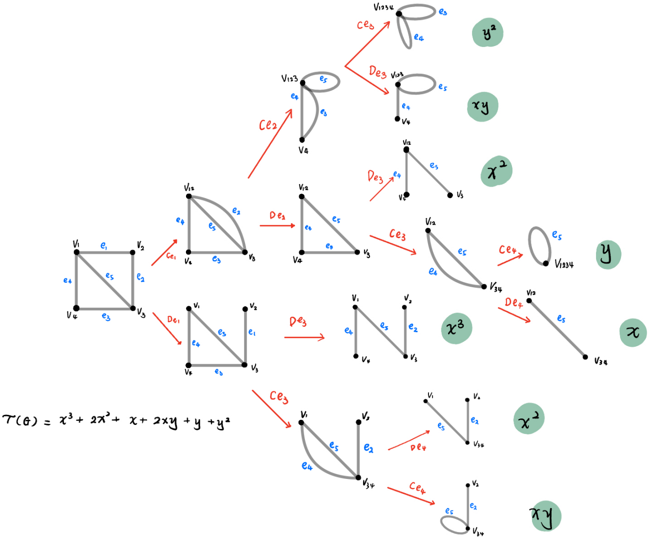
Activity Version

Theorem

Let $G = (V, E)$ be a graph with a fixed total ordering on the edges and $\mathcal{F}$ be the set of all maximal spanning forests of G. For any $F \in \mathcal{F}$ let $IA(F)$ be the set of internally active edges and $EA(F)$ the set of externally active edges. Then the Tutte Polynomial of G is:

$$T(x, y) = \sum_{F \in \mathcal{F}} x^{|IA(F)|} y^{|EA(F)|}$$

Example



Terminal Minor Version

Theorem

Let G be a graph the Tutte Polynomial is

$$T_G(x,y) = \begin{cases} 1, & \text{if } E = \varnothing \\ x^n y^m, & \text{if } G \text{ has } n \text{ bridge and } m \text{ loops} \\ T_{G \setminus e}(x,y) + T_{G/e}(x,y) & \text{if } e \in E(G) \text{ is neither a bridge nor a loop} \end{cases}$$

Flow Polynomial

Background

Definition

- For a graph G with edge set E let $\overrightarrow{E}$ be an orientation of the edges where $e = xy$ has orientation (e, x, y) and (e, y, x) describing moving from x to y and y to x respectively along e.
- A *capacity function* is $c : \overrightarrow{E} \to \mathbb{N}$

Flows

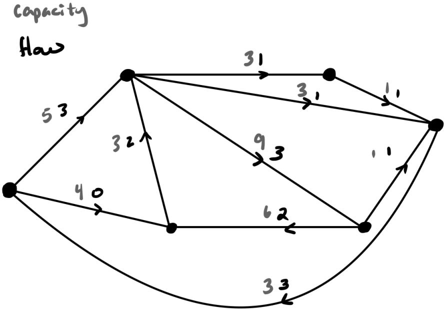
Definition

Let (G, c) be a graph with capacity function c. Then $f : \overrightarrow{E} \to \mathbb{R}$ is a *flow* if it satisfies the following

1. (Energy Reversal) $f(e, x, y) = -f(e, y, x)$ for all $(e, x, y) \in \overrightarrow{E}$ with $x \neq y$,
2. (Capacity Conservation) $f(\overrightarrow{e}) \leq c(\overrightarrow{e})$ for all $e \in \overrightarrow{E}$
3. (Vertex Conservation) The flow into a vertex is equal to the flow out of a vertex for all vertices.

A flow is *nowhere zero* if no edges receives a zero flow value.
A flow is a *k-flow* if every edge has $-k < f(\overrightarrow{e}) < k$.

Example



The Flow Polynomial

Definition

The *Flow Polynomial* of a graph G denoted $F(G:k)$ is the polynomial that evaluated at k is the number of nowhere zero k-flows

Theorem

The flow polynomial of a graph G satisfies

$$F(G;k) = \begin{cases} 1, & \text{if } E = \emptyset \\ 0, & \text{if } e \text{ is a bridge} \\ (k-1)F(G \setminus e;k), & \text{if } e \text{ is a loop} \\ F(G/e;k) - F(G \setminus e;k) & \text{if } e \in E(G) \text{ is neither a bridge nor a loop} \end{cases}$$

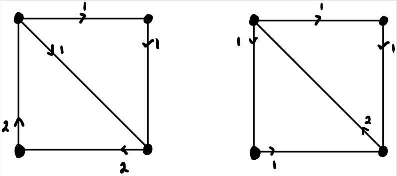
Relation to the Tutte Polynomial

Theorem

The flow polynomial is a specialization of the Tutte polynomial. For a graph G, $F(G;k) = (-1)^{n(G)}T(G;0,1-k)$ *where* $n(G) = |E(G)| - |V(G)| + |\{C \text{ is a connected component of } G\}|$.

Example

We found $T(G;x,y) = x^3 + 2x^2 + x + 2xy + y + y^2$ so
$F(G;k) = (-1)^{5-4+1}T(G;0,1-k) = k^2 - 3k + 2$.
Thus G has two nowhere zero 3-flows.



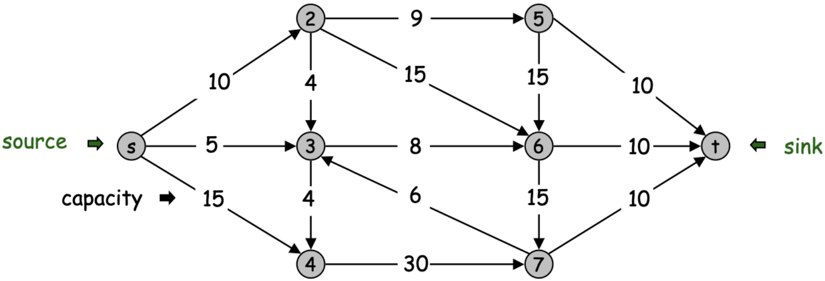
Applications

Maximum Flow Problem

Question

How to assign flows to edge as to

- *Equalize inflow and outflow at every intermediate vertex.*
- *Maximize flow sent from s to t.*



Airline Scheduling

Description:

- Manage flight crews by reusing them over multiple flights
- Consider each city as a vertices on the graph
- Flights time are flows

Running time:

- $O(k)$ nodes
- $O(k^2)$ edges
- At most k crews needed → Solve k max flow problems
- Overall time = $O(k^4)$

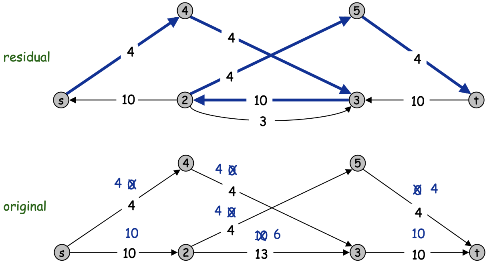
Augmenting Path Theorem

Definition

Augmenting path is the path in residual graph with capacity.

Theorem

A flow f is a max flow if and only if there are no augmenting paths.



Other Related Applications

- Network connectivity
- Bipartite matching
- Data mining
- Open-pit mining
- Image processing
- Project selection
- Baseball elimination
- Network reliability
- Security of statistical data
- Distributed computing
- Egalitarian stable matching
- Distributed computing

Reference

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Thank you for listening!